*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR CNRS 5509

Linear (and nonlinear) wave propagation in fluids

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http://acoustique.ec-lyon.fr

- **Models for linear wave propagation in fluids**

  Introduction, surface gravity waves, internal waves, acoustic waves, waves in rotating flows, ...

  Longitudinal and transverse waves, dispersion relation, phase velocity, group velocity

- **Theories for linear wave propagation**

  Fourier integral solution, asymptotic behaviour (stationary phase)

  Propagation of energy, ray theory (high-frequency approximation for inhomogeneous medium)

- **Introduction to nonlinear wave propagation**

  Euler equations, N-waves, weak shocks, Burgers

  Solitary waves (Korteweg - de Vries)

- **Schedule**

friday 08/12/2017	CM1
friday 15/12/2017	TD1 (2h homework)
friday 22/12/2017	TD2 (2h homework)
monday 08/01/2018	CM2 & TD1
friday 12/01/2018	CM3
monday 15/01/2018	CM4 & TD2
friday 19/01/2018	TD3
friday xx/01/2018	Exam xx

Laptop required for small classes !

Textbooks

Guyon, E., Hulin, J.P. & Petit, L., 2001, Hydrodynamique physique, *EDP Sciences / Editions du CNRS*, Paris - Meudon.

Lighthill, J., 1978, *Waves in fluids*, Cambridge University Press, Cambridge.

Johnson, R. H., 1997, *A modern introduction to the mathematical theory of water waves*, Cambridge University Press, Cambridge,

Morse, P.M. & Ingard, K.U., 1986, *Theoretical acoustics*, Princeton University Press, Princeton, New Jersey.

Ockendon, H. & Ockendon, J. R., 2000, *Waves and compressible flow*, Springer-Verlag, New York, New-York.

Pierce, A.D., 1994, *Acoustics*, Acoustical Society of America, third edition.

Rayleigh, J. W. S., 1877, *The theory of sound*, Dover Publications, New York, 2nd edition (1945), New-York.

Temkin, S., 2001, *Elements of acoustics*, Acoustical Society of America through the American Institute of Physics.

Thual, O., 2005, Des ondes et des fluides, *Cépaduès-éditions*, Toulouse.

Whitham, G.B., 1974, *Linear and nonlinear waves*, Wiley-Interscience, New-York.

Waves in fluids : models for linear wave propagation

Introduction

Acoustic waves (in a homogeneous medium at rest)

hyperbolic wave equation

$$\frac{\partial^2 p'}{\partial t^2} - c_\infty^2 \frac{\partial^2 p'}{\partial x_1^2} = 0$$

General one dimensional solution $p'(x_1, t) = p_l(x_1 + c_\infty t) + p_r(x_1 - c_\infty t)$
known as d'Alembert's solution

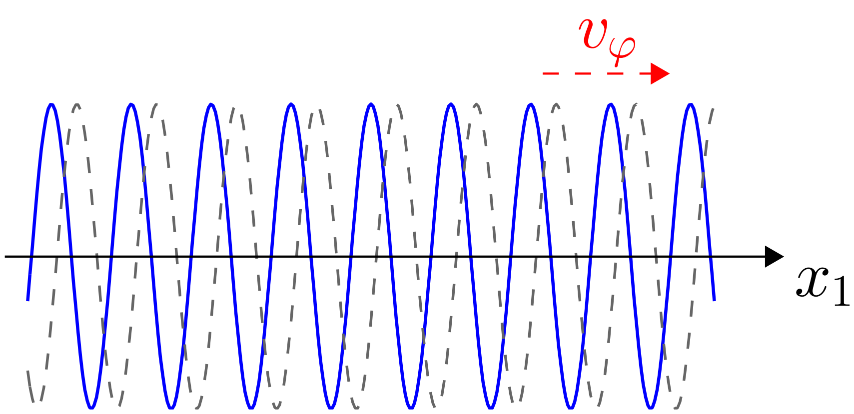
Dispersion relation

For a plane wave (*i.e.* a particular Fourier component) $\sim e^{i(k_1 x_1 - \omega t)}$

$\omega = \pm c_\infty k_1$ non dispersive waves

$p_r \sim e^{ik_1(x_1 - c_\infty t)}$

phase velocity $v_\varphi = \omega / k_1 = c_\infty$



Introduction (cont'd)

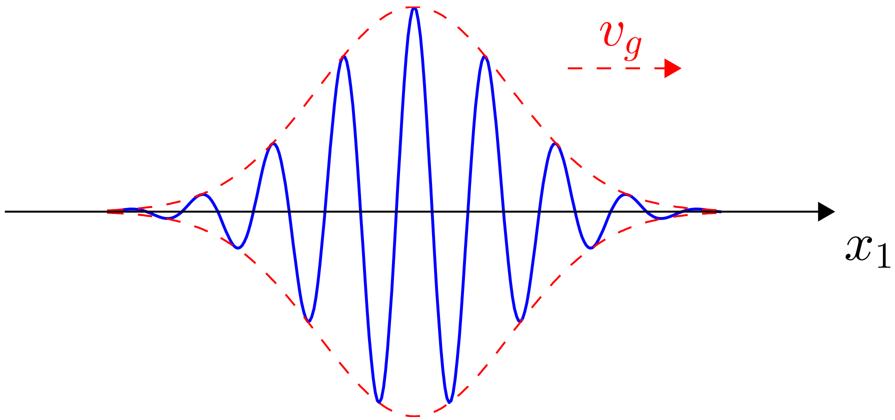
1-D dispersive waves

$$\eta(x_1, t) = Ae^{i(k_1 x_1 - \omega t)} = Ae^{ik_1(x_1 - v_\varphi t)} \quad \text{with} \quad \omega = \Omega(k_1)$$

The phase speed $v_\varphi = \Omega(k_1)/k_1$ generally depends on k_1

The dispersion relation $\omega = \Omega(k_1)$ or $\mathcal{D}(k_1, \omega) = 0$, is obtained by requiring the plane waves to be solution of the linearized equations of motion.

The general solution is a superposition of modes $\propto e^{i(k_1 x_1 - \omega t)}$ through a Fourier integral : wave packet characterized by a group velocity $v_g = \partial\omega/\partial k_1$



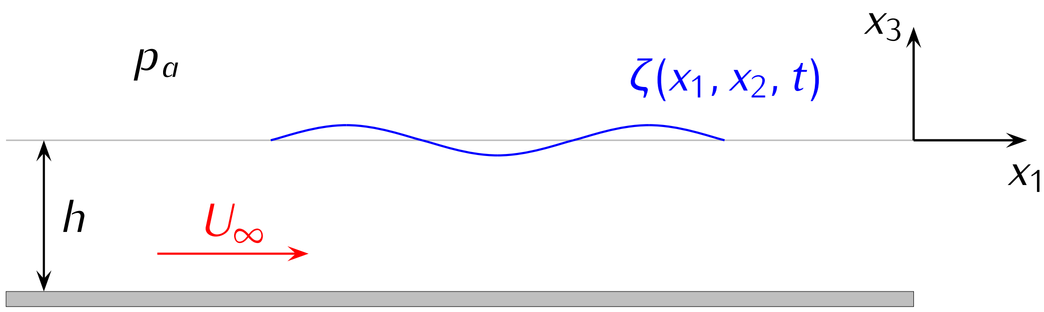
- Body moving steadily in deep water

USS John F. Kennedy aircraft carrier and accompanying destroyers

Ducks swimming across a lake

Kelvin's angle of the wake $2\alpha = 2\text{asin}(1/3) \simeq 39\,\text{deg}$!

Formulation



- Flow velocity $\boldsymbol{u} = (u_1, u_2, u_3)$, potential flow $\boldsymbol{u} = \nabla\phi$
  incompressibility $\nabla \cdot \boldsymbol{u} = 0$, Laplace's equation $\nabla^2\phi = 0$

- Euler's equation for the potential function

$$\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\nabla\phi\cdot\nabla\phi + \rho g x_3 + p = \mathrm{cst} \qquad (1)$$

- Boundary conditions

$$u_3 = \frac{\partial\phi}{\partial x_3} = 0 \quad \text{on} \quad x_3 = -h \quad \text{(impermeable wall)}$$

and free boundary problem on $x_3 = \zeta(x_1, x_2, t)$

Surface tension

which introduces a pressure difference across a curved surface

Surface tension prevents the paper clip (denser than water) from submerging.

Capillary waves
(ripples – short waves $\lambda \leq 2$ cm)
produced by a dropplet of wine !
Courtesy of Olivier Marsden (2010)

Surface tension

Work δW_t needed to increase the surface area of a mass of liquid by an amount dS, $\delta W_t = \gamma_t dS$ (surface tension γ_t in J.m^{-2} = N.m^{-1})

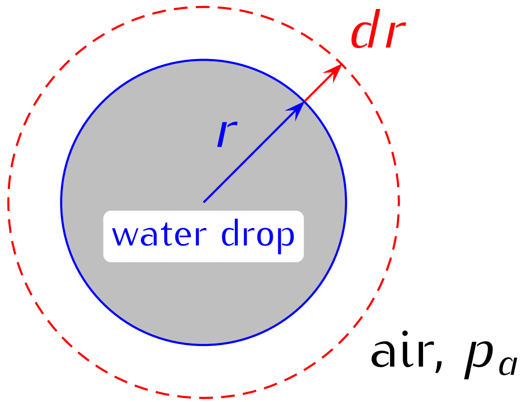
Total energy variation δW

$$\delta W = -p_w dV_w - p_a dV_a + \gamma_t dS$$

$$dV_w = d(4\pi r^3/3) = 4\pi r^2 dr \quad dV_a = -dV_w$$

$$dS = d(4\pi r^2) = 8\pi r dr$$

balance $\delta W = 0 \implies \boxed{p_w - p_a = \dfrac{2\gamma_t}{r}}$



$\gamma_{t\ \text{air-water}} \simeq 0.0728$ N.m^{-1} (20°C)

