Homework #5

Turbulent boundary layer model

The simplified equation governing the mean velocity profile of a turbulent boundary layer (introduced in classroom) is given by

$$(1+\nu_t^+)\,\frac{d\overline{U}_1^+}{dx_2^+} = 1 \tag{1}$$

1. Solve numerically this equation for the case of a mixing-length model $l_m^+ = \kappa x_2^+$ to compute the turbulent viscosity ν_t^+. The value of the von Kármán constant is chosen to be equal to $\kappa \simeq 0.38$. In order to integrate Eq. (1), we can analytically derive the expression for the derivative $d\overline{U}_1^+/dx_2^+$, and then numerically integrate this equation from $x_2^+ = 0$ using a Runge-Kutta algorithm (with Matlab, see reverse side).

2. Estimate the constant B of the log-law by computing the expression

$$B = \overline{U}_1^+ - \frac{1}{\kappa}\ln x_2^+$$

from your numerical solution for $x_2^+ = 200, 300, 400$ and 500. Comment on this result.

3. Compare your numerical solution to the exact analytical solution of Eq. (1), provided by Hinze[1]

$$U_1^+ = \frac{1}{\kappa}\,\frac{1-\sqrt{1+4\,(\kappa x_2^+)^2}}{2\,\kappa x_2^+} + \frac{1}{\kappa}\ln\left[2\kappa x_2^+ + \sqrt{1+4\,(\kappa x_2^+)^2}\right]$$

4. Repeat the numerical integration with the following mixing-length model including a dumping function, proposed by Van Driest[2]

$$l_m^+ = \kappa x_2^+\left(1-e^{-x_2^+/A_0^+}\right) \qquad \text{with} \qquad A_0^+ = 26$$

5. Plot on the same graph both numerical solutions, as well as the viscous sublayer and the inner log laws. Comment.

6. Perform a Taylor series of the functions l_m^+ and $-\overline{u_1' u_2'}^+$ for both mixing-length models, and provide the behaviour of these two functions as $x_2^+ \to 0$.

7. By considering the incompressibility condition $\nabla\cdot\boldsymbol{u}' = 0$, provide the theoretical behaviour of $-\overline{u_1' u_2'}^+$ as $x_2^+ \to 0$.

8. Repeat the numerical integration using the Van Driest model for the following equation,

$$(1+\nu_t^+)\,\frac{d\overline{U}_1^+}{dx_2^+} = 1 - \frac{x_2^+}{\mathrm{Re}^+} \qquad 0 \le x_2^+ \le \mathrm{Re}^+$$

where the term x_2/δ is now included, to describe the mean velocity profile up the edge of the boundary layer. The following numerical values, determined from experiments, can be used : $U_{e1} = 100$ m.s^{-1}, $\delta = 2$ cm, $u_\tau = 4$ m.s^{-1} and $\nu = 1.5\times10^{-5}$ m^2.s^{-1}.

Références

[1] Hinze, J. O., 1975, *Turbulence*, McGraw-Hill International Book Company, New York.

[2] Van Driest, E. R., 1956, On turbulent flow near a wall, *Journal of Aeronautical Sciences*, **23**(11), 1007-1011.

Annexe

In order to integrate ordinary differential equation, you can directly used the *ode45.m* function from Matlab. You can also write your own script from the following 4th-order Runge-Kutta algorithm. To integrate the first-order differential equation $\partial \boldsymbol{U}/\partial t = \boldsymbol{F}(\boldsymbol{U}, t)$, consider

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \Delta t \left(b_1 \boldsymbol{K}^1 + b_2 \boldsymbol{K}^2 + b_3 \boldsymbol{K}^3 + b_4 \boldsymbol{K}^4 \right)$$

where

$$\begin{cases} \boldsymbol{K}^1 = \boldsymbol{F}(\boldsymbol{U}^n, t^n) \\ \boldsymbol{K}^2 = \boldsymbol{F}(\boldsymbol{U}^n + a_{21}\boldsymbol{K}^1, t^n + c_2 \Delta t) \\ \boldsymbol{K}^3 = \boldsymbol{F}(\boldsymbol{U}^n + a_{32}\boldsymbol{K}^2, t^n + c_3 \Delta t) \\ \boldsymbol{K}^4 = \boldsymbol{F}(\boldsymbol{U}^n + a_{43}\boldsymbol{K}^3, t^n + c_4 \Delta t) \end{cases}$$

and with

$$\begin{array}{c|c} c_i & a_{ij} \\ \hline & b_i \end{array} \qquad\qquad \begin{array}{l|cccc} c_1 = 0 & 0 & & & \\ c_2 = 1/2 & 1/2 & & & \\ c_3 = 1/2 & 0 & 1/2 & & \\ c_4 = 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$