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On an initial-boundary value problem for a wide-angle parabolic equation in a waveguide with a variable bottom

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SUMMARY

We consider the third-order Claerbout-type wide-angle parabolic equation (PE) of underwater acoustics in a cylindrically symmetric medium consisting of water over a soft bottom B of range-dependent topography. There is strong indication that the initial-boundary value problem for this equation with just a homogeneous Dirichlet boundary condition posed on B may not be well-posed, for example when B is downsloping. We impose, in addition to the above, another homogeneous, second-order boundary condition, derived by assuming that the standard (narrow-angle) PE holds on B, and establish *a priori* H^2 estimates for the solution of the resulting initial-boundary value problem for any bottom topography. After a change of the depth variable that makes B horizontal, we discretize the transformed problem by a second-order accurate finite difference scheme and show, in the case of upsloping and downsloping wedge-type domains, that the new model gives stable and accurate results. We also present an alternative set of boundary conditions that make the problem exactly energy conserving; one of these conditions may be viewed as a generalization of the Abrahamsson–Kreiss boundary condition in the wide-angle case. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS: wide-angle parabolic equation; underwater acoustics; initial-boundary value problems; variable bottom topography; finite difference methods; partial differential equations; numerical analysis

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1. INTRODUCTION

In this paper we shall study an initial-boundary value problem (ibvp) for the third-order partial differential equation (pde) that represents a wide-angle parabolic approximation to the Helmholtz equation, when the latter is written in cylindrical coordinates, in the presence of azimuthal symmetry. The problem will be posed in a domain whose boundary varies with the time-like independent variable.

Parabolic equations (PEs) have long been in use as approximations of outgoing solutions of the Helmholtz equation in waveguides in the far-field, paraxial regime [1, 2]. (In this paper we have in mind their applications in underwater acoustics.) We shall consider a simple wide-angle extension of the 'standard' PE of [2]. Wide-angle equations [1, 3] provide a more accurate description of the long-range acoustic field, in that they suffer from smaller discrepancies in approximating the Helmholtz equation and provide better simulation of propagating modes that interact strongly with bottom layers.

The wide-angle PE that will be studied here is

$$\left[1+q\beta(z,r)+\frac{q}{k_0^2}\partial_z^2\right]v_r=\mathrm{i}(p-q)k_0\left[\frac{1}{k_0^2}v_{zz}+\beta(z,r)v\right] \tag{1}$$

where $v=v(z,r)$ is a complex-valued function of the depth z and range r (distance from the source), representing the acoustic field generated in a single-layer fluid medium ('water') by a time-harmonic point source of frequency f placed on the z-axis, cf. Figure 1(a). In (1) $k_0=2\pi f/c_0$ is a reference wave number associated with a constant reference speed of sound c_0, and $\beta=\beta(z,r)$ is a smooth function given by $\beta(z,r)=n^2(z,r)-1$, where $n(z,r)$ is the range-dependent index of refraction defined as $(c_0/c(z,r))(1+\mathrm{i}\delta)$, where $c(z,r)$ is the speed of sound in the medium and δ is a small nonnegative quantity intended to model attenuation of sound in the water column. Consequently, we will assume that the function β is complex-valued with nonnegative imaginary part. The coefficients p, q are such that the rational function $(1+px)/(1+qx)$ is an approximation to $\sqrt{1+x}$ near $x=0$. The choice $p=1/2$, $q=0$ yields the standard (narrow-angle) PE [2], while the $(1,1)$-Padé approximation to $\sqrt{1+x}$, given by $p=\frac{3}{4}$, $q=\frac{1}{4}$, yields the Claerbout equation [3]. In general, we shall take p and q complex [4], $p-q=\frac{1}{2}$ and $\mathrm{Im}\,q\leqslant 0$; such a choice has certain theoretical and numerical advantages, as will be seen in the sequel.

The pde (1) is obtained as an approximation to the pseudo-differential operator equation, whose formal steps are outlined e.g. in [1, 5], wherein the expression $\sqrt{1+x}$ is approximated by the

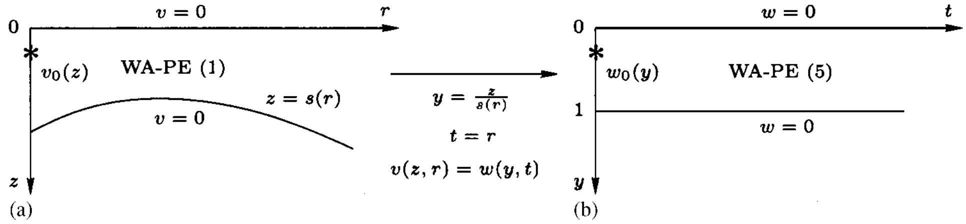

Figure 1. Domains of the ibvp's for (a) the wide-angle PE (1) and (b) the transformed wide-angle PE (5).

rational function $(1+px)/(1+qx)$. It may be viewed as the first-order member of a family of *higher-order* wide-angle equations obtained by approximating $\sqrt{1+x}$ by rational functions with numerator and denominator of higher degree [1, 6], and which have found widespread use in underwater acoustics [7]. In addition, three-dimensional extensions of (1) and its higher-order analogs for use in problems with azimuthal dependence have received a lot of attention in recent years [7–10].

We shall pose (1) in a single fluid layer of constant density occupying the domain $0\leqslant z\leqslant s(r)$, $r\geqslant 0$, where $z=0$ is the free surface and $z=s(r)$ is a positive smooth function representing a bottom of variable topography, cf. Figure 1(a). We supplement (1) by an initial condition modelling the source

$$v(z,0)=v_0(z), \quad 0\leqslant z\leqslant s(0) \tag{2}$$

where $v_0(0)=v_0(s(0))=0$, and the homogeneous Dirichlet boundary conditions

$$v(0,r)=0, \quad v(s(r),r)=0, \quad r\geqslant 0 \tag{3}$$

modelling, respectively, the behavior of the acoustic field at the free surface and at the bottom (considered to be acoustically soft). In the case of a horizontal bottom $s(r)=s(0)$, $r\geqslant 0$, it is known that the ibvp (1)–(3) is well-posed, provided that the operator $1+q\beta+(q/k_0^2)\partial_z^2$, acting on v_r in the left-hand side of (1), is invertible under the boundary conditions (3), cf. [11, 12]. (As will be seen in Lemma 2.3 in the sequel, invertibility of this operator is guaranteed e.g. if $\mathrm{Im}\, q<0$ or if $\mathrm{Im}\,\beta\neq 0$ or if $k_0 s(0)$ is sufficiently small.)

When the problem is posed in a waveguide with range-dependent bottom $z=s(r)$, one may perform the change of variables

$$y=z/s(r), \quad t=r, \quad w(y,t)=v(z,r) \tag{4}$$

which maps the bottom onto the horizontal line $y=1$, cf. Figure 1(b), and the domain $0\leqslant z\leqslant s(r)$, $r\geqslant 0$, onto the strip $0\leqslant y\leqslant 1$, $t\geqslant 0$. The wide-angle PE (1) is then transformed to

$$\begin{aligned}&\left[1+q\gamma(y,t)+\frac{q}{k_0^2 s^2(t)}\partial_y^2\right]\left(w_t-y\frac{\dot{s}(t)}{s(t)}w_y\right)\\ &\quad =\mathrm{i}(p-q)k_0\left[\frac{1}{k_0^2 s^2(t)}w_{yy}+\gamma(y,t)w\right], \quad 0\leqslant y\leqslant 1, \quad t\geqslant 0\end{aligned} \tag{5}$$

where $\dot{s}=\mathrm{d}s/\mathrm{d}t$, $\gamma(y,t):=\beta(ys(t),t)$. The initial condition (2) becomes

$$w(y,0)=w_0(y):=v_0(ys(0)), \quad 0\leqslant y\leqslant 1 \tag{6}$$

while the homogeneous Dirichlet boundary conditions (3) are preserved

$$w(0,t)=0, \quad w(1,t)=0, \quad t\geqslant 0 \tag{7}$$

The pde (5) has now, in addition to w_{yyt}, the third-order derivative term w_{yyy} and, since the problem is posed in the finite y-interval $[0,1]$, an extra boundary condition may be required for well-posedness. We shall propose in the sequel one such extra boundary condition at the bottom. We point out that the need for analogous extra conditions at *interfaces* for the wide-angle equation

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posed in multi-layered waveguides with range-dependent interfaces has already been recognized by Godin [13]. (No extra conditions are required when the interfaces are horizontal; in this case the standard interface conditions of acoustics suffice for well-posedness [12].) To motivate the particular extra boundary condition that we shall use, consider for the moment the associated standard (narrow-angle) PE, i.e. (1) with $p=\frac{1}{2}$, $q=0$, in the original, variable domain $0\leqslant z\leqslant s(r)$, $r\geqslant 0$, i.e.