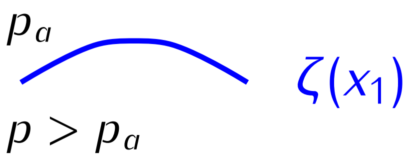
$r = 1$ mm, $\Delta p / p_a \simeq 0.14\%$

Young-Laplace equation (1805)

$p_a - p = \gamma_t C_f$ where $C_f = -\nabla \cdot \boldsymbol{n}_{\to a}$ is the mean curvature in fluid mechanics (the curvature is positive if the surface curves "towards" the normal, convex)

For a sphere, $\boldsymbol{n} = \boldsymbol{e}_r$ and $\nabla \cdot \boldsymbol{n} = \dfrac{1}{r^2}\dfrac{\partial(r^2 \times 1)}{\partial r} = \dfrac{2}{r}$

1-D interface $\zeta(x_1)$



$$C_f = \frac{\zeta_{x_1x_1}}{\left(1 + \zeta_{x_1}^2\right)^{3/2}}$$

2-D interface $\zeta(x_1, x_2)$

$$C_f = \frac{(1+\zeta_{x_2}^2)\zeta_{x_1x_1} + (1+\zeta_{x_1}^2)\zeta_{x_2x_2} - 2\zeta_{x_1}\zeta_{x_2}\zeta_{x_1x_2}}{(1+\zeta_{x_1}^2+\zeta_{x_2}^2)^{3/2}}$$

$\simeq \zeta_{x_1x_1} + \zeta_{x_2x_2}$ by linearization

$\zeta_{x_1} \equiv \partial\zeta/\partial x_1$, ...

Formulation in incompressible flow : free boundary problem

Kinematic condition for the surface deformation ζ (Kelvin, 1871)
interface defined by $f \equiv \zeta(x_1, x_2, t) - x_3 = 0$

Fluid particles on the boundary always remain part on this free surface (the free surface moves with the fluid), that is $Df/Dt = 0$

$$\frac{Df}{Dt} = 0 \quad \Longrightarrow \quad \frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x_1} + u_2 \frac{\partial \zeta}{\partial x_2} - u_3 = 0$$

$$u_3 = \frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x_1} + u_2 \frac{\partial \zeta}{\partial x_2} = \frac{D\zeta}{Dt} \tag{2}$$

Formulation in incompressible flow : free boundary problem

Kinematic condition for the surface deformation ζ, Eqs (1) - (2)

$$\begin{cases} \rho \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}\rho \nabla \phi \cdot \nabla \phi + \rho g \zeta + p_a - \gamma_t C_f = p_a \\ \dfrac{\partial \phi}{\partial x_3} = \dfrac{\partial \zeta}{\partial t} + \dfrac{\partial \phi}{\partial x_1}\dfrac{\partial \zeta}{\partial x_1} + \dfrac{\partial \phi}{\partial x_2}\dfrac{\partial \zeta}{\partial x_2} \end{cases} \qquad \text{on } x_3 = \zeta(x_1, x_2, t)$$

Linearization, velocity potential $\phi = U_\infty x_1 + \phi'$

$$\begin{cases} \rho \dfrac{\partial \phi'}{\partial t} + \rho U_\infty \dfrac{\partial \phi'}{\partial x_1} + \rho g \zeta - \gamma_t \left(\dfrac{\partial^2 \zeta}{\partial x_1^2} + \dfrac{\partial^2 \zeta}{\partial x_2^2} \right) = 0 \\ \dfrac{\partial \phi'}{\partial x_3} = \dfrac{\partial \zeta}{\partial t} + U_\infty \dfrac{\partial \zeta}{\partial x_1} \end{cases} \qquad \text{on } x_3 = 0$$

Wave equation obtained by applying D_∞/Dt to eliminate ζ

$$\frac{D_\infty}{Dt}\left[\frac{D_\infty \phi'}{Dt} + g\zeta - \frac{\gamma_t}{\rho}\left(\frac{\partial^2 \zeta}{\partial x_1^2} + \frac{\partial^2 \zeta}{\partial x_2^2} \right) \right] = 0 \qquad \frac{D_\infty}{Dt} \equiv \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x_1}$$

In summary : 2-D surface waves

$$\nabla^2 \phi' = 0 \tag{3}$$

$$\frac{\partial \phi'}{\partial x_3} = 0 \quad \text{on } x_3 = -h \text{ (bottom)} \tag{4}$$

$$\frac{D_\infty^2 \phi'}{Dt^2} + g\frac{\partial \phi'}{\partial x_3} - \frac{\gamma_t}{\rho}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\frac{\partial \phi'}{\partial x_3} = 0 \quad \text{on } x_3 = 0 \text{ (surface)} \tag{5}$$

« Phare des Baleines » (Lighthouse of the Whales, Île de Ré, by Vauban in 1682)

Photography taken by Michel Griffon

cross sea : two wave systems traveling at oblique angles

The dispersion relation for surface waves

Let us try to find a normal mode solution in Eq. (3), of the form
$\phi' = \psi(x_3) e^{i(k_1 x_1 + k_2 x_2 - \omega t)}$

$$\nabla^2 \phi' = 0 \quad \Longrightarrow \quad \frac{d^2\psi}{dx_3^2} - (k_1^2 + k_2^2)\psi = 0 \qquad k \equiv \sqrt{k_1^2 + k_2^2}$$

Waves on water of a finite (constant) depth h
$\psi(x_3) = A_0 \cosh[k(x_3 + h)] + B_0 \sinh[k(x_3 + h)]$, $B_0 = 0$ with Eq. (4)

The dispersion relation is provided by Eq. (5)

$$-(k_1 U_\infty - \omega)^2 + \left(gk + \frac{\gamma_t}{\rho} k^3 \right) \tanh(kh) = 0 \tag{6}$$

The dispersion relation for surface waves (cont'd)

with $U_\infty = 0$, no running stream to simplify the discussion

$$\omega^2 = \left(1 + \frac{\gamma_t}{\rho g} k^2\right) gk \tanh(kh) \qquad \text{dispersive waves (Kelvin, 1871)}$$

- Capillary waves

$$l_c \equiv \sqrt{\frac{\gamma_t}{\rho g}} \quad \text{capillary length} \qquad l_c \simeq 2.7 \text{ mm for air-water interface}$$

$kl_c = 1 \implies \lambda = 2\pi l_c \simeq 1.7$ cm

Only important for short waves ('ripples')

$\lambda \geq l_c, kh \gg 1 \qquad \omega^2 \simeq \left[1 + (kl_c)^2\right] gk$

Phase velocity $v_\varphi = \left\{(gl_c)/(kl_c)\left[1 + (kl_c)^2\right]\right\}^{1/2}$

and minimum reached for $kl_c = 1$, $v_\varphi = \sqrt{2gl_c} \simeq 0.23$ m.s^{-1}

- **Properties of the dispersion relation for surface waves**

  phase velocity $v_\varphi^2 = \left[1 + (k l_c)^2\right] \frac{gh}{kh} \tanh(kh)$

long waves ($k \to 0$)
$v_\varphi = \sqrt{gh}$
(non-dispersive waves)

