Homework #6

Some properties of the $k_t - \epsilon$ turbulence model

We consider some properties of the $k_t - \epsilon$ turbulence model, widely used in industry for solving the averaged Navier-Stokes equations. The governing equations are first recalled. For an incompressible flow, the conservation of mass and of momentum are given by

$$\frac{\partial \overline{U}_i}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial(\rho\overline{U}_i)}{\partial t} + \frac{\partial(\rho\overline{U}_i\,\overline{U}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(\overline{P} + \frac{2}{3}\rho k_t\right) + \frac{\partial}{\partial x_j}\left[2(\mu+\mu_t)\overline{S}_{ij}\right] \tag{2}$$

where the mean deformation tensor reads $\overline{S}_{ij} = (\partial\overline{U}_i/\partial x_j + \partial\overline{U}_j/\partial x_i)/2$. The Reynolds tensor is closed thanks to the introduction of a turbulent viscosity μ_t,

$$-\rho\overline{u_i'u_j'} = 2\mu_t\overline{S}_{ij} - \frac{2}{3}\rho k_t\delta_{ij} \tag{3}$$

and this flow dependent viscosity is calculated from the turbulent kinetic energy k_t and its dissipation rate ϵ as follows,

$$\mu_t = \rho C_\mu \frac{k_t^2}{\epsilon} \tag{4}$$

Two additional transport equations are solved to compute k_t and $\epsilon \simeq \epsilon^h$ (the superscript h will be disregard to simplify notations) for a high Reynolds number flow,

$$\left\{\begin{array}{l} \dfrac{\partial(\rho k_t)}{\partial t} + \dfrac{\partial(\rho k_t\overline{U}_i)}{\partial x_i} = \dfrac{\partial}{\partial x_j}\left[\left(\mu + \dfrac{\mu_t}{\sigma_k}\right)\dfrac{\partial k_t}{\partial x_j}\right] + \mathcal{P} - \rho\epsilon \\ \dfrac{\partial(\rho\epsilon)}{\partial t} + \dfrac{\partial(\rho\epsilon\overline{U}_i)}{\partial x_i} = \dfrac{\partial}{\partial x_j}\left[\left(\mu + \dfrac{\mu_t}{\sigma_\epsilon}\right)\dfrac{\partial \epsilon}{\partial x_j}\right] + \dfrac{\epsilon}{k_t}\left(C_{\epsilon 1}\mathcal{P} - C_{\epsilon 2}\rho\epsilon\right) \end{array}\right. \tag{5}$$

where $\mathcal{P}$ is the production term of the turbulent kinetic energy k_t

$$\mathcal{P} = -\rho\overline{u_i'u_j'}\frac{\partial\overline{U}_i}{\partial x_j} \tag{6}$$

In this model, five calibration constants must be determined. For the standard formulation, they are given by

$$C_\mu = 0.09 \quad C_{\epsilon 1} = 1.44 \quad C_{\epsilon 2} = 1.92 \quad \sigma_k = 1.0 \quad \sigma_\epsilon = 1.3 \tag{7}$$

1. Production term

1. Show that the production term can be expressed as $\mathcal{P} = 2\mu_t\bar{S}_{ij}^2$

2. By considering the function $f(\lambda) = \overline{(u_i' + \lambda u_j')^2}$, show that the Reynolds tensor must satisfy Schwarz's inequality, that is

$$\overline{u_i'u_j'}^2 \leq \overline{u_i'^2}\;\overline{u_j'^2}$$

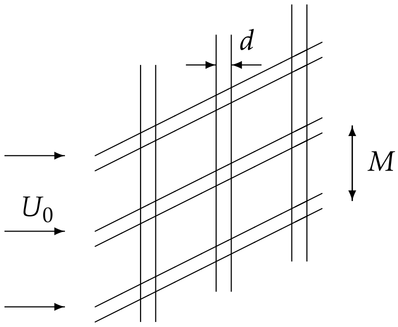
3. For a two-dimensional mean shear flow, $\overline{U}_1 = \overline{U}_1(x_2)$ and $\overline{U}_2 = \overline{U}_3 = 0$, show that the $k_t - \epsilon$ model imposes the following relationship

$$9C_\mu^2\,\bar{S}_{12}^2 \le \frac{\epsilon^2}{k_t^2} \sim \omega_t^2$$

How can one interpret ω_t ?