$$u_r=\frac{\mathrm{i}}{2k_0}u_{zz}+\frac{\mathrm{i}k_0}{2}\beta(z,r)u \tag{8}$$

posed with the bottom boundary condition

$$u(s(r),r)=0, \quad r\geqslant 0 \tag{9}$$

Assuming, using continuity, that (8) holds at the bottom $z=s(r)$, differentiating with respect to r the boundary condition (9) to obtain $u_r(s(r),r)+\dot{s}(r)u_z(s(r),r)=0$, and using this last equation in (8) at $z=s(r)$, yields that

$$\dot{s}(r)u_z(s(r),r)+\frac{\mathrm{i}}{2k_0}u_{zz}(s(r),r)=0$$

holds for $r\geqslant 0$. This is not, of course, an additional boundary condition for the PE (8), but merely a compatibility relation that holds at the smooth boundary $z=s(r)$. However, it may serve as an additional boundary condition for the higher-order wide-angle PE (1). Under the change of variables (4), this condition becomes

$$w_{yy}(1,t)=2\mathrm{i}k_0 s(t)\dot{s}(t)w_y(1,t), \quad t\geqslant 0 \tag{10}$$

In Section 2 of the paper at hand we study the ibvp's (5)–(7) and (5)–(7), (10) on finite 'temporal' intervals $[0,T]$. For the first ibvp, i.e. in the presence of only the two boundary conditions (7), we show, if q and γ are real-valued, $\dot{s}(t)\leqslant 0$ in $[0,T]$, i.e. in the case of *upsloping* bottoms, and if $k_0\max_{0\leqslant t\leqslant T}s(t)$ is sufficiently small (i.e. under a small-frequency/'shallow-water' assumption), that an *a priori* H^1 estimate holds for the solution of (5)–(7) on $[0,T]$. This implies uniqueness of solutions of this ibvp under the stated assumptions. When the third boundary condition is added, we show, for complex-valued q and γ, and general bottom profiles $s(t)$, that an *a priori* H^2 estimate holds on $[0,T]$ for the solution of the ibvp (5)–(7), (10), provided the operator $1+q\gamma(y,t)+(q/(k_0^2 s^2(t)))\partial_y^2$ (that occurs in the left-hand-side of (5)) is invertible under the boundary conditions (7). In Lemma 2.3 we prove that this holds if, for example, $\mathrm{Im}\,q<0$ or $\mathrm{Im}\,\gamma\neq 0$ or, in case q and γ are real, if $k_0\max_{0\leqslant t\leqslant T}s(t)$ is sufficiently small. This *a priori* H^2 estimate is the basic step on which a proof of well-posedness of this ibvp may be constructed. In Section 3 we derive two finite difference schemes of second order of accuracy in y and t for discretizing the two ibvp's mentioned above. We demonstrate, by means of numerical experiment, that the finite difference scheme approximating the solution of the ibvp (5)–(7) (called FDI) is stable and convergent in domains with upsloping bottom but that it does not converge in downsloping bottom cases. On the other hand, if the third boundary condition (10) is also discretized and incorporated into the numerical method, then the new scheme (called FDII) is stable and convergent in the presence of downsloping and also for general bottom profiles. In Section 4 we consider some realistic underwater acoustic problems, including the upsloping and downsloping ASA wedge

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benchmarks [14], and compare the results of FDII with those obtained by a wide-angle finite difference PE code that approximates the solutions of the ibvp (1)–(3) in the original physical domain $0\leqslant z\leqslant s(r)$, $r\geqslant 0$, using a 'staircase' (piecewise constant) discretization of the bottom. We observe good agreement between the two codes; this is a strong indication that the new boundary condition put forth in this paper furnishes a physically acceptable solution to the ibvp under consideration. We close the paper with a section of conclusions, in which, among others, we point out alternative boundary conditions that may be used in place of (10) to some advantage, and which are the subject of ongoing investigation. A preliminary version of the present paper appeared in the proceedings of the *Eighth European Conference on Underwater Acoustics* [15].

2. *A PRIORI* ESTIMATES

For ease of exposition, we re-write here the ibvp's under consideration. (In (5) we take $p-q=\frac{1}{2}$.) We seek a complex-valued function $w=w(y,t)$, defined for $(y,t)\in[0,1]\times[0,T]$ for some $0<T<\infty$, satisfying the wide-angle PE

$$\left[1+q\gamma(y,t)+\frac{q}{k_0^2 s^2(t)}\partial_y^2\right]\left(w_t-y\frac{\dot{s}(t)}{s(t)}w_y\right) = \frac{\mathrm{i}k_0}{2}\left[\frac{1}{k_0^2 s^2(t)}w_{yy}+\gamma(y,t)w\right],\quad 0\leqslant y\leqslant 1,\ \ 0\leqslant t\leqslant T \tag{11}$$

We supplement (11) with the initial condition

$$w(y,0)=w_0(y),\quad 0\leqslant y\leqslant 1 \tag{12}$$

and the homogeneous Dirichlet boundary conditions

$$w(0,t)=0,\quad w(1,t)=0,\quad 0\leqslant t\leqslant T \tag{13}$$

We also consider the additional boundary condition

$$w_{yy}(1,t)=2\mathrm{i}k_0 s(t)\dot{s}(t)w_y(1,t),\quad 0\leqslant t\leqslant T \tag{14}$$

We write $q=q_\mathrm{R}+\mathrm{i}q_\mathrm{I}$, $\gamma(y,t)=\gamma_\mathrm{R}(y,t)+\mathrm{i}\gamma_\mathrm{I}(y,t)$ to denote the real and imaginary parts of q and γ. We assume that $q_\mathrm{R}>0$, $q_\mathrm{I}\leqslant 0$, $\gamma_\mathrm{I}\geqslant 0$, that $s(t)>0$, $t\in[0,T]$, and that the functions γ, w_0, s are smooth enough as required by the estimates to be derived below. In the sequel we let $(f,g)=\int_0^1 f\overline{g}\,\mathrm{d}y$ denote the inner product on $L^2=L^2(0,1)$ and $\|\cdot\|$ the associated norm. For positive integers k, we denote by $H^k=H^k(0,1)$ the usual (complex) Sobolev spaces of order k with associated norm $\|\cdot\|_k$. The symbol C will denote generic constants, not necessarily the same in any two places. In the present section we shall derive, under certain restrictive hypotheses on the data, an *a priori* H^1 estimate for the solution of the ibvp (11)–(13), and, under minimal hypotheses, an H^2 estimate for the ibvp (11)–(14). We start by establishing an energy identity that the solution of (11)–(12) satisfies when only the two boundary conditions (13) are present.

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Lemma 2.1
If w satisfies (11)–(13), then, for $0\leqslant t\leqslant T$,

$$\frac{|q|^2}{2k_0^2s^2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_y\|^2-\frac{1}{2k_0^2s^3}[|q|^2\dot{s}|w_y(1,t)|^2+(|q|^2\dot{s}+q_{\mathrm{I}}k_0s)\|w_y\|^2]$$
$$-\int_0^1(q_{\mathrm{I}}-|q|^2\gamma_{\mathrm{I}})\,\mathrm{Im}(w_t\overline{w})\,\mathrm{d}y-\int_0^1(q_{\mathrm{R}}+|q|^2\gamma_{\mathrm{R}})\,\mathrm{Re}(w_t\overline{w})\,\mathrm{d}y$$
$$=-\mathrm{Re}\left((1+q\gamma)y\frac{\dot{s}}{s}w_y,qw\right)-\mathrm{Re}\left(\frac{\mathrm{i}k_0}{2}\gamma w,qw\right)\tag{15}$$

Proof
Take the real part of the L^2 inner product of both sides of (11) with qw to obtain

$$\mathrm{Re}((1+q\gamma)w_t,qw)+\frac{|q|^2}{k_0^2s^2}\mathrm{Re}(w_{yyt},w)-\mathrm{Re}\left((1+q\gamma)y\frac{\dot{s}}{s}w_y,qw\right)$$
$$-\frac{|q|^2\dot{s}}{k_0^2s^3}\mathrm{Re}(\partial_y^2(yw_y),w)-\mathrm{Re}\left[\frac{\mathrm{i}\overline{q}}{2k_0s^2}(w_{yy},w)\right]-\mathrm{Re}\left(\frac{\mathrm{i}k_0}{2}\gamma w,qw\right)=0\tag{16}$$

We have

$$\mathrm{Re}((1+q\gamma)w_t,qw)=\mathrm{Re}\int_0^1(\overline{q}+|q|^2\gamma)w_t\overline{w}\,\mathrm{d}y$$
$$=\int_0^1(q_{\mathrm{R}}+|q|^2\gamma_{\mathrm{R}})\,\mathrm{Re}(w_t\overline{w})\,\mathrm{d}y+\int_0^1(q_{\mathrm{I}}-|q|^2\gamma_{\mathrm{I}})\,\mathrm{Im}(w_t\overline{w})\,\mathrm{d}y$$

In addition, using the boundary conditions (13) we obtain by integrating by parts

$$(w_{yy},w)=-\|w_y\|^2$$
$$\mathrm{Re}\,(w_{yyt},w)=-\mathrm{Re}\,(w_{yt},w_y)=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_y\|^2$$
$$\mathrm{Re}\,(\partial_y^2(yw_y),w)=-\mathrm{Re}\,(\partial_y(yw_y),w_y)=-\tfrac{1}{2}(\|w_y\|^2+|w_y(1,t)|^2)$$

Substituting these expressions in (16), we obtain (15). □

The presence of the $\mathrm{Im}\,(w_t\overline{w})$ term in (15) prevents deriving an H^1 estimate of w from (15). However, if $q_{\mathrm{I}}=0$ and $\gamma_{\mathrm{I}}=0$, the integral containing this term vanishes and one may obtain an H^1 estimate in the upsloping bottom case, under a small-frequency/shallow-water assumption, as the following proposition shows.