short waves ($k \to \infty$)
$v_\varphi = (k l_c)^2 \, g/k$

gravity waves

$k l_c = 1$

capillary waves

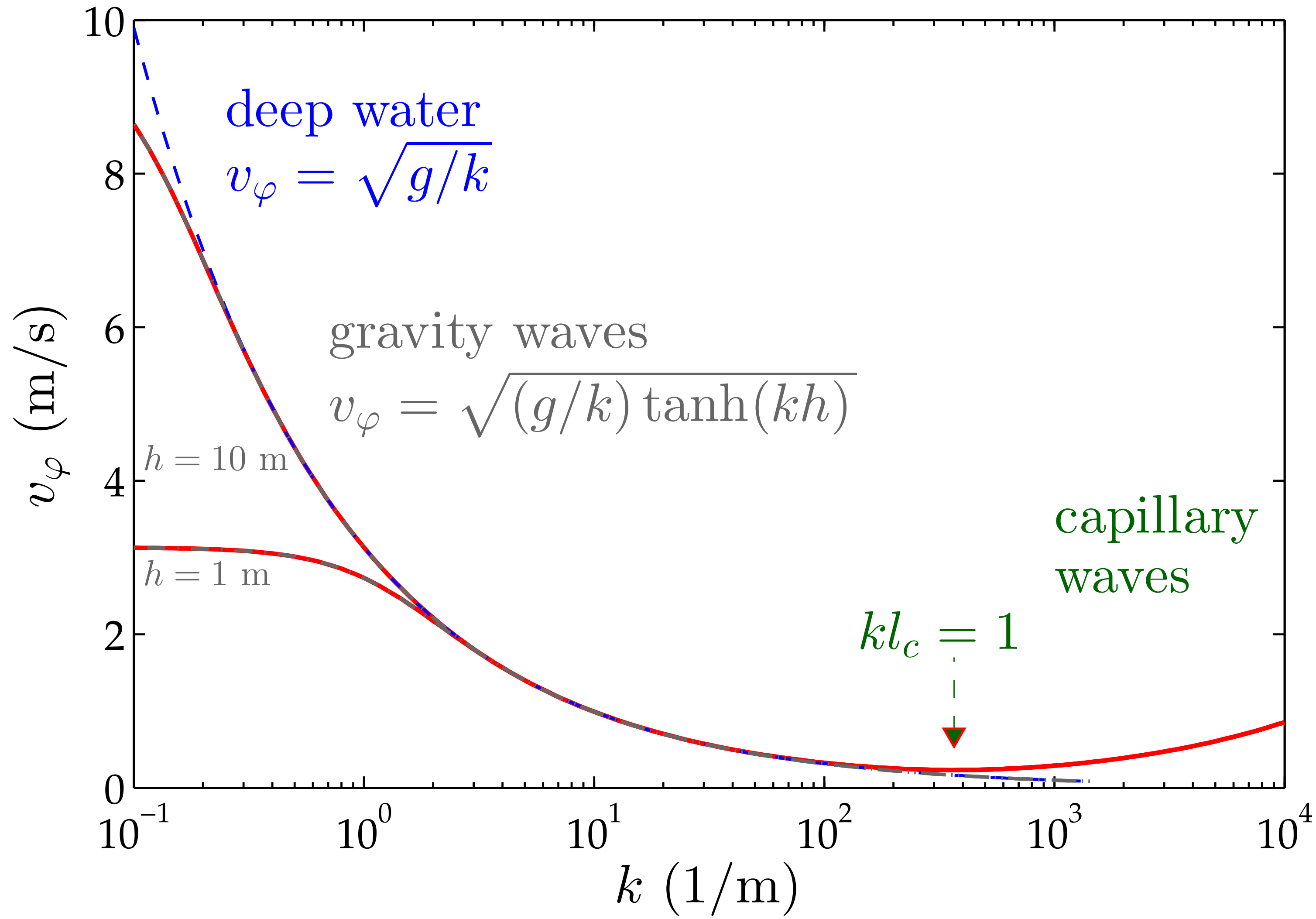
wavenumber k

negligible surface tension
$v_\varphi^2 = (g/k) \tanh(kh)$

deep water
$\lambda \ll h$ or $kh \gg 1$
$v_\varphi = \sqrt{g/k}$

shallow water
$\lambda \gg h$ or $kh \ll 1$
$v_\varphi = \sqrt{gh}$

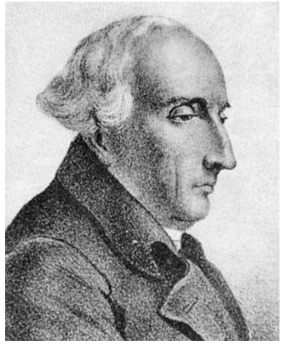
- **The dispersion relation for surface waves**
  $v_\varphi = \left\{\left[1 + (k l_c)^2\right](g/k)\tanh(kh)\right\}^{1/2}$



The dispersion relation for surface waves

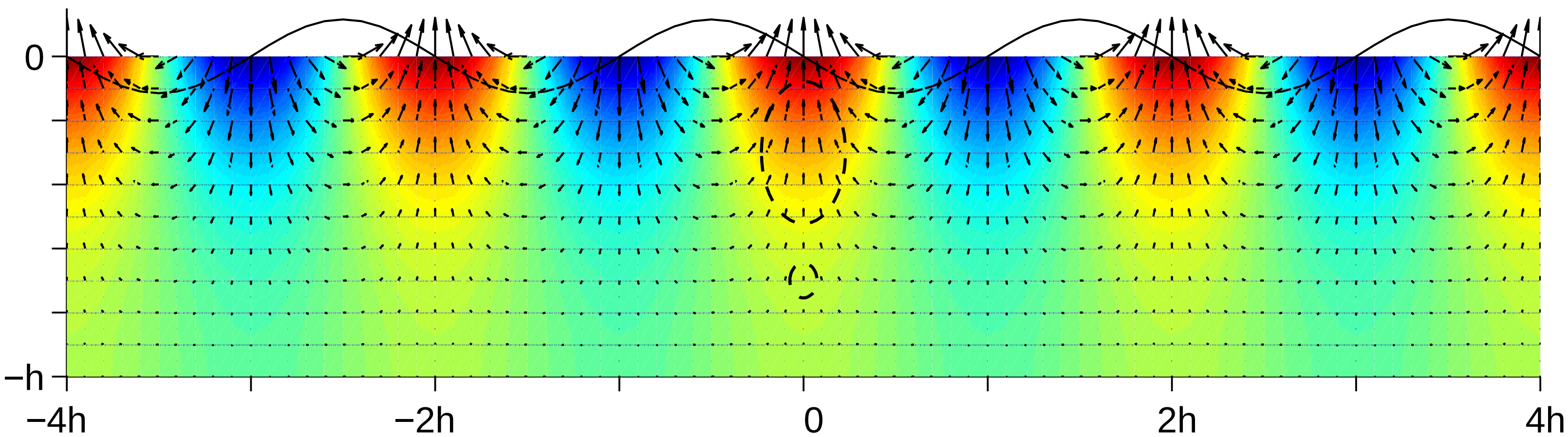
When surface tention effects are negligible, the dispersion relation for gravity waves was obtained by Lagrange

$$\omega^2 \simeq gk \tanh(kh)$$



Joseph Louis Lagrange
(1736–1813)

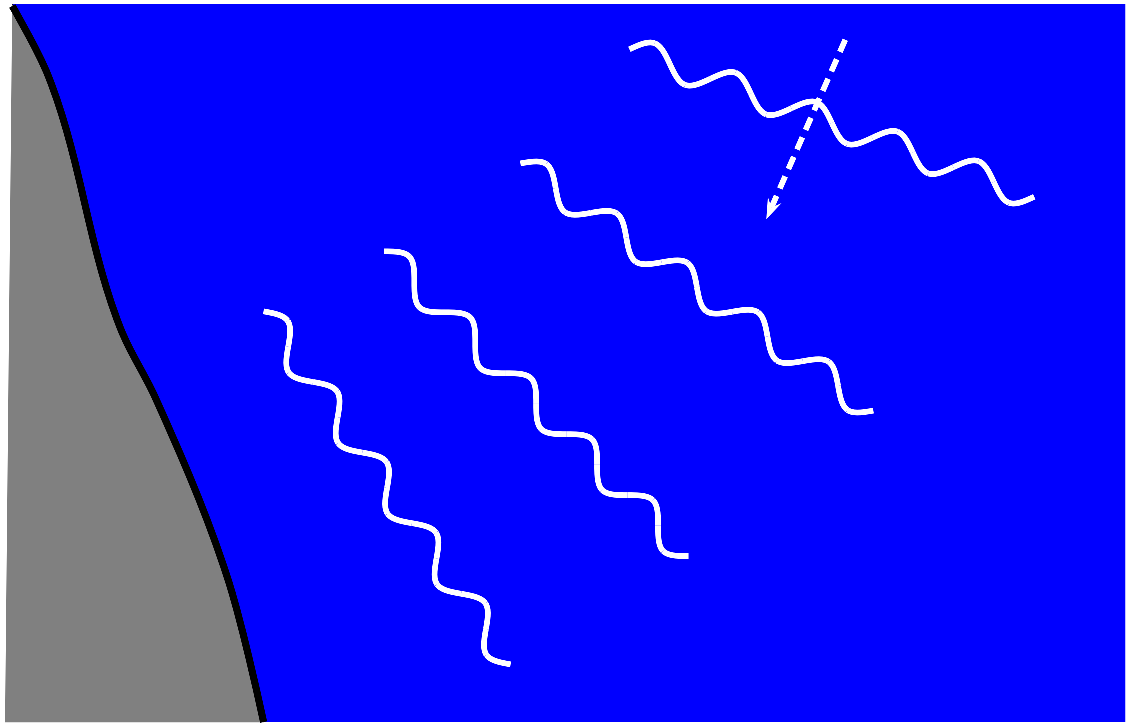
$T = 8$ s, $kh = \pi$, $\lambda \simeq 100$ m, $v_\varphi \simeq 12.5$ m.s^{-1}
(propagation of crests)



Snapshot of the solution obtained for $\lambda = 2h$ $(kh = \pi)$

Wave refraction

Alignment of wave crests arriving near the shore



In deep water,

$\omega_0 = \sqrt{gk} \quad v_\varphi = \sqrt{g/k}$

Near the coast, $h \searrow$

$\omega_0^2 = gk \tanh(kh)$

$\Longrightarrow \quad \lambda, v_\varphi \searrow$

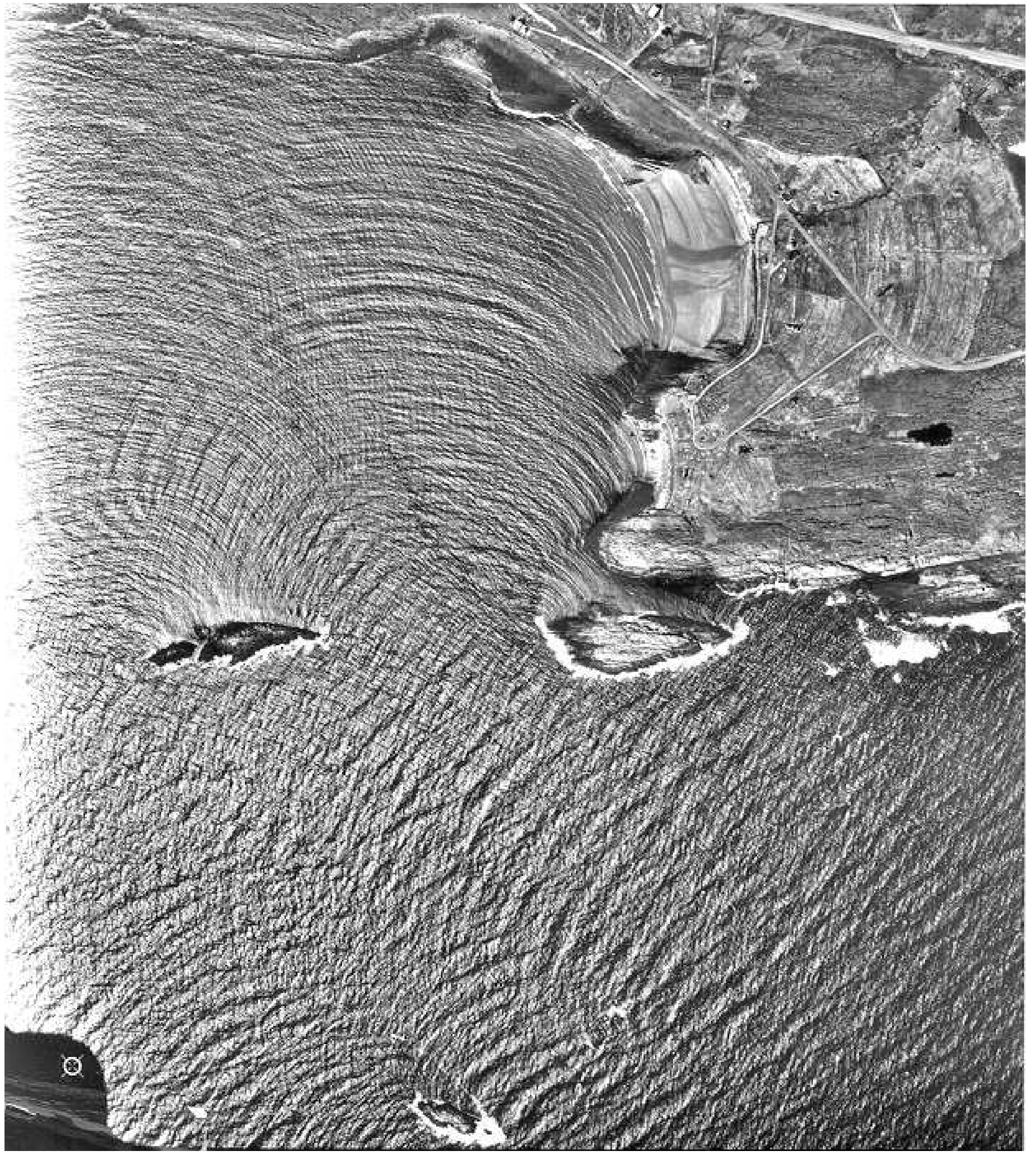
Reduction of both the wavelength and the wave speed near coasts by shallow-water effects

