Homework #8

We consider some mathematical properties of the spectral tensor ϕ_{ij}, as well as some remarkable expressions for the dissipation ϵ for homogeneous and isotropic turbulence.

Properties of the spectral tensor ϕ_{ij} for homogenous turbulence

1. Recall the definition of the spectral tensor ϕ_{ij}

2. Show that this tensor satisfies the hermitian symmetry relation, that is

$$\phi_{ij}(-\boldsymbol{k}) = \phi^{\star}_{ij}(\boldsymbol{k}) \qquad \phi_{ij}(\boldsymbol{k}) = \phi^{\star}_{ji}(\boldsymbol{k})$$

where * stands for the complex conjugate. Hint – Note that correlation functions are real functions, and that turbulence is homogeneous.

3. Show that for incompressible homogeneous turbulence,

$$k_i \phi_{ij}(\boldsymbol{k}) = k_j \phi_{ij}(\boldsymbol{k}) = 0$$

or equivalently in the physical space,

$$\frac{\partial R_{ij}(\boldsymbol{r})}{\partial r_i} = \frac{\partial R_{ij}(\boldsymbol{r})}{\partial r_j} = 0$$

4. By taking into account the previous properties, provide the general form of the spectral tensor ϕ_{ij}. Show that this tensor can be expressed with only four independent scalar real functions.

Expressions of the dissipation for homogeneous and isotropic turbulence

The dissipation rate of the turbulent kinetic energy is defined as

$$\rho\epsilon = \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}}$$

In this exercice, we consider alternative expressions of the dissipation, to easily estimate numerically or experimentally this quantity, or to provide particular physical meaning for ϵ. Assumptions used at each step of the calculations must be carefully specified.

1. Show that dissipation can be expressed as follows,

$$\rho\epsilon = \overline{\tau'_{ij}s'_{ij}} = \frac{1}{2}\overline{\tau'_{ij}\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)}$$

2. Show that for incompressible turbulence, the dissipation can be recast as

$$\epsilon = \frac{1}{2}\nu\overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)^2} = \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \nu\frac{\partial^2 \overline{u_i' u_j'}}{\partial x_i \partial x_j}$$

Deduce that for homogeneous turbulence, the dissipation is exactly equal to ϵ^h

$$\epsilon = \epsilon^h \equiv \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

3. Show that for homogeneous incompressible turbulence, the dissipation can be written as a function of vorticity components,

$$\epsilon = \frac{1}{2}\nu\overline{\left(\frac{\partial u_i'}{\partial x_j} - \frac{\partial u_j'}{\partial x_i}\right)^2} = \nu\overline{\omega_i'^2}$$

4. Show that for isotropic turbulence, the incompressibility condition and the Kàrmàn & Howarth relation leads to $g = f + (r/2)f'$. Deduce the following relation for Taylor scales, $\lambda_f = \sqrt{2}\lambda_g$.

Then, show that for isotropic turbulence,

$$\epsilon = 3\nu\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} + 6\nu\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2}$$

and finally, deduce that

$$\epsilon = \frac{15}{2}\nu\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2} = 15\nu\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2} \qquad \text{and} \qquad \epsilon = 15\nu\frac{\overline{u'^2}}{\lambda_g^2} = 30\nu\frac{\overline{u'^2}}{\lambda_f^2}$$

5. Show that for isotropic turbulence, the dissipation can be expressed from the one-dimensional velocity spectrum as

$$\epsilon = 30\nu\int_0^\infty k_1^2 E_{11}^{(1)}(k_1)\,dk_1$$