Proposition 2.1
Suppose w satisfies (11)–(13), and that $q_{\mathrm{I}}=0$, $\gamma_{\mathrm{I}}=0$, $\dot{s}(t)\leqslant 0$ for $t\in[0,T]$, and $k_0\max_{t\in[0,T]}s(t)$ is sufficiently small. Then there exists a constant C such that for $0\leqslant t\leqslant T$ we have

$$\|w_y\|\leqslant C\|w_0'\|\tag{17}$$

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Proof
Under our assumptions, (15) yields

$$\frac{q_{\mathrm{R}}}{2k_0^2 s^2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_y\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(1+q_{\mathrm{R}}\gamma_{\mathrm{R}})|w|^2\,\mathrm{d}y+\frac{1}{2}\int_0^1 q_{\mathrm{R}}\gamma_{\mathrm{R},t}|w|^2\,\mathrm{d}y$$
$$\leqslant -\frac{\dot{s}}{s}\int_0^1(1+q_{\mathrm{R}}\gamma_{\mathrm{R}})y\,\mathrm{Re}\,(w_y\overline{w})\,\mathrm{d}y$$

Therefore, using Poincaré's inequality $\|w\|\leqslant\|w_y\|$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_y\|^2-\frac{k_0^2 s^2}{q_{\mathrm{R}}}\int_0^1(1+q_{\mathrm{R}}\gamma_{\mathrm{R}})|w|^2\,\mathrm{d}y\right]\leqslant C\|w_y\|^2,\quad 0\leqslant t\leqslant T$$

Integrating both sides of the above inequality with respect to t, and using Poincaré's inequality yields, for $0\leqslant t\leqslant T$,

$$(1-Ck_0^2s^2)\|w_y\|^2\leqslant C\left[\|w_0'\|^2+\int_0^t\|w_y(\tau)\|^2\,\mathrm{d}\tau\right]$$

Hence, if $k_0\max_{t\in[0,T]}s(t)$ is sufficiently small, (17) follows from the above inequality and Gronwall's lemma. □

The estimate (17) and Poincaré's inequality show, in particular, that the solution of (11)–(13) for real q and γ in upsloping, low-frequency/shallow-water problems is unique.

We proceed now to establish a basic identity involving w_{yy} that the solution of the ibvp (11)–(14), i.e. with all three boundary conditions, satisfies.

Lemma 2.2
If w satisfies (11)–(14), then, for $0\leqslant t\leqslant T$,

$$\frac{|q|^2}{2k_0^2s^2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{yy}\|^2-\frac{2|q|^2\dot{s}^3}{s}|w_y(1,t)|^2-\frac{1}{2k_0^2s^3}(3|q|^2\dot{s}+q_{\mathrm{I}}k_0 s)\|w_{yy}\|^2$$
$$+\mathrm{Re}\left((1+q\gamma)\left(w_t-y\frac{\dot{s}}{s}w_y\right),qw_{yy}\right)+\frac{k_0}{2}\mathrm{Im}(\gamma w,qw_{yy})=0 \quad (18)$$

Proof
Take the L^2 inner product of both sides of (11) with qw_{yy} and then real parts. Observe that

$$\mathrm{Re}\left(\frac{q}{k_0^2s^2}w_{yyt},qw_{yy}\right)=\frac{|q|^2}{k_0^2s^2}\mathrm{Re}(w_{yyt},w_{yy})=\frac{|q|^2}{2k_0^2s^2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{yy}\|^2 \quad (19)$$

In addition, note that

$$(\partial_y^2(yw_y),w_{yy})=(yw_{yyy}+2w_{yy},w_{yy})=2\|w_{yy}\|^2+I$$

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where $I := (yw_{yyy}, w_{yy})$. Integration by parts yields

$$I = |w_{yy}(1,t)|^2 - \int_0^1 w_{yy}(\overline{w_{yy}} + y\overline{w_{yyy}})\,\mathrm{d}y = |w_{yy}(1,t)|^2 - \|w_{yy}\|^2 - \overline{I}$$

From these relations it follows that

$$\mathrm{Re}(\partial_y^2(yw_y), w_{yy}) = \tfrac{3}{2}\|w_{yy}\|^2 + \tfrac{1}{2}|w_{yy}(1,t)|^2$$

Hence, taking into account the third boundary condition (14), we conclude that

$$\begin{aligned}\mathrm{Re}\left(-\frac{q}{k_0^2 s^2}\partial_y^2\left(y\frac{\dot{s}}{s}w_y\right), q w_{yy}\right) &= -\frac{|q|^2\dot{s}}{k_0^2 s^3}\left(\frac{3}{2}\|w_{yy}\|^2 + 2k_0^2 s^2 \dot{s}^2 |w_y(1,t)|^2\right)\\ &= -\frac{3}{2}\frac{|q|^2\dot{s}}{k_0^2 s^3}\|w_{yy}\|^2 - \frac{2|q|^2\dot{s}^3}{s}|w_y(1,t)|^2 \qquad (20)\end{aligned}$$

Finally, since

$$\mathrm{Re}\left\{\frac{\mathrm{i}k_0}{2}\left(\frac{1}{k_0^2 s^2}w_{yy}, q w_{yy}\right)\right\} = \frac{q_\mathrm{I}}{2k_0 s^2}\|w_{yy}\|^2$$

we conclude from (11), (19), and (20) that (18) holds. □

Our aim is to use the energy identities (15) and (18) to derive an *a priori* estimate for $\|w\|_2$. To do this, we need to control the terms involving w_t in these equations. For this reason we write (11) in the form $-\mathscr{L}w_t = \mathscr{M}w$, where $\mathscr{L}$ is the linear operator $\mathscr{L}: H^2 \cap H_0^1 \to L^2$ defined for $0 \leqslant t \leqslant T$ by

$$\mathscr{L}u := -\left(1 + q\gamma + \frac{q}{k_0^2 s^2}\partial_y^2\right)u, \quad u \in H^2 \cap H_0^1 \qquad (21)$$

We first study the *invertibility* of $\mathscr{L}$ on its domain $H^2 \cap H_0^1$, i.e. the question whether $\mathscr{L}w = 0$, $w \in H^2 \cap H_0^1$, implies $w = 0$. The lemma that follows gives sufficient conditions for $\mathscr{L}$ to be invertible:

Lemma 2.3
If one of the following conditions holds, then $\mathscr{L}$ is invertible for $0 \leqslant t \leqslant T$:

(i) $q_\mathrm{I} < 0$,
(ii) $q_\mathrm{I} = 0$, and for each $t \in [0,T]$, $\gamma_\mathrm{I} \not\equiv 0$,
(iii) $q_\mathrm{I} = 0$, $\gamma_\mathrm{I} = 0$, and $k_0 \max_{t\in[0,T]} s(t)$ is sufficiently small.

Proof
Fix $t \in [0,T]$ and let $\mathscr{L}w = 0$ for some $w \in H^2 \cap H_0^1$. Then $(\mathscr{L}w, w) = 0$. Taking the real and imaginary part of this equation we obtain, respectively,

$$\|w\|^2 + \int_0^1 (q_\mathrm{R}\gamma_\mathrm{R} - q_\mathrm{I}\gamma_\mathrm{I})|w|^2\,\mathrm{d}y - \frac{q_\mathrm{R}}{k_0^2 s^2}\|w'\|^2 = 0 \qquad (22)$$

$$\int_0^1 (q_\mathrm{I}\gamma_\mathrm{R} + q_\mathrm{R}\gamma_\mathrm{I})|w|^2\,\mathrm{d}y - \frac{q_\mathrm{I}}{k_0^2 s^2}\|w'\|^2 = 0 \qquad (23)$$

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(i) If $q_I<0$, eliminating $\|w'\|^2$ from the above yields

$$q_I\|w\|^2=|q|^2\int_0^1 \gamma_I|w|^2\,dy$$

from which, since $\gamma_I\geqslant 0$, it follows that $w=0$.