$$\text{for } h = 10 \text{ m}, \lambda \simeq 70.9 \text{ m}, v_\varphi \simeq 8.9 \text{ m.s}^{-1}$$
$$\text{for } h = 1 \text{ m}, \lambda \simeq 24.8 \text{ m}, v_\varphi \simeq 3.1 \text{ m.s}^{-1}$$

- Tsunamis


(Kamakura, south of Tokyo, August 2016)

Wave refraction and diffraction



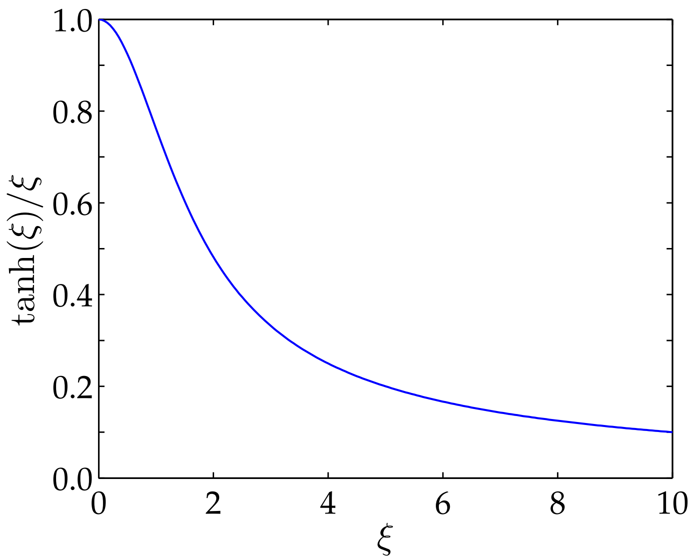
Aerial photo of an area near Kiberg on the coast of Finnmark in Norway (taken 12 June 1976 by Fjellanger Widerøe A.S.)

- **The dispersion relation for surface waves on a running stream**

From Eq. (6), $(k_1 U_\infty - \omega)^2 = gk \tanh(kh)$, stationnary waves as $\omega \to 0$

2-D case (x_1, x_3), $k_3 = k = k_1$

$$U_\infty^2 = gh \, \frac{\tanh(kh)}{kh}$$



Only solutions for $U_\infty^2 \leq gh$,
corresponding to a Froude number Fr

$$\mathrm{Fr} \equiv \frac{U_\infty}{\sqrt{gh}} < 1$$

The Froude number is the ratio of the flow velocity U_∞ to the phase velocity $v_\varphi = \sqrt{gh}$. The flow is subcritical for $\mathrm{Fr} < 1$ (analogous to subsonic in gasdynamics)

William Froude
(1810 – 1879)

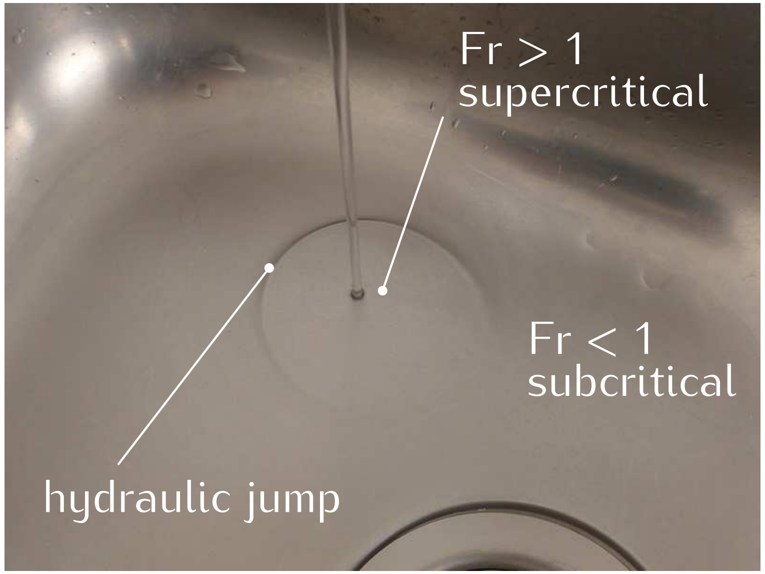
- **The dispersion relation for surface waves on a running stream**

stationary waves ($\omega \to 0$)

3-D case, with now $k_3 = k = \sqrt{k_1^2 + k_2^2}$

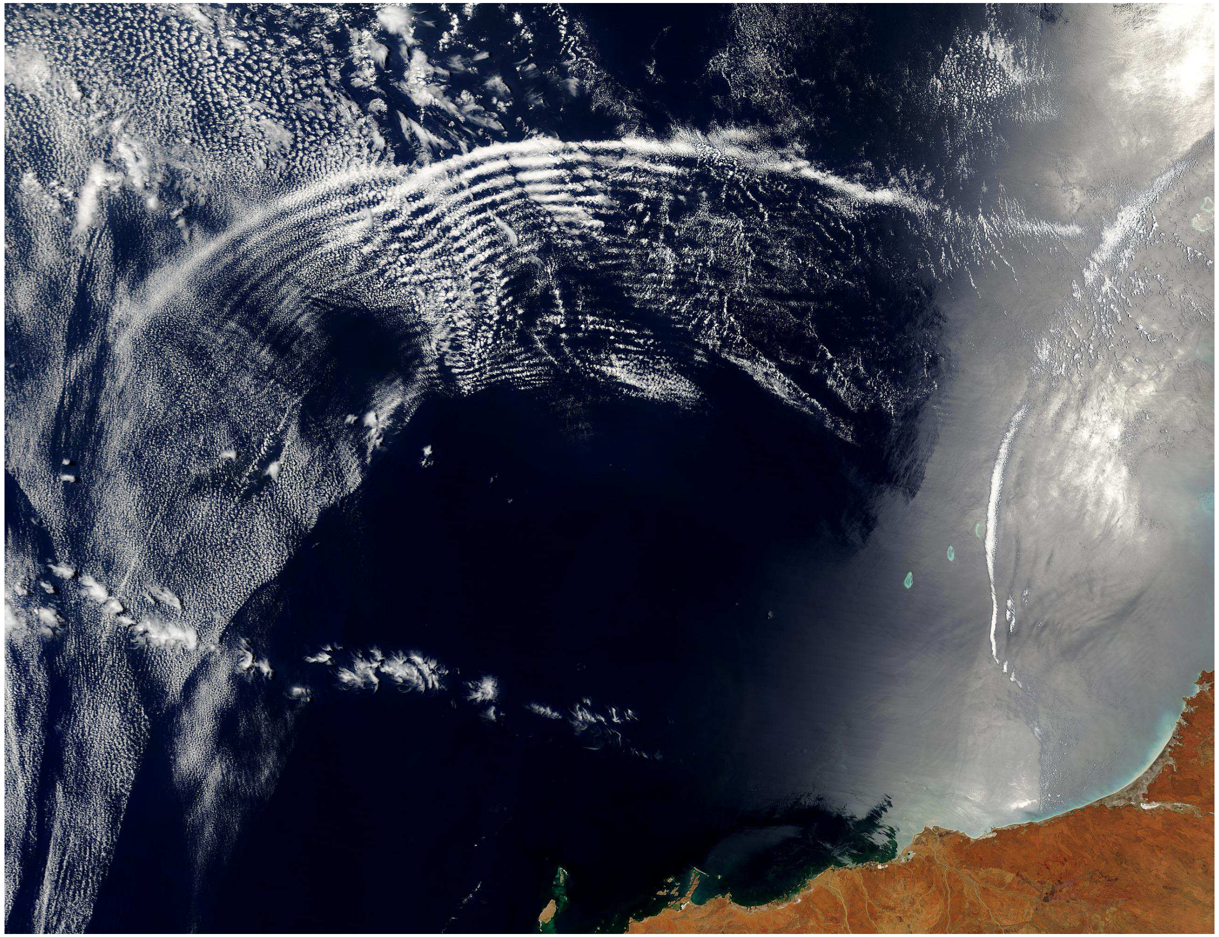
$$U_\infty^2 k_1^2 = gk \tanh(kh)$$

For a shallow water flow, $h \to 0$, $(U_\infty^2 - gh)k_1^2 = ghk_2^2$
only solutions $k_2 \neq 0$ for $\mathrm{Fr} > 1$, supercritical flow



