Homework #2

Temperature fluctuations in the lowest layer of Earth's atmosphere

The turbulent kinetic energy budget, which has been established during the course, is revisited to include temperature effects. The Navier-Stokes equation, and the energy conservation written for the specific energy e, are given by

$$\begin{cases} \dfrac{\partial(\rho \boldsymbol{u})}{\partial t} + \nabla\cdot(\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla\cdot\boldsymbol{\tau} + \rho \boldsymbol{g} \\ \dfrac{\partial(\rho e)}{\partial t} + \nabla\cdot(\rho e \boldsymbol{u}) = -\nabla\cdot\boldsymbol{q} + \rho\dot{\boldsymbol{q}}_v + \boldsymbol{\tau} : \nabla\boldsymbol{u} - p\nabla\cdot\boldsymbol{u} \end{cases}$$

where $de = c_v dT$ for a perfect gas, $\boldsymbol{q} = -\lambda\nabla T$ (Fourier's law) and $\dot{\boldsymbol{q}}_v$ denotes a possible volumetric heat source, that is assumed to be zero here, $\dot{\boldsymbol{q}}_v \equiv 0$. We recall that the term $\boldsymbol{\tau} : \nabla\boldsymbol{u} = \tau_{ij}\, \partial u_i/\partial x_j$ represents the viscous dissipation.

1. Derive the equation for the mean temperature $\bar{T}$ by introducing the Reynolds decomposition $T = \bar{T} + \theta$ for the temperature. Show that the total mean heat flux $\bar{\boldsymbol{q}}^t$ can be written as

$$\bar{q}_j^t = -\lambda\frac{\partial\bar{T}}{\partial x_j} + \rho c_v \overline{\theta u_j'} = -\rho c_v\left(\alpha\frac{\partial\bar{T}}{\partial x_j} - \overline{\theta u_j'}\right) \qquad \text{with} \qquad \alpha = \frac{\lambda}{\rho c_v}$$

   What does the coefficient α represent ? How could you simplified the expression of the total heat flux in turbulent regime ?

2. Temperature fluctuations θ are involved in the Navier-Stokes equation through the buoyancy force $\rho\boldsymbol{g}$, which can be developed from a Taylor series of the density around the mean temperature,

$$\rho\boldsymbol{g} \simeq (\bar{\rho} + \rho')\boldsymbol{g} \simeq \bar{\rho}(1 - \beta\theta)\boldsymbol{g} \qquad \text{with} \qquad \beta = \left(-\frac{1}{\rho}\frac{\partial\rho}{\partial T}\bigg|_p\right)_{\bar{\rho},\bar{T}}$$

   where β is the thermal expansion coefficient, $g_i = -g\delta_{3i}$ is the gravity with $g \simeq 9.81$ m.s^{-2}. *In the following, the Boussinesq approximation will be applied to the Navier-Stokes equation. The density fluctuations will be thus taken into account in the buoyancy force only.*

   Calculate the expression of the thermal expansion coefficient for a perfect gas, the additional term in the Navier-Stokes equation written for the fluctuating velocity component u_i' and also the additional term in the turbulent kinetic energy budget.

   Remark – Note that you can reuse without demonstration all the results mentioned in the slides of the course (by only citing the considered slide number for instance)

3. In order to quantify the importance of the buoyancy force effects, we can compare the additional term in the transport equation for k_t with the production term $\mathcal{P}$, by introducing the Richardson number Ri,

$$\text{Ri} = -\frac{\bar{\rho} g \overline{\theta u_3'}/\bar{T}}{\mathcal{P}} \qquad \text{où} \quad \mathcal{P} = -\bar{\rho}\, \overline{u_1' u_3'}\, \frac{\partial \bar{U}_1}{\partial x_3}$$

Show that if $d\bar{T}/dx_3 < 0$, the additional term $\bar{\rho}\ g\overline{\theta u_3'}/\bar{T}$ acts as a production term for k_t. Whats is the sign of the Richardson number ? Comment the dispersion of a pollutant for an unstable thermal stratification, that is when the temperature decreases with the altitude.

4. Repeat your analysis for the case of a stable thermal stratification.

5. A temperature inversion over the first few hundred meters is regularly observed in the atmospheric layer, leading to the trap of air pollution. Find a recent example of this phenomenon (internet, scientific paper for a large audience), where the temperature inversion is documented and comment the study in a few sentences.