(ii) If $q_I=0$, (23) yields that $\int_0^1 \gamma_I|w|^2\,dy=0$. If γ_I is positive on an interval in $(0,1)$, then w vanishes in that interval and we may continue it by zero to $[0,1]$ by solving the initial-value problem for the ordinary differential equation $\mathscr{L}w=0$ with zero initial conditions. Hence (ii) follows.

(iii) If $q_I=0$, $\gamma_I=0$, (22) yields

$$\|w\|^2+q_R\int_0^1 \gamma_R|w|^2\,dy-\frac{q_R}{k_0^2s^2}\|w'\|^2=0$$

from which, by Poincaré's inequality and the hypothesis that k_0s is sufficiently small, it follows that $w=0$. □

We remark that if $\mathscr{L}$ is invertible for $0\leqslant t\leqslant T$, then, from standard elliptic pde theory and the Fredholm alternative [16], we may infer that given $f\in L^2(0,1)$, then $\mathscr{L}^{-1}f\in H^2\cap H_0^1$ and

$$\|\mathscr{L}^{-1}f\|_2\leqslant C\|f\| \tag{24}$$

for some positive quantity $C=C(t)$ independent of f. Since the coefficients of $\mathscr{L}$ are smooth, we may take C to be a continuous function on $[0,T]$.

The next result is the required estimate of $\|w_t\|$ in terms of $\|w\|_2$. This estimate holds if we only assume the boundary conditions (13).

Lemma 2.4
If w satisfies (11)–(13), and $\mathscr{L}$ is invertible for $0\leqslant t\leqslant T$, then there exists a constant C such that for $0\leqslant t\leqslant T$

$$\|w_t\|\leqslant C\|w\|_2 \tag{25}$$

Proof
In addition to the operator $\mathscr{L}$ defined by (21), we consider its 'maximal' extension $\mathscr{T}:H^2\to L^2$ defined by

$$\mathscr{T}u:=-\left(1+q\gamma+\frac{q}{k_0^2s^2}\partial_y^2\right)u,\quad u\in H^2$$

With this notation, taking into account (13) we rewrite the wide-angle PE (11) as

$$-\mathscr{L}w_t+\frac{\dot{s}}{s}\mathscr{T}(yw_y)=\frac{\mathrm{i}}{2k_0s^2}w_{yy}+\frac{\mathrm{i}k_0}{2}\gamma w$$

i.e. as

$$w_t=\frac{\dot{s}}{s}\mathscr{L}^{-1}\mathscr{T}(yw_y)-\frac{\mathrm{i}}{2k_0s^2}\mathscr{L}^{-1}w_{yy}-\frac{\mathrm{i}k_0}{2}\mathscr{L}^{-1}(\gamma w) \tag{26}$$

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Because of the estimate (24) we immediately have that there exists a constant C such that for $0\leqslant t\leqslant T$

$$\|\mathscr{L}^{-1}w_{yy}\|\leqslant C\|w_{yy}\| \tag{27}$$

and

$$\|\mathscr{L}^{-1}(\gamma w)\|\leqslant C\|w\| \tag{28}$$

In order to handle the $\mathscr{L}^{-1}\mathscr{T}(yw_y)$ term in the right-hand side of (26), fix $t\in[0,T]$ and for $f\in L^2$, consider the problem

$$\mathscr{L}u=f \tag{29}$$

which, because of our hypothesis, has a unique solution $u\in H^2\cap H_0^1$. Define for $g\in L^2$ its H^{-2} norm as the quantity

$$\|g\|_{-2}=\sup_{v\in H^2\cap H_0^1, v\neq 0}\frac{|(g,v)|}{\|v\|_2}$$

which is obviously well-defined. We will show that from (29) it follows that there exists a constant C such that

$$\|u\|\leqslant C\|f\|_{-2}=C\|\mathscr{L}u\|_{-2} \tag{30}$$

To see (30), consider the problem

$$\mathscr{L}^*v=u \tag{31}$$

where u is the solution of (29) and $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$ given on $H^2\cap H_0^1$ by $\mathscr{L}^*v=-(1+\overline{q}\,\overline{\gamma}+(\overline{q}/(k_0^2s^2))\partial_y^2)v$. It is easy to check that $\mathscr{L}^*$ is invertible whenever $\mathscr{L}$ is, and that as in (24) $\|v\|_2\leqslant C\|\mathscr{L}^*v\|$ for $v\in H^2\cap H_0^1$. Therefore, from (31), (29), and the definition of the H^{-2} norm we have

$$\begin{aligned}\|u\|^2&=(\mathscr{L}^*v,u)=(v,\mathscr{L}u)=(v,f)\\&\leqslant \|v\|_2\|f\|_{-2}\leqslant C\|\mathscr{L}^*v\|\|f\|_{-2}=C\|u\|\|f\|_{-2}\end{aligned}$$

from which the desired estimate (30) follows. We may take again C in (30) to depend only on T for $0\leqslant t\leqslant T$.

Returning to (26) and using the definition of $\mathscr{T}$ we have

$$\mathscr{T}(yw_y)=-(1+q\gamma)yw_y-\frac{2q}{k_0^2s^2}w_{yy}-\frac{q}{k_0^2s^2}yw_{yyy}$$

Therefore,

$$\|\mathscr{L}^{-1}\mathscr{T}(yw_y)\|\leqslant\|\mathscr{L}^{-1}[(1+q\gamma)yw_y]\|+C\|\mathscr{L}^{-1}w_{yy}\|+C\|\mathscr{L}^{-1}(yw_{yyy})\| \tag{32}$$

Because of (24), we have

$$\|\mathscr{L}^{-1}[(1+q\gamma)yw_y]\|\leqslant C\|(1+q\gamma)yw_y\|\leqslant C\|w_y\| \tag{33}$$

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For the second term in the right-hand side of (32) we have already noted that (27) holds. To treat the third term, consider the problem

$$\mathscr{L}\phi = y w_{yyy}, \quad \phi \in H^2 \cap H_0^1$$

Using the estimate (30) we have

$$\|\phi\| = \|\mathscr{L}^{-1}(y w_{yyy})\| \leqslant C \|y w_{yyy}\|_{-2} = C \sup_{v \in H^2 \cap H_0^1, v \neq 0} \frac{|(y w_{yyy}, v)|}{\|v\|_2}$$

But for $v \in H^2 \cap H_0^1$ using integration by parts we obtain

$$|(y w_{yyy}, v)| = |-(w_{yy}, (yv)_y)| \leqslant C \|w_{yy}\| \|v\|_1$$

We conclude therefore that

$$\|\mathscr{L}^{-1}(y w_{yyy})\| \leqslant C \|w_{yy}\| \tag{34}$$

Hence, (32), (33), (27), and (34) give

$$\|\mathscr{L}^{-1}\mathscr{T}(y w_y)\| \leqslant C(\|w_y\| + \|w_{yy}\|) \tag{35}$$

which, combined with (26), (27), and (28), yields (25). □

We are now in a position to prove the main result of this section.

Theorem 2.1
If w satisfies (11)–(14), and $\mathscr{L}$ is invertible for $0 \leqslant t \leqslant T$, then there exists a constant C such that

$$\|w\|_2 \leqslant C \|w_0\|_2 \tag{36}$$

holds for $0 \leqslant t \leqslant T$.

Proof
The first energy identity (15) yields by Sobolev's theorem, the Cauchy–Schwarz inequality and (25),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_y\|^2 \leqslant C \|w\|_2^2, \quad 0 \leqslant t \leqslant T \tag{37}$$

Using now analogous estimates in the second energy identity (18) we obtain for $0 \leqslant t \leqslant T$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_{yy}\|^2 \leqslant C \|w\|_2^2 \tag{38}$$

Of course, by (25) we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 = 2\,\mathrm{Re}\,(w_t, w) \leqslant C \|w\|_2^2, \quad 0 \leqslant t \leqslant T \tag{39}$$

From (37)–(39) we get (36) using Gronwall's lemma. □

In summary, Proposition 2.1 gives us an H^1 *a priori* estimate for the solution of the ibvp (11)–(13) for real q and γ in the case of upsloping domains with $k_0 \max_{0 \leqslant t \leqslant T} s(t)$ sufficiently

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small. Addition of the boundary condition (14) yields an *a priori* H^2 estimate for the solution of (11)–(14) for any type of bottom provided $\mathscr{L}$ is invertible for $0\leqslant t\leqslant T$. As Lemma 2.3 shows, this latter condition is fulfilled for $q_{\mathrm{I}}<0$ or in the presence of absorption or for $k_0 \max_{0\leqslant t\leqslant T} s(t)$ sufficiently small in the case of real q and γ.