Hydraulic (laminar) jump – analogous to a shock wave in gasdynamics – when tap water spreads on the horizontal surface of a sink
$\rightsquigarrow$ nonlinear problem

- **Atmospheric** **internal gravity waves** off Australia
  (taken by Terra Satellite on Nov. 2003 - NASA)

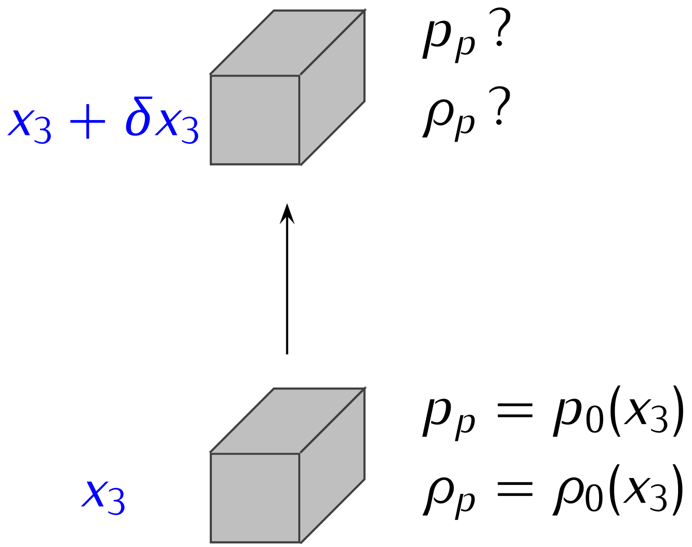


- **Oscillations in the presence of gravity (atmosphere, ocean)**

  Stratified medium at rest, $\rho_0(x_3)$, $p_0(x_3)$ satisfying the hydrostatic equation

$$\frac{dp_0}{dx_3} = -\rho_0 g$$

Fluid particle moving from altitude x_3 to $x_3 + \delta x_3$



- The pressure of the fluid particle at $x_3 + \delta x_3$ is $p_p = p_0(x_3 + \delta x_3) \simeq p_0(x_3) - \rho_0(x_3) g \delta x_3$
- Assuming a reversible (adiabatic) process, the density of the particle at $x_3 + \delta x_3$ is

$$\frac{\rho_p(x_3 + \delta x_3)}{\rho_p(x_3)} = \left(\frac{p_p(x_3 + \delta x_3)}{p_p(x_3)} \right)^{1/\gamma} \simeq 1 - \frac{\rho_0 g \delta x_3}{\gamma p_0}$$

$$\Longrightarrow \rho_p(x_3 + \delta x_3) \simeq \rho_0(x_3) - \rho_0(x_3) \frac{g}{c_0^2(x_3)} \delta x_3$$

Oscillations in the presence of gravity (atmosphere, ocean)

To observe wave propagation (oscillations : restoring force from the principle of Archimedes), the density of the surrounding fluid at $x_3 + \delta x_3$ must be smaller than the density of the fluid particle, that is $\rho_0(x_3 + \delta x_3) < \rho_p(x_3 + \delta x_3)$

$$\rho_0(x_3) + \frac{d\rho_0}{dx_3}(x_3)\delta x_3 < \rho_0(x_3) - \rho_0(x_3)\frac{g}{c_0^2(x_3)}\delta x_3$$

$$-\frac{d\rho_0}{dx_3} - \rho_0\frac{g}{c_0^2} \geq 0$$

The restoring gravitational force per unit volume may be written $\rho_0 N^2 \delta x_3$ where $N(x_3)$ has the dimension of a frequency, known as the Väisälä-Brunt frequency

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dx_3} - \frac{g^2}{c_0^2} \qquad (7)$$

Very low frequency – in the atmosphere, typically $T = 2\pi/N \sim 10^2$ s

$N^2 > 0$ for a stable stratified fluid

Oscillations in the presence of gravity (atmosphere, ocean)

Stratified fluid at rest $\rho_0(x_3)$, incompressible perturbations governed by the linearized Euler equations

$$\nabla \cdot \boldsymbol{u}' = 0 \qquad \frac{\partial \rho'}{\partial t} + \boldsymbol{u}' \cdot \nabla \rho_0 = 0 \qquad \rho_0 \frac{\partial \boldsymbol{u}'}{\partial t} = -\nabla p' + \rho' \boldsymbol{g}$$

By cross-differentiation to eliminate ρ', p', u_1' and u_2', the following equation can be derived for u_3'

$$\frac{\partial^2}{\partial t^2} \nabla^2 u_3' = -N_0^2 \nabla_\perp^2 u_3' + \frac{N_0^2}{g} \frac{\partial^3 u_3'}{\partial t^2 \partial x_3}$$

where $\nabla_\perp^2 \equiv \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2}$ is the horizontal Laplacian, and N_0^2 is the approximation of N^2 for incompressible perturbations, see Eq. (7),

$$N_0^2(x_3) = -\frac{g}{\rho_0}\frac{d\rho_0}{dx_3} - \frac{g^2}{c_0^2} \qquad (c_0 \to \infty) \tag{8}$$

- **Oscillations in the presence of gravity (atmosphere, ocean)**

  By assuming that $N_0 \simeq$ cte to simplify calculations (*e.g.* isothermal atmosphere), the following dispersion relation is obtained with $u_3' \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$

$$\omega^2 = \frac{N_0^2 k_\perp^2}{k^2 + ik_3 N_0^2/g} \qquad\qquad k_\perp^2 \equiv k_1^2 + k_2^2$$

  Furthermore, with $N_0^2 \sim g/H$ where H is a characteristic scale of the stratified atmosphere, a classic assumption is $kH \gg 1$ (high-frequency approximation, background medium varies slowly over a wave cycle)

$$\omega^2 \simeq N_0^2 \frac{k_\perp^2}{k^2}$$

  Waves are only possible in the case $\omega \leq N_0$, and more surprisingly, the wavelength is not determined by the dispersion relation

Oscillations in the presence of gravity (atmosphere, ocean)

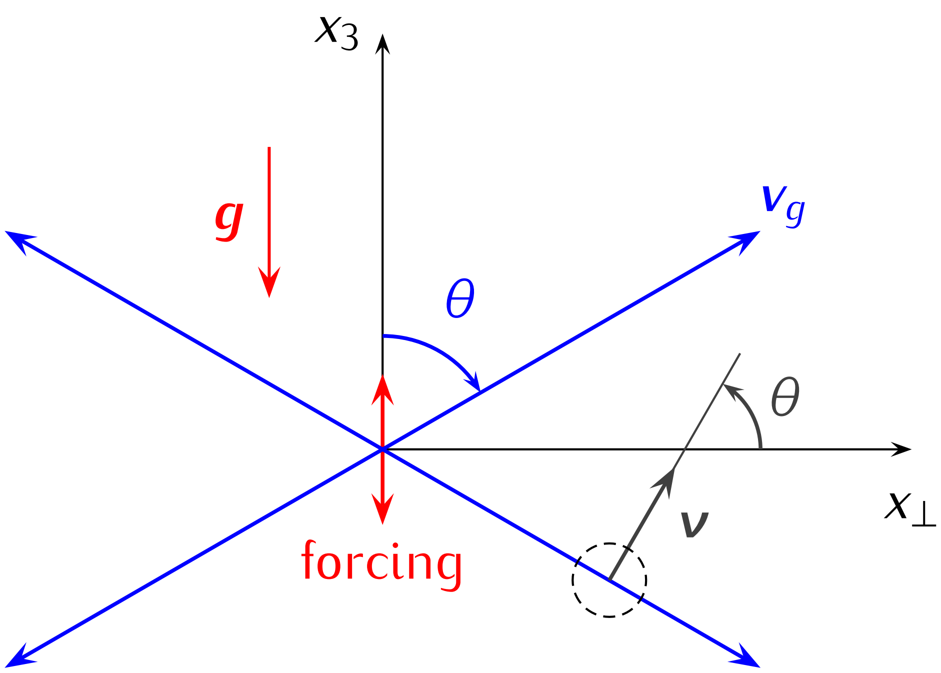
With $k_\perp \equiv k\cos\theta$ and $k_3 \equiv k\sin\theta$, the dispersion relation reads

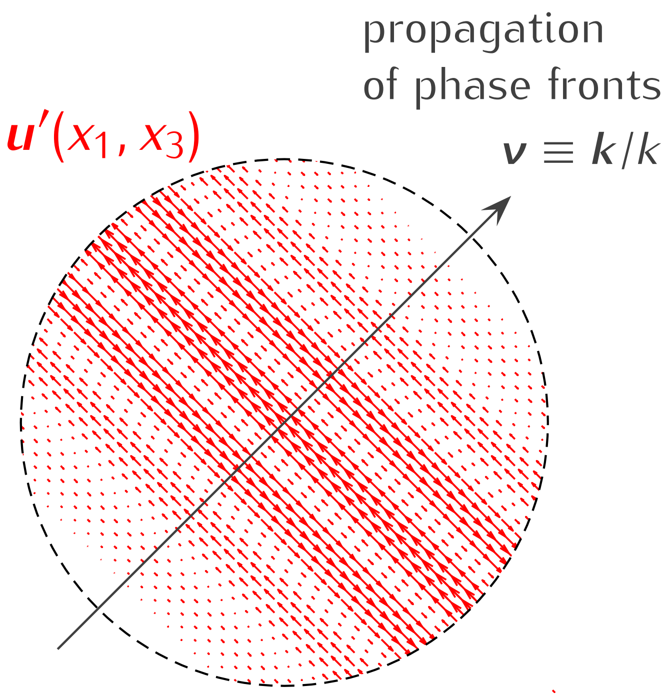
$$\omega = N_0 \frac{k_\perp}{k} = N_0|\cos\theta|$$

$$\boldsymbol{u}' = \hat{\boldsymbol{u}}(\boldsymbol{k})e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

$$\nabla\cdot\boldsymbol{u}' = 0 \quad \Longrightarrow \quad \boldsymbol{k}\cdot\hat{\boldsymbol{u}} = 0$$