2. Determination of the C_{ϵ_2} constant

For grid-generated homogeneous turbulence, the production term $\mathcal{P} \equiv 0$ is zero and turbulence is simply convected at a constant velocity $\overline{U}_1 = U_0$ and decays behind the grid. The two transport equations of the system (5) are reduced to



$$\begin{cases} U_0 \dfrac{\partial k_t}{\partial x_1} \simeq -\epsilon \\ U_0 \dfrac{\partial \epsilon}{\partial x_1} \simeq -C_{\epsilon 2}\dfrac{\epsilon^2}{k_t} \end{cases}$$

4. Show that the solution to this system of ordinary differential equations can be recast in the following form,

$$k_t = k_{t0}\left[1 + (C_{\epsilon_2} - 1)\frac{\epsilon_0}{k_0}\frac{x_1}{U_0}\right]^{-\frac{1}{C_{\epsilon_2}-1}} \qquad \text{and} \qquad \frac{\epsilon}{\epsilon_0} = \left(\frac{k_t}{k_{t0}}\right)^{C_{\epsilon_2}}$$

where $k_t = k_{t0}$ and $\epsilon = \epsilon_0$ at $x_1 = 0$.

5. Experimentally, refer to the measurements by Comte-Bellot & Corrsin (1966) among others, the following decaying law is observed $(k/k_0) \sim (t/t_0)^{-n}$ with $n \simeq 1.3$, in a frame convected at the velocity vitesse U_0, consequently $t = x_1/U_0$. Deduce a numerical estimate of the constant $C_{\epsilon 2}$.

6. Provide the evolution of the length scale $L = k_t^{3/2}/\epsilon$? and give an interpretation of this scale.

3. Calibration of the C_μ and C_{ϵ_1} constants

For turbulent shear flow, in particular in the log-law of a turbulent boundary layer where $\mathcal{P} \simeq \rho\epsilon$, refer to Bradshaw *et al.* (1967), one also observes that

$$-\frac{\overline{u_1' u_2'}}{k_t} \simeq 0.30$$

To determine the value of C_μ, consider the turbulent boundary layer established in a plane channel flow, with $\overline{U}_1 = \overline{U}_1(x_2), \overline{U}_2 = \overline{U}_3 = 0$.

6. By examining the balance $\mathcal{P} \simeq \rho\epsilon$, show that

$$-\frac{\overline{u_1' u_2'}}{k_t} = C_\mu^{1/2}$$

and provide a numerical estimate of C_μ.

7. Provide the expressions of $d\overline{U}_1/dx_2$ and ϵ inside the log-law of a turbulent boundary layer. Show also that $\nu_t \simeq \kappa u_\tau x_2$ and that $k_t \simeq u_\tau^2/\sqrt{C_\mu}$.

8. By noting that the two differential equations of the system (5) simplify in the considered flow to

$$0 = \mathcal{P} - \rho\epsilon \qquad \text{and} \qquad 0 = \frac{d}{dx_2}\left[\left(\frac{\mu_t}{\sigma_\epsilon}\right)\frac{d\epsilon}{dx_2}\right] + \frac{\epsilon}{k_t}\left(C_{\epsilon 1}P - C_{\epsilon 2}\rho\epsilon\right)$$

show that the compatibility relation must be satisfied,

$$\sigma_\epsilon C_\mu^{1/2}\left(C_{\epsilon_2} - C_{\epsilon_1}\right) = \kappa^2$$

where κ is the von Kármán constant. Comment this result.