3. FINITE DIFFERENCE SCHEMES

In this section we shall construct and test numerically two finite difference schemes that we call FDI and FDII. Both schemes are Crank–Nicolson-type discrete analogs of the wide-angle PE (11) supplemented with the initial condition (12) and the surface boundary condition $w(0,t)=0$. Furthermore, scheme FDI takes into account only the homogeneous Dirichlet bottom boundary condition $w(1,t)=0$, while FDII incorporates, in addition to $w(1,t)=0$, a discretization of the third boundary condition (14) as well.

Let J and N be positive integers and consider a uniform partition of $[0,1]$ defined by the nodes $\{y_j\}_{j=0}^{J+1}$ given by $y_j:=jh$, for $j=0,\ldots,J+1$, where $h:=1/(J+1)$. Let $\{t^n\}_{n=0}^{N}$ be a partition of the interval $[0,T]$ defined by $t^n:=nk$, $n=0,\ldots,N$, where $k:=T/N$. Let $t^{n+1/2}:=t^n+k/2$, $0\leqslant n\leqslant N-1$, and define

$$X_h:=\{z=(z_0,\ldots,z_{J+1})^T\in\mathbb{C}^{J+2}: z_0=z_{J+1}=0\}$$

We shall construct $W^n\in X_h$ for $n=0,\ldots,N$ such that for each $n\in\{0,\ldots,N\}$ W_j^n approximate the values of $w(y_j,t^n)$, $0\leqslant j\leqslant J+1$, of the solution of (11) under the given initial and boundary conditions. To this end, we define the difference operator $\delta_h^2: X_h\to X_h$ by

$$\delta_h^2 v_j := \frac{v_{j+1}-2v_j+v_{j-1}}{h^2},\quad j=1,\ldots,J,\ \ v\in X_h$$

and the difference operator $\delta_h^1: X_h\to X_h$ by

$$\delta_h^1 v_j := \frac{v_{j+1}-v_{j-1}}{2h},\quad j=1,\ldots,J,\ \ v\in X_h$$

Thus, if v is a smooth function on $[0,1]$ and $v_j:=v(y_j)$, then $\delta_h^1 v_j$ and $\delta_h^2 v_j$ are the usual centered difference quotient approximations to, respectively, the first derivative $v'(y_j)$ and the second derivative $v''(y_j)$ at the interior nodes y_j, $j\in\{1,\ldots,J\}$. We also define the difference operator $\delta_{\mathrm{I},h}^3: X_h\to X_h$ by

$$\delta_{\mathrm{I},h}^3 v_j := \begin{cases} \dfrac{v_3-3v_2+3v_1-v_0}{h^3}, & j=1 \\[2mm] \dfrac{v_{j+2}-2v_{j+1}+2v_{j-1}-v_{j-2}}{2h^3}, & j=2,\ldots,J-1 \\[2mm] \dfrac{v_{J+1}-3v_J+3v_{J-1}-v_{J-2}}{h^3}, & j=J \end{cases}$$

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for $v\in X_h$. In addition we let $\delta^3_{\mathrm{II},h}(t): X_h\to X_h$ be defined for $t\in[0,T]$ and $v\in X_h$ by

$$\delta^3_{\mathrm{II},h}(t)v_j := \delta^3_{\mathrm{I},h}v_j, \quad j=1,\dots,J-1$$

$$\delta^3_{\mathrm{II},h}(t)v_J := \frac{1}{2h^3}\left[\frac{h\tau(t)}{1-\frac{h}{2}\tau(t)}v_{J+1}-\left(\frac{h\tau(t)}{1-\frac{h}{2}\tau(t)}+1\right)v_J+2v_{J-1}-v_{J-2}\right]$$

where $\tau(t):=2\mathrm{i}k_0\dot{s}(t)s(t)$. For $\omega=(\omega_0,\dots,\omega_{J+1})^T\in\mathbb{C}^{J+2}$ and $z\in X_h$, define $\omega\otimes z\in X_h$ by $(\omega\otimes z)_j:=\omega_j z_j$, $j=1,\dots,J$. Let $\gamma_j^{n+1/2}:=\gamma(y_j,t^{n+1/2})$ for $n=0,\dots,N-1$ and $j=1,\dots,J$. Finally, given $\{V^n\}_{n=0}^N\subset X_h$, let $V^{n+1/2}:=\frac{1}{2}(V^n+V^{n+1})$ and $\partial V^n:=(1/k)(V^{n+1}-V^n)$ for $n=0,\dots,N-1$.

With this notation in place, we are now in a position to define the finite difference schemes FDI and FDII

(i) FDI
Step 1: Set $W_j^0=w_0(y_j)$, $j=0,\dots,J+1$.
Step 2: For $n=0,\dots,N-1$ compute $W^{n+1}\in X_h$, such that

$$[1+q\gamma_j^{n+1/2}]\left(\partial W_j^n-y_j\frac{\dot{s}(t^{n+1/2})}{s(t^{n+1/2})}\delta_h^1 W_j^{n+1/2}\right)$$
$$+\frac{q}{k_0^2s^2(t^{n+1/2})}\left\{\partial\delta_h^2W_j^n-\frac{\dot{s}(t^{n+1/2})}{s(t^{n+1/2})}[\delta^3_{\mathrm{I},h}(Y\otimes W^{n+1/2})_j-\delta_h^2W_j^{n+1/2}]\right\}$$
$$=\mathrm{i}(p-q)k_0\left[\frac{1}{k_0^2s^2(t^{n+1/2})}\delta_h^2W_j^{n+1/2}+\gamma_j^{n+1/2}W_j^{n+1/2}\right],\quad j=1,\dots,J$$

where $Y_j:=y_j$ for $j=0,\dots,J+1$.

(ii) FDII
Step 1: Set $W_j^0=w_0(y_j)$, $j=0,\dots,J+1$

Step 2: For $n=0,\dots,N-1$ compute $W^{n+1}\in X_h$, such that

$$[1+q\gamma_j^{n+1/2}]\left(\partial W_j^n-y_j\frac{\dot{s}(t^{n+1/2})}{s(t^{n+1/2})}\delta_h^1W_j^{n+1/2}\right)+\frac{q}{k_0^2s^2(t^{n+1/2})}$$
$$\times\left\{\partial\delta_h^2W_j^n-\frac{\dot{s}(t^{n+1/2})}{s(t^{n+1/2})}\left[y_j\frac{\delta^3_{\mathrm{II},h}(t^{n+1})W_j^{n+1}+\delta^3_{\mathrm{II},h}(t^n)W_j^n}{2}+2\delta_h^2W_j^{n+1/2}\right]\right\}$$
$$=\mathrm{i}(p-q)k_0\left[\frac{1}{k_0^2s^2(t^{n+1/2})}\delta_h^2W_j^{n+1/2}+\gamma_j^{n+1/2}W_j^{n+1/2}\right],\quad j=1,\dots,J$$

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Both schemes are implicit, of Crank–Nicolson-type in t, and require solving pentadiagonal complex systems of linear equations for each n. If v is a smooth function on $[0,1]$, we may check, using straightforward Taylor expansions, that the difference quotient $\delta^3_{\mathrm{I},h} v_j$ used in FDI approximates (if $v_j := v(y_j)$) the third derivative $v'''(y_j)$ to $O(h^2)$ at the nodes y_j, $2 \leqslant j \leqslant J-1$, and to $O(h)$ at y_1 and y_J. Note that the difference quotient $\delta^3_{\mathrm{II},h}(t) v_j$ used in FDII is identical to $\delta^3_{\mathrm{I},h} v_j$ at $j=1,\ldots,J-1$. At $j=J$ it is designed to incorporate the boundary condition (14), i.e. the condition $v_{yy}(1)=\tau(t) v_y(1)$, and approximate $v'''(y_J)$ to $O(h)$. Thus, both schemes are formally $O(k^2+h^2)$ accurate at (y_j, t^n) for $2 \leqslant j \leqslant J-1$, and $O(k^2+h)$ at (y_1, t^n) and (y_J, t^n). A further difference between the two schemes lies in the way that the term $\partial^2_y (y w_y)$ (cf. (11)) is handled. In FDI we view this term as $\partial^3_y (yw) - w_{yy}$ and discretize the third derivative by $\delta^3_{\mathrm{I},h}$, while in FDII we discretize the analog of $\partial^2_y (y w_y) = y w_{yyy} + 2 w_{yy}$ using $\delta^3_{\mathrm{II},h}$ for ∂^3_y. This was done for stability and accuracy purposes.