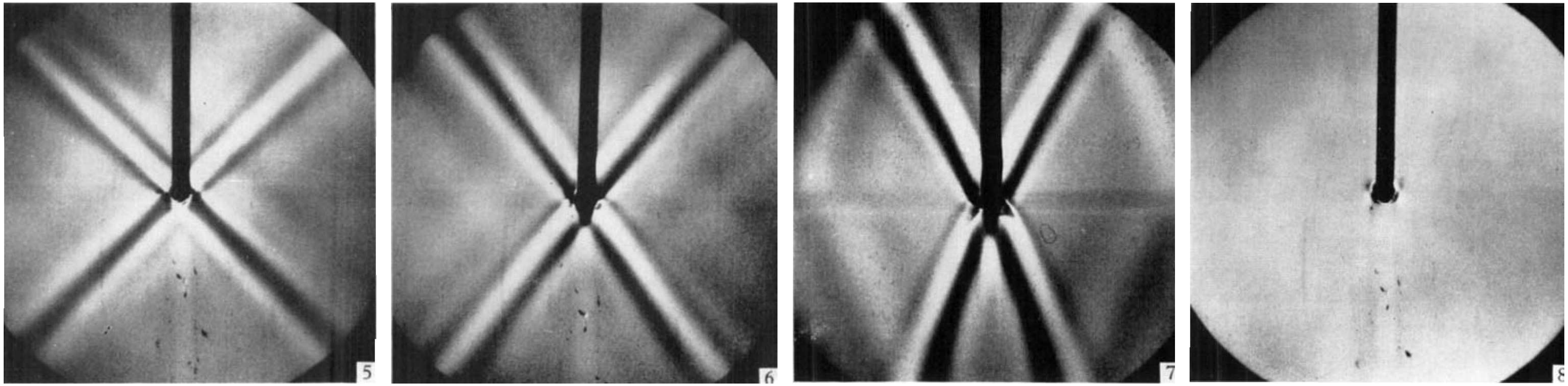
transverse waves





phase velocity $v_\varphi = \omega/k \equiv$ propagation of constant phase lines in the $\boldsymbol{k}$ direction

- **Oscillations in the presence of gravity (atmosphere, ocean)**
  Mowbray & Rarity, *J. Fluid Mech.*, 1967



Source : vertically oscillating cylinder ($D = 2$ cm) normal to the pictures

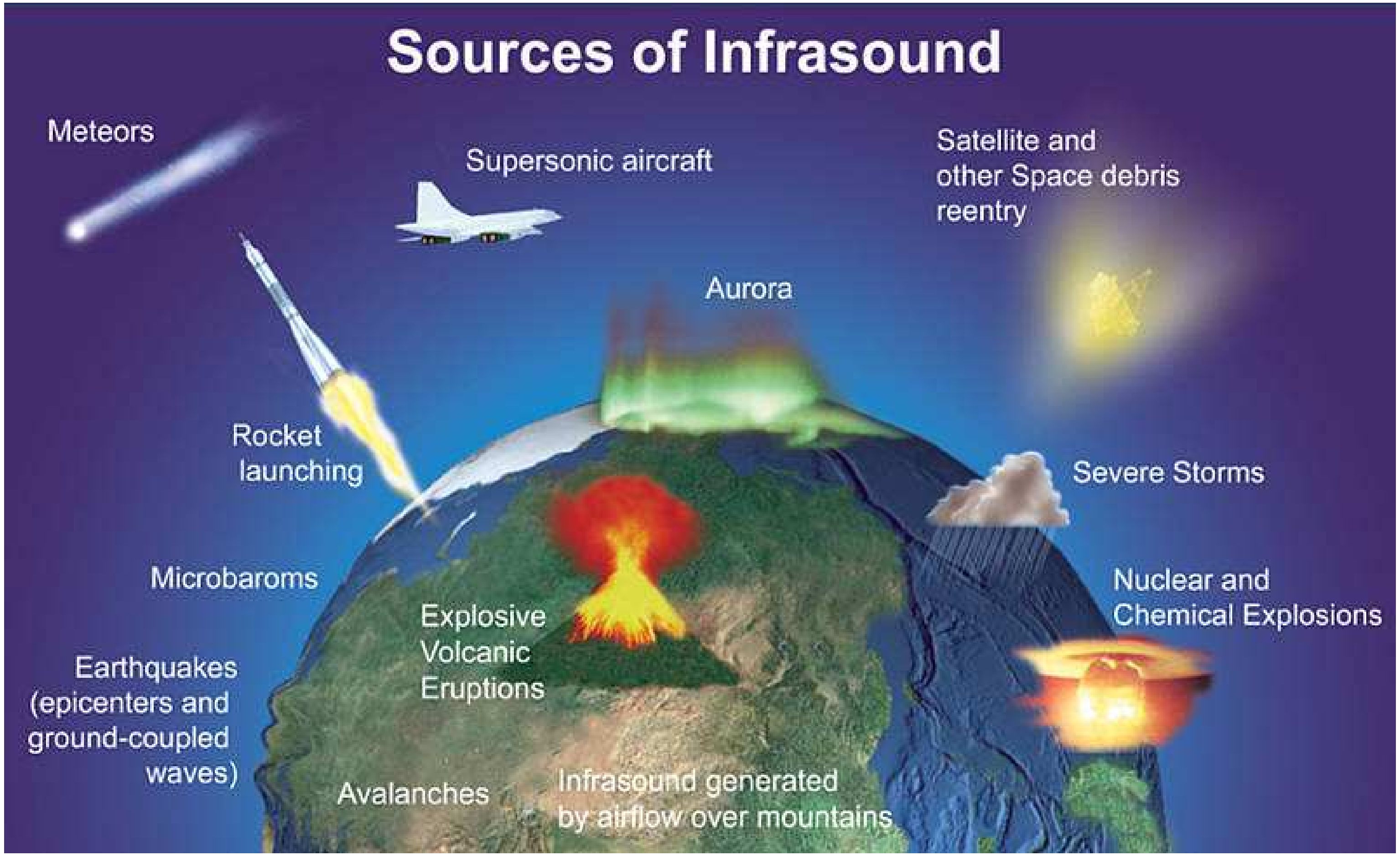
$\omega/N_0 \simeq 0.615, 0.699, 0.900 \qquad \Longrightarrow \qquad \theta \simeq 52, 46, 26$ deg

No gravity waves for $\omega/N_0 \simeq 1.11$

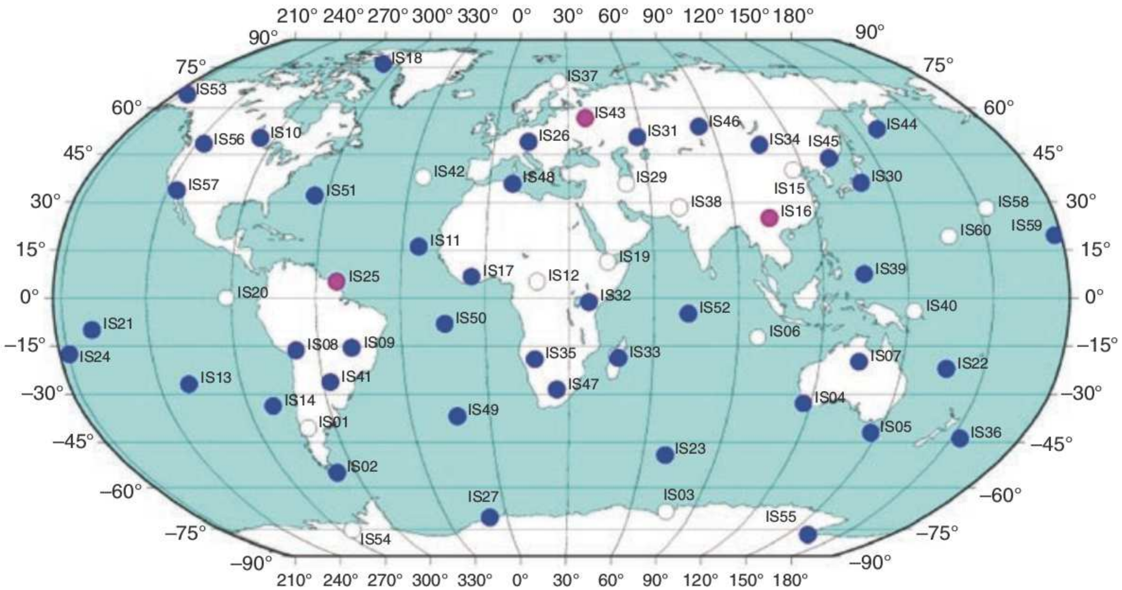
($\theta = 56$ deg)

Motivations for monitoring infrasound

Infrasound : academic definition, $0.01 \leq f < 20$ Hz. These low-frequency waves can propagate over long distances (several hundreds of km) in the Earth's atmosphere. In practice, the relevant passband is closer to $0.02 \leq f \leq 4$ Hz



- **Worldwide infrasound monitoring network** developed to verify compliance with the Comprehensive Nuclear-Test-Ban Treaty (CTBT)



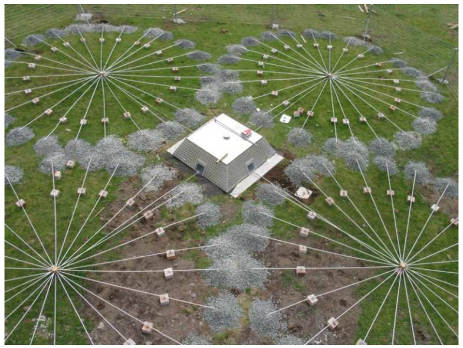
Headquarter : Vienna, Austria

60 stations with 4 to 8 micro-barometers over an area of 1 - 9 km^2

- Certified and sending data to the International Data Centre (IDC)
- under construction, ○ planned

