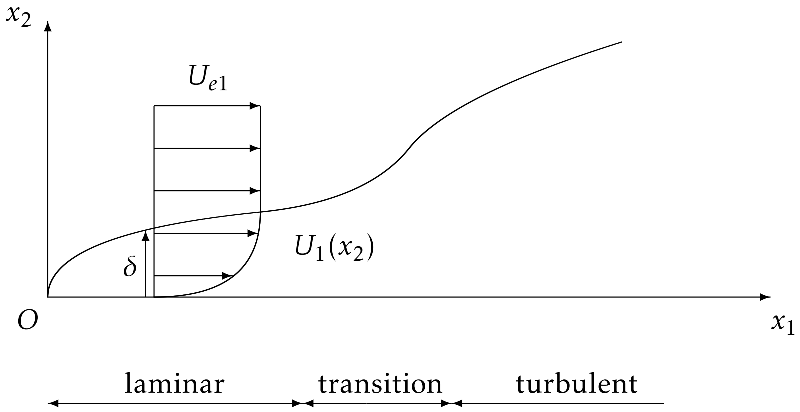
Homework #4

Velocity profile in laminar boundary layer

Consider a laminar two-dimensional boundary layer developing along a flat plate in the presence of a uniform and steady external flow U_{e1} without a pressure gradient.



Development of a zero-pression-gradient boundary layer on a flate plate. The transition occurs for $\mathrm{Re}_{x_1} = x_1 U_{e1}/\nu \simeq 3.2 \times 10^5$, that is $\mathrm{Re}_\delta = \delta U_{e1}/\nu \simeq 2800$.

1. Recall the governing equations for a laminar boundary layer (without demonstration) and the underlying assumptions.

2. Introduce the stream function ψ defined by $U_1 = \partial\psi/\partial x_2$ and $U_2 = -\partial\psi/\partial x_1$, and show that the problem consists in solving the equation,

$$\frac{\partial\psi}{\partial x_2}\frac{\partial^2\psi}{\partial x_1 \partial x_2} - \frac{\partial\psi}{\partial x_1}\frac{\partial^2\psi}{\partial x_2^2} = \nu\frac{\partial^3\psi}{\partial x_2^3}$$

associated with the following boundary conditions,

$$\left.\frac{\partial\psi}{\partial x_2}\right|_{x_2=0} = 0 \qquad \left.\frac{\partial\psi}{\partial x_1}\right|_{x_2=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial\psi}{\partial x_2}\right|_{x_2\to\infty} = U_{e1}$$

3. Recall briefly the reasoning that leads to $\delta \sim \sqrt{\nu x_1/U_{e1}}$. The self similarity variable η is then introduced,

$$\eta = x_2\sqrt{\frac{U_{e1}}{\nu x_1}} \qquad \text{and consequently,} \qquad \psi(x_1, x_2) = \sqrt{U_{e1}\nu x_1}\, f(\eta)$$

Show that the equation satisfied by f, known as the Blasius equation (1908), reads

$$2f''' + ff'' = 0$$

and provide the associated boundary conditions.

4. Solve numerically the previous equation by integrating the corresponding first-order system (using Matlab with *ode45.m* for example)

$$\begin{cases} f' = g \\ g' = h \\ h' = -\frac{1}{2} f h \end{cases} \tag{1}$$

over the interval $0 \leq \eta \leq 10$, with the following initial conditions at $\eta = 0$: $f(0) = 0$, $g(0) = 0$ and $h(0) = \alpha$. We then seek to determine the value of α in order to satisgy the boundary condition at $\eta = 10$, that is $N(\alpha) \equiv g - 1 = 0$. A simple and efficiency method is to use a Newton algorithm,

$$\alpha^{n+1} = \alpha^n - N(\alpha^n) / \left. \frac{\partial N}{\partial \alpha} \right|_{\alpha^n}$$

and to observe that the derivative $\partial N / \partial \alpha$ can be obtained by solving, in parallel to (1), the following variational system

$$\begin{cases} F' = G \\ G' = H \\ H' = -\frac{1}{2}(Fh + fH) \end{cases} \qquad \text{where} \qquad \begin{cases} F = \partial f / \partial \alpha \\ G = \partial g / \partial \alpha \\ H = \partial h / \partial \alpha \end{cases}$$

with the boundary conditions $F(0) = 0, G(0) = 0$ and $H(0) = 1$.

5. From the computation of the f function, deduce the following properties for a zero-pressure-gradient laminar boundary layer,

$$\delta \simeq 4.92 \sqrt{\frac{\nu x_1}{U_{e1}}} \qquad \delta^\star \simeq 1.72 \sqrt{\frac{\nu x_1}{U_{e1}}} \qquad \delta_\theta \simeq 0.664 \sqrt{\frac{\nu x_1}{U_{e1}}} \qquad c_f = \frac{\tau_w}{\frac{1}{2}\rho U_{e1}^2} = \frac{0.664}{\mathrm{Re}_{x_1}^{1/2}}$$

We recall that the displacement thickness $\delta^\star$ and the momentum thickness δ_θ are defined by

$$\delta^\star = \int_0^\infty \left(1 - \frac{U1}{U_{e1}}\right) dx_2 \qquad \delta_\theta = \int_0^\infty \frac{U_1}{U_{e1}} \left(1 - \frac{U_1}{U_{e1}}\right) dx_2$$

6. Calculate the expression of the transverse velocity U_2. What value does U_2 take when $\eta \to \infty$?

7. The following approximation has been used in class to model the developement of a turbulent boundary layer,

$$\int_0^{x_2} \left(U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} \right) dx_2 \simeq -\frac{x_2}{\delta} u_\tau^2$$

Assess this approximation for the Blasius solution.

8. Consider the development of a boundary layer on a commercial aircraft (Airbus A320 for instance) in cruise condition at an altitude of 10000 m

$$\mathrm{M} = 0.76 \qquad T = -50^{\mathrm{o}}\ \mathrm{C} \qquad P = 26500\ \ \mathrm{Pa}$$

Estimate the transition location and the characteristics of the boundary layer at this point.