It is not our intention in this paper to prove rigorous error estimates for these difference schemes. Instead, we shall check their accuracy and stability by means of numerical experiments. As a test problem we employ an artificial, nonhomogeneous version of (1), i.e. an equation, which in the original variable domain $0 \leqslant r \leqslant R$, $0 \leqslant z \leqslant s(r)$, is of the form

$$\left[1 + q\beta(z,r) + \frac{q}{k_0^2}\partial^2_z\right] v_r = \frac{\mathrm{i}(p-q)}{k_0} v_{zz} + \mathrm{i}(p-q) k_0 \beta(z,r) v + f(z,r) \tag{40}$$

where we take $q = 1+\mathrm{i}$, $p-q=1$, $k_0=1$, $\beta(z,r) = 1 + (1/2)\sin(zr/s(r))$, and define f so that the exact solution of (40) is $v(z,r) = (1+\mathrm{i})\sin(zr/s(r))$. (We will make below several choices for the bottom profile $s(r)$.) We supplement (40) with the initial condition

$$v_0(z) = 0, \quad 0 \leqslant z \leqslant s(0) \tag{41}$$

the free surface boundary condition

$$v(0,r) = 0, \quad r \geqslant 0 \tag{42}$$

and the bottom boundary conditions

$$v(s(r),r) = g_1(r), \quad r \geqslant 0 \tag{43}$$

and

$$\dot{s}(r) v_z(s(r),r) + \frac{\mathrm{i}}{2k_0} v_{zz}(s(r),r) = g_2(r), \quad r \geqslant 0 \tag{44}$$

These conditions are now nonhomogeneous and the functions $g_1(r)$, $g_2(r)$ are compatible with the chosen exact solution. After the change of variables $y = z/s(r)$, $t = r$ we solve the problem numerically using the nonhomogeneous analog of the scheme FDI to discretize the ibvp obtained by transforming (40)–(43) and (the nonhomogeneous analog of) FDII for the ibvp that we get by transforming (40)–(44).

In all the examples below we took $T=1$ and used $h = 1/(J+1)$ and $k = T/N$ with $J=N$. In the tables the values of J and the maximum norm $\|\varepsilon\|_\infty$ of the error of the numerical solution computed over all points (y_j, t^n), $0 \leqslant j \leqslant J+1$, $0 \leqslant n \leqslant N$ are shown. Also shown are the observed rates of convergence α of the error as h decreases, assuming that $\|\varepsilon\|_\infty \approx Ch^\alpha$.

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Table I. Example 1 (left) and Example 2 (right).

Upsloping, FDI			Downsloping, FDI		
J	$\|\varepsilon\|_\infty$	Rate	J	$\|\varepsilon\|_\infty$	Rate
10	1.212851e−03	—	10	7.323060e−04	—
20	2.999388e−04	2.015	20	1.216307e−03	
40	7.610642e−05	1.978	40	2.656963e−04	
80	1.910357e−05	1.994	80	2.038290e−04	
160	4.777279e−06	1.999	160	1.874397e−04	
320	1.194376e−06	1.999	320	1.778468e−04	
640	2.985521e−07	2.000	640	1.727810e−04	
			6000	2.991996e−04	

Table II. Example 3 (left) and Example 4 (right).

Upsloping, FDII			Downsloping, FDII		
J	$\|\varepsilon\|_\infty$	Rate	J	$\|\varepsilon\|_\infty$	Rate
10	2.180255e−03	—	10	7.841872e−04	—
20	3.820216e−04	2.512	20	2.148323e−04	1.867
40	8.411472e−05	2.183	40	5.244569e−05	2.034
80	1.957920e−05	2.103	80	1.334829e−05	1.974
160	4.809866e−06	2.025	160	3.369580e−06	1.986
320	1.196289e−06	2.007	320	8.568883e−07	1.975
640	2.986845e−07	2.001	640	2.133469e−07	2.005

- Example 1 (upsloping, FDI). Here we take $s(r)=10-3r$. The errors and rates of convergence appear in Table I. FDI is apparently stable and second-order accurate in this upsloping case.
- Example 2 (downsloping, FDI). In this example, $s(r)=10+3r^3$. It is clear that FDI is not convergent in this case, cf. Table I.
- Example 3 (upsloping, FDII). In this example, $s(r)=10-3r$. Table II shows that FDII is apparently stable and of second-order of accuracy for this example.
- Example 4 (downsloping, FDII). For this example we took $s(r)=10+3r^3$. The results of Table II show that FDII, i.e. the finite difference scheme incorporating the second bottom boundary condition as well, is apparently stable and second-order accurate.
- Example 5 (general $s(r)$, FDII). In this example we took the oscillating bottom profile $s(r)=5+\sin(10r)$ and computed with FDII. The scheme is apparently stable and second-order accurate as shown in Table III.

We conclude from these examples, and from other similar ones that we ran, that FDII, i.e. the scheme incorporating the pressure-release bottom boundary condition $w(1,t)=0$ and, in addition to it, the bottom boundary condition (14), appears to be unconditionally stable and of second-order of accuracy for any type of bottom profile. On the other hand, FDI appears to be stable and convergent only in the presence of upsloping bottoms. It is interesting to note that in upsloping environments and for sufficiently small values of k and h, the two schemes FDI and FDII yield numerical approximations that are very close to each other. This may be seen e.g. in the results of

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Table III. Example 5.

General $s(r)$, FDII		
J	$\|\varepsilon\|_\infty$	Rate
10	1.403489e−03	—
20	4.615283e−04	1.604
40	1.309232e−04	1.817
80	3.310404e−05	1.983
160	8.757484e−06	1.918
320	2.047834e−06	2.096
640	5.025428e−07	2.026

Examples 1 and 3 (Tables I and II), where for $J\geqslant 320$, the errors of the two schemes are equal to three significant digits.

4. NUMERICAL EXPERIMENTS ON UNDERWATER ACOUSTICS PROBLEMS

In this section we consider three more realistic problems in the context of which we test the validity of the ibvp (11)–(14), discretized by the finite difference scheme FDII, as a wide-angle underwater sound propagation model. In (11) we used the complex coefficients $q=(0.252252311, -1.35135138\mathrm{e}-02)$ (this value was used by Collins in [4] to obtain a stable higher-order elastic PE model), and $p=q+\frac{1}{2}$. Our first test case was the standard ASA upsloping wedge example [14], with a pressure-release bottom given by $s(r)=200(1-r/4000)$ (distances in meters). The bottom thus decreases linearly from $s(0)=200\,\mathrm{m}$ at $r=0$ to zero at $r=4000\,\mathrm{m}$, and makes an angle with a constant value of 2.86° with respect to the horizontal surface. The time-harmonic point source emits at a frequency $f=25\,\mathrm{Hz}$ and is located at $r=0$ at a depth $z_S=100\,\mathrm{m}$. We assume that there is no attenuation in the water layer, i.e. that $\mathrm{Im}\,\gamma=0$, and that $c_0=c=1500\,\mathrm{m/s}$, so that $\mathrm{Re}\,\gamma=0$ as well. The computation was carried out up to a maximum range value of 3300 m using the FDII scheme with 4000 intervals of equal length in y and $k=0.83475\,\mathrm{m}$. As initial condition at $r=0$ we used a normal mode starter corresponding to the sum of the propagating modes present at the source location, i.e.

$$v_0(z)=\mathrm{i}\sqrt{2\pi}\sum_{m=1}^{M}\frac{\psi_m(z_S)\psi_m(z)}{\sqrt{k_{r,m}}},\quad 0\leqslant z\leqslant s(0) \tag{45}$$

where ψ_m, $1\leqslant m\leqslant M$, denote the depth-dependent mode-shaped functions defined for each $m\in\{1,\ldots,M\}$ by $\psi_m(z):=\eta_m\sin(m\pi z/s(0))$, $0\leqslant z\leqslant s(0)$, with η_m such that $\int_0^{s(0)}|\psi_m(z)|^2\,\mathrm{d}z=1$. In (45) $k_{r,m}$ is the horizontal wave number associated with mode m, defined by $k_{r,m}:=\sqrt{(2\pi f/c)^2-(m\pi/s(0))^2}$. The test case studied here is characterized by the presence of six propagating modes ($M=6$) at the source range. Note that the source is placed at mid-depth, which corresponds exactly to a null of modes 2, 4, and 6. Only three propagating modes are thus excited at $r=0$ (mode 1, mode 3, and mode 5). Figure 2 shows the resulting transmission loss (TL) contour plot (referred back to the original z, r coordinates). The TLs were computed by using the formula $\mathrm{TL}=-20\log_{10}(|v(z,r)|/\sqrt{r})$. The minima and maxima that appear distinguishably in Figure 2 are due to interference patterns of the (only) three propagating modes present in