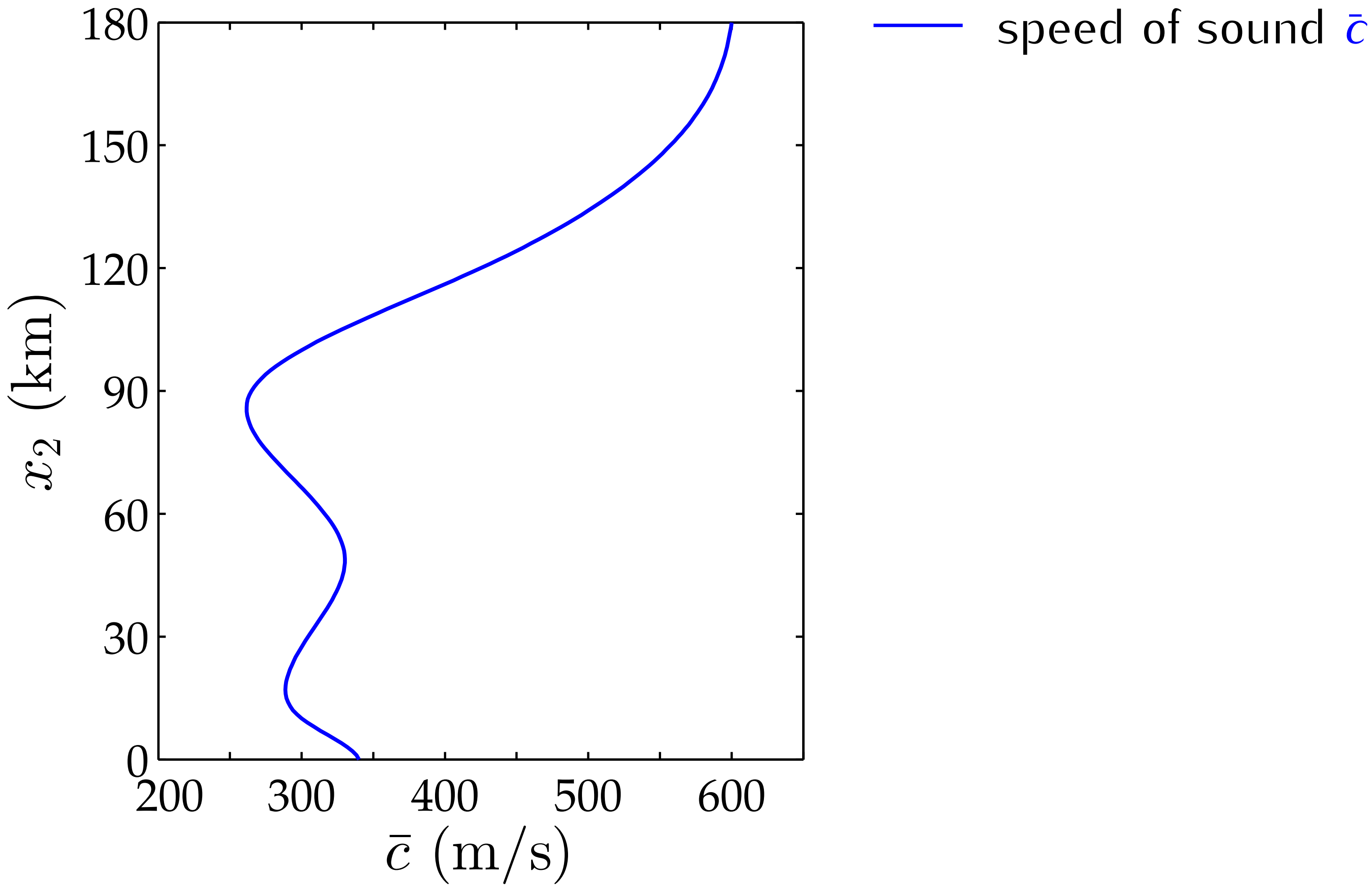
(Christie & Campus, 2010)






Propagation in the Earth's atmosphere

Stratified atmosphere extending up to 180 km altitude



Propagation in the Earth's atmosphere

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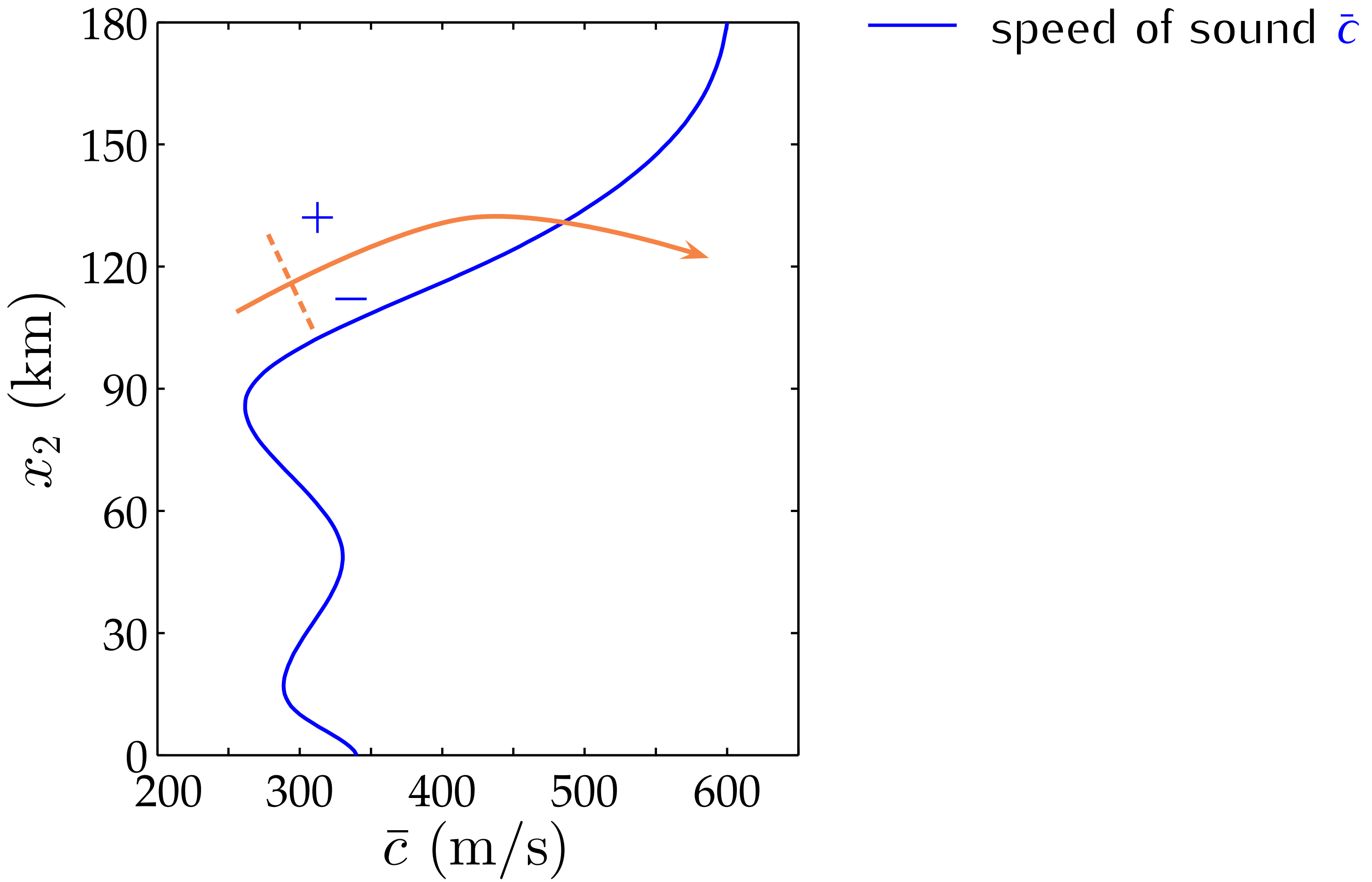
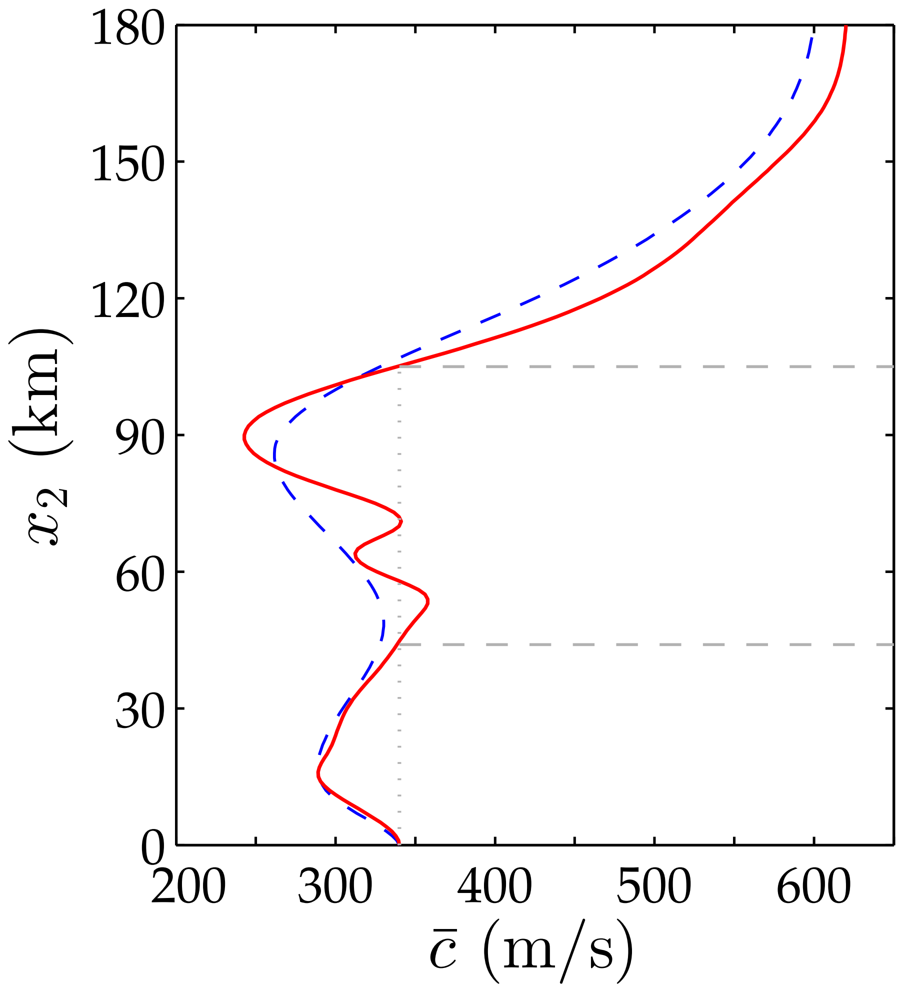