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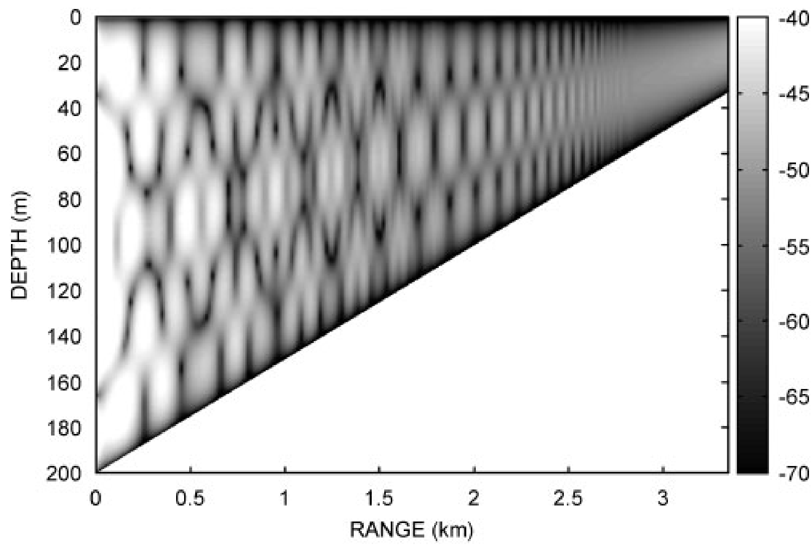

Figure 2. Transmission loss as a function of z and r. Upsloping ASA wedge test case, FDII.

the waveguide. The interference patterns exhibit three different sections in the contour plot, corresponding mainly to the propagation of three modes (modes 1, 3, and 5), two modes (modes 1 and 3) and only one single mode (mode 1).

For this example, we compared the results of FDII with those obtained by a code that solves the wide-angle equation (1) in the original variables using a 'staircase' approximation for the bottom and pressure-matching across the vertical part of the stairs, as is quite customary in underwater acoustics. (In this 'staircase' PE code we simulated the pressure-release bottom boundary condition $v=0$ by placing a homogeneous, artificial fluid sediment layer under the bottom; this layer was assumed to have a very high sound speed of $10^{10}\,\mathrm{m/s}$ and a very low density of $0.1\,\mathrm{kg/m^3}$ as suggested in [14]. The false bottom was placed at a depth of 1000 m. The boundary-value problem was then treated as an interface problem with two layers.) Figure 3 shows the one-dimensional TL versus r plots for both codes at depth $z=30\,\mathrm{m}$. We observe overall a good agreement between the two codes; we shall comment below on the discrepancy observed. In Figure 4 we compare the results of FDII for the same test problem with those of COUPLE implemented with a pressure-release bottom. COUPLE is a standard code in underwater acoustics that solves the full Helmholtz equation by a coupled normal mode technique [17]. (We compare here with the *outgoing* component of the field given by COUPLE as the wide-angle equation (1) is valid for one-way propagation only.) We observe a rather good overall agreement between the results of the two models. (We used different axis scales in Figure 4 in order to fit the whole field in the picture.) The small-amplitude oscillatory behavior of FDII in the vicinity of 2.5 km is due to the fact that the wide-angle equation (1) represents a low-order formal approximation of the outgoing component of the full Helmholtz equation corresponding to a rational approximation $(1+px)/(1+qx)$ of $\sqrt{1+x}$ with linear numerator and denominator. Use of higher-order accurate rational approximations [1, 6], suppresses these wiggles. (Of course, rational approximations with higher-order polynomials lead to higher-order wide-angle pde's for which additional boundary conditions have to be provided so

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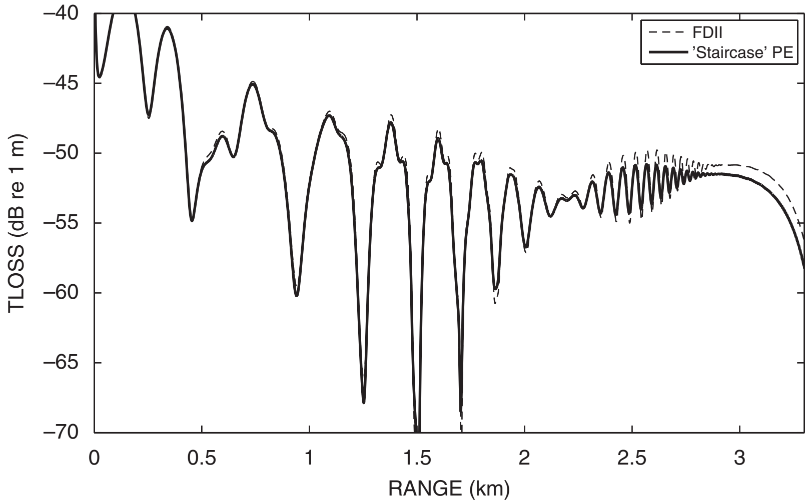

Figure 3. Transmission loss as a function of r at receiver depth $z=30\,\mathrm{m}$. Upsloping ASA wedge test case, FDII versus 'staircase' PE.

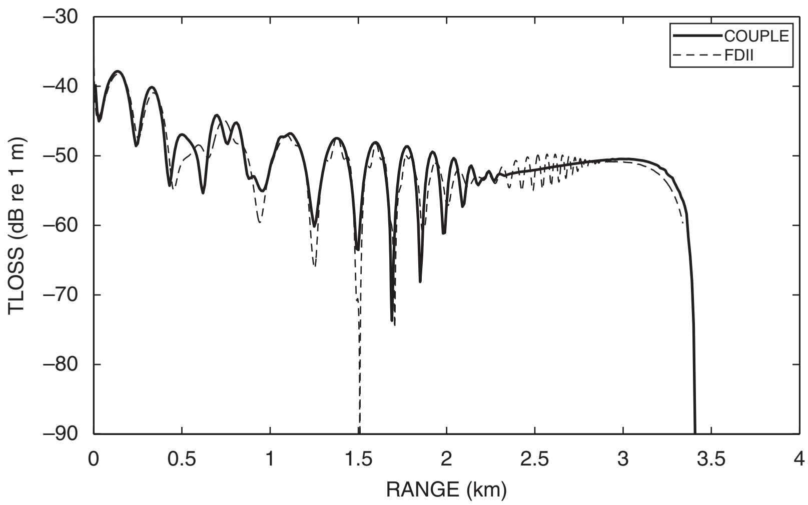

Figure 4. Transmission loss as a function of r at receiver depth $z=30\,\mathrm{m}$. Upsloping ASA wedge test case, FDII versus outgoing field of COUPLE.

that they yield well-posed variable-bottom problems. In practice, one may suppress the wiggles by using a higher-order analog of the 'staircase' PE.)

We now examine a second example, that of a 'downsloping wedge', corresponding to $s(r)=200(1+r/4000)$. The properties of the medium, the coefficients of (1), the initial condition, and the discretization parameters of the FDII scheme were taken as before. In Figure 5 we show the TL field computed by the FDII scheme for this example, while in Figure 6 we compare the FDII

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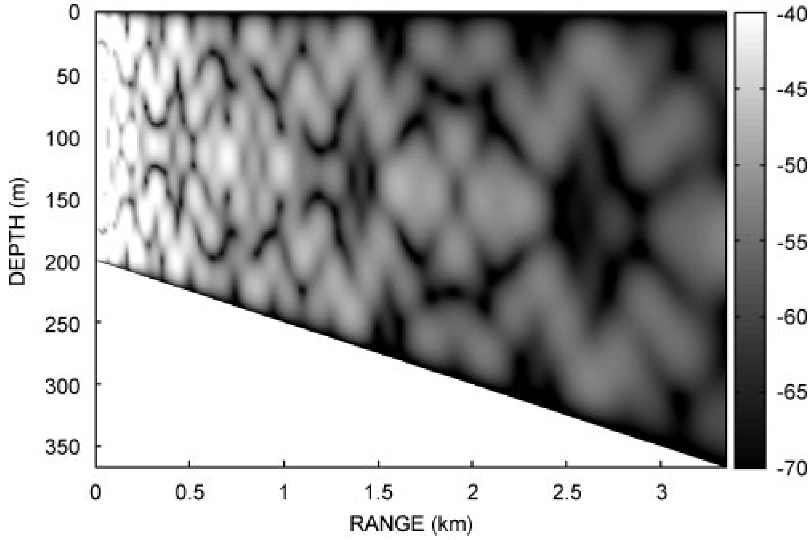

Figure 5. Transmission loss as a function of z and r. Downsloping wedge, FDII.

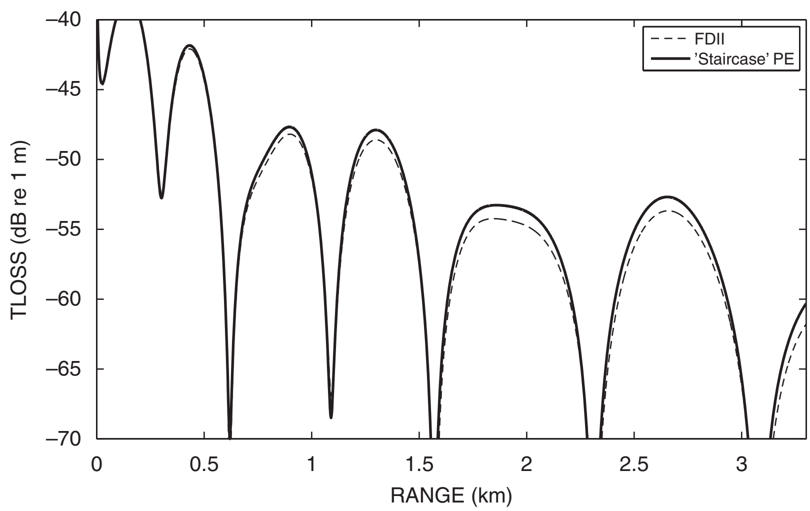

Figure 6. Transmission loss as a function of r at receiver depth $z=30\,\mathrm{m}$. Downsloping wedge, FDII versus 'staircase' PE.

results, in the case of a typical one-dimensional TL plot, with those obtained by the 'staircase' PE code. Again the observed agreement is very good.

Finally, in Figure 7 we show the TL field obtained by FDII in a similar waveguide having a sinusoidal bottom given by $s(r)=200-50\sin(2\pi r/2000)$. The change-of-variable technique enables one to easily discretize problems with general variable bottoms like this.

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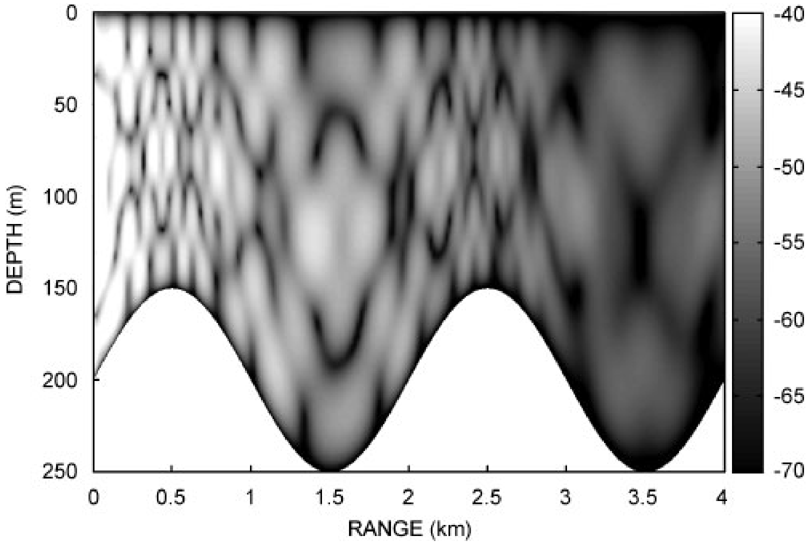

Figure 7. Transmission loss as a function of z and r. Sinusoidal bottom, FDII.

As a final note of interest, we remark that the proposed new ibvp, as discretized by FDII, seems to furnish physically improved results, when compared with the staircase discretization for the wide-angle equation (1). This is evident from the TL graphs of Figures 3 and 6. Indeed, it is well-known that a model that is not energy conserving tends to lose energy in upsloping and gain energy in downsloping environments. It has also been documented that any 'staircase' PE model employing a pressure-matching condition across the vertical step of the staircase is not energy conserving [18]. Examination of the two curves in Figures 3 and 6 reveals that the new model is more energy conserving than the staircase approximation.

5. CONCLUSION

In this paper we formulated an ibvp for the wide-angle PE (1) in a domain with variable bottom $z=s(r)$. The solution of (1) satisfies homogeneous Dirichlet boundary conditions at the water surface and at the 'pressure-release' bottom. The problem was transformed by a range-dependent change of variables into an equivalent one posed in a horizontal strip, cf. (11)–(13). In addition to the zero Dirichlet boundary condition at the bottom, it was found that it is necessary in general to pose another boundary condition there. The specific additional condition used in this paper, i.e. (14) in the transformed variables, is obtained by assuming that the lower-order standard PE (8) holds along the bottom surface. For upsloping bottoms, i.e. when $\dot{s}(t)<0$, uniqueness of solutions and *a priori* H^1 estimates thereof may be proved under some additional assumptions for the original ibvp if we assume only the original Dirichlet condition at the bottom. However, in the presence of general bottom topography, adding the new boundary condition allows one to derive an *a priori* H^2 estimate under some additional assumptions (e.g. if q in (1) is complex with $\operatorname{Im} q<0$).

We also discretized the two ibvp's by simple finite difference schemes that were experimentally found to be of second order of accuracy. The scheme FDII that approximates the ibvp (11)–(14), i.e.

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with all three boundary conditions present, appears to be unconditionally stable and second-order accurate in domains with general bottom topographies and yields accurate results in the case of realistic examples of underwater sound propagation. On the other hand, the finite difference scheme FDI, which approximates the ibvp (11)–(13), fails to converge in downsloping environments. (Indeed, in numerical examples, we have on occasion observed that in such environments FDI is unstable.) It is worthwhile to note that the numerical results that FDI yields in problems with an *upsloping* bottom are very close to those produced by FDII. (We observed this not only in 'artificial' problems like the test cases of Section 3, but also in more realistic underwater sound propagation problems posed over upsloping bottoms.)

The numerical experiments and theoretical estimates presented in this paper lead us to the conjecture that the pde (11) is probably of 'hyperbolic' character, in the sense that when accompanied by only the two boundary conditions (13) it yields a well-posed ibvp in domains where $\dot{s}(t)\leqslant 0$, while it seems to need an additional boundary condition in case where $\dot{s}$ is positive in some interval of t. It also seems, at least in the cases of the numerical examples that we have tried, that the additional boundary condition (14) is apparently 'compatible' with the equation and the rest of the data in upsloping domains, and does not contribute much to the solution of (11)–(13) in such cases.

We conclude the paper with two remarks.

(i) Computing with a *complex* q with $\mathrm{Im}\, q<0$ was found to be essential for stabilizing the numerical results of FDII. Indeed, when we took real-valued q in the underwater sound propagation problems of Section 4 (where $\gamma=0$) we observed small oscillations that grew as t increased. (Recall that $\mathrm{Im}\, q<0$ is also a sufficient condition for the invertibility of $\mathscr{L}$, cf. Lemma 2.3, and hence, for the validity of the estimate of the solution of (11)–(14).)

(ii) The additional boundary condition (14) is just an example of a third boundary condition needed, in conjunction with (13), for well-posedness in a general domain. Other sets of boundary conditions may be derived that could also render well-posed problems. For example, consider (1) in the original variable domain $0\leqslant z\leqslant s(r)$, $r\geqslant 0$, and pose it as an ibvp with a given initial condition $v(z,0)=v_0(z)$, $0\leqslant z\leqslant s(0)$, with the pressure-release surface boundary condition $v(0,r)=0$ and the boundary condition $v_r(s(r),r)=0$ for $r\geqslant 0$. Assume for example that the bottom is upsloping and, for simplicity, that q and β are real. Then, under the usual low-frequency/shallow-water assumption, one may prove that for some constant C, the H^1 estimate

$$\int_0^{s(r)} |v_z(z,r)|^2\,\mathrm{d}z \leqslant C\int_0^{s(0)} |v_0'(z)|^2\,\mathrm{d}z \tag{46}$$

holds for $0\leqslant r\leqslant R$. This implies the uniqueness of v for such problems. It is interesting to note that if, in addition, we also impose the boundary condition

$$v_z - \frac{\mathrm{i}k_0\dot{s}(r)}{2(p-q)}v + \frac{\mathrm{i}q}{k_0(p-q)}v_{rz} = 0 \quad \text{at } z=s(r), \quad r\geqslant 0 \tag{47}$$

then, we may show, under no sign assumptions on $\dot{s}(r)$, that

$$\int_0^{s(r)} |v(z,r)|^2\,\mathrm{d}z = \int_0^{s(0)} |v_0(z)|^2\,\mathrm{d}z \tag{48}$$

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i.e. that the associated ibvp (with three boundary conditions) conserves the L^2 norm (the 'energy') of the solution. When $p=1/2$, $q=0$, (47) reduces to the well-known Abrahamsson–Kreiss bottom boundary condition $v_z - \mathrm{i}k_0\dot{s}(r)v=0$ [19], which is a 'parabolized' Neumann boundary condition for the standard PE (8). Research for studying the well-posedness, the physical relevance, and numerical approximations of these new ibvp's is under way.

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