Propagation in the Earth's atmosphere

Stratified atmosphere extending up to 180 km altitude



- - - speed of sound $\bar{c}$

— effective speed of sound

$\bar{c}_e = \bar{c} + \bar{u}_1$

Waves naturally refracted towards stratospheric and thermospheric wave guides ($x_2 \simeq 44$ km and $x_2 \simeq 105$ km) according to geometrical acoustics through the Snell-Descartes law

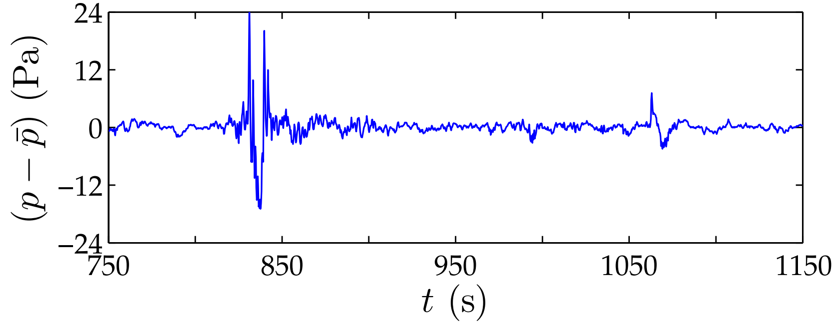
Measured signals from *Misty picture* event

High chemical explosion experiment at White Sands Missile Range, New Mexico, USA, May 14, 1987 (US Defense Nuclear Agency)

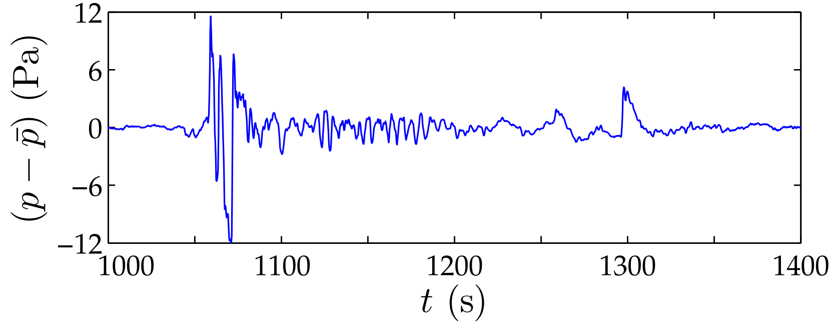
Signals recorded by 3 laboratories up to 1200 km from the source (4.7 kt AFNO)

(Gainville *et al.*, 2010)

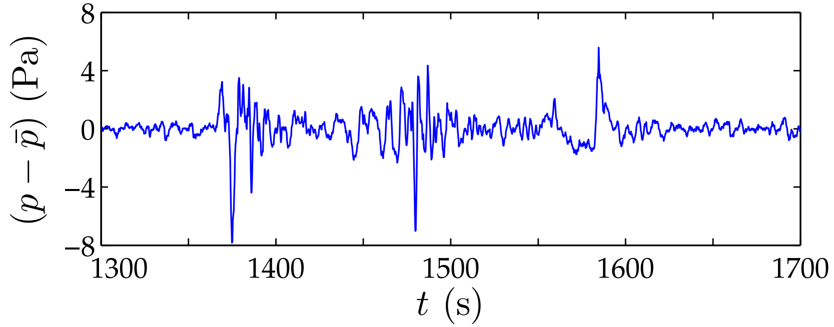
Alpine 248 km W



White River 324 km W

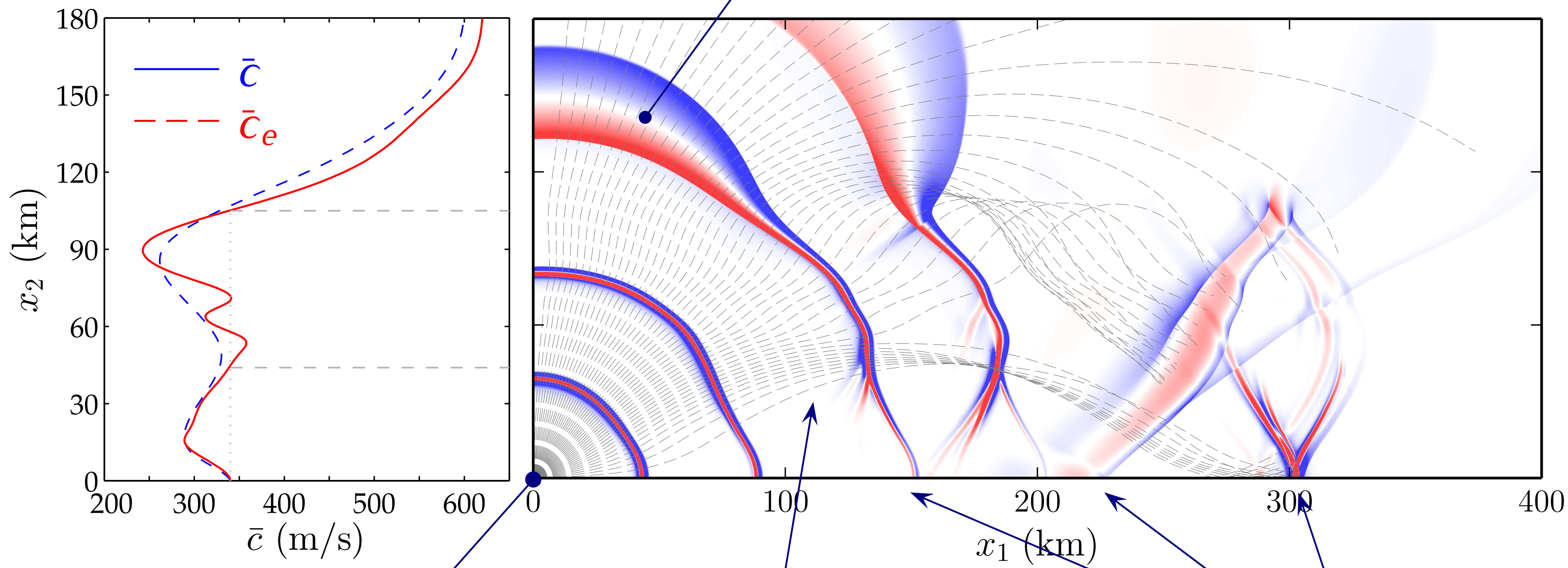


Roosevelt 431 km W



- Nonlinear propagation with wind (NLW) : global view

nonlinear effects
$p'/\bar{p} \sim \bar{\rho}^{-1/2}$ + absorption



refraction induced by
$\bar{c}(\boldsymbol{x})$ and $\bar{u}_1(\boldsymbol{x})$

source
$f_s \sim 0.1$ Hz

Various arrivals
at ground level

Sabatini *et al.* (2015)

In summary

Linear wave equation with constant coefficients $\mathcal{L}(\phi) = 0$
We assume the elementary solution (plane wave) has the form $\phi \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$,
leading to the dispersion relation $\mathcal{D}(\boldsymbol{k}, \omega) = 0$

Sound waves in a homogeneous medium at rest, $\partial_{tt}p' - c_\infty^2 \nabla^2 p' = 0$
$\omega = \pm\Omega(\boldsymbol{k})$ with $\Omega(\boldsymbol{k}) = k c_\infty$ (2 modes)

Advection equation $\partial_t u + c_\infty \partial_{x_1} u = 0$
$\omega = \Omega(\boldsymbol{k})$ with $\Omega(\boldsymbol{k}) = c_\infty k_1$ (1 mode)

Surface gravity waves (without surface tension effects)
$\omega = \pm\Omega(\boldsymbol{k})$ with $\Omega(\boldsymbol{k}) = \sqrt{gk \tanh(kh)}$ (2 modes) $k = k_3 = \sqrt{k_1^2 + k_2^2}$

Internal gravity waves (Boussinesq approximation)
$\omega = \pm\Omega(\boldsymbol{k})$ with $\Omega(\boldsymbol{k}) = N_0 \, |\cos\theta|$ (2 modes) $\cos\theta = k_\perp / k$

Waves in Fluids : models for linear wave propagation
$\rightsquigarrow$ theories for linear dispersive waves I

Acoustic wave equation : d'Alembert's solution

Solution of Cauchy's initial value problem
$p'(x_1) = g_0(x_1)$ and $\partial_t p' = g_1(x_1)$ at time $t = 0$

$$\frac{\partial^2 p'}{\partial t^2} - c_\infty^2 \frac{\partial^2 p'}{\partial x_1^2} = 0$$

From the general solution $p'(x,t) = p_l(x_1 + c_\infty t) + p_r(x_1 - c_\infty t)$, obtained by introducing the characteristic variables $\eta^+ = x_1 - c_\infty t$ and $\eta^- = x_1 + c_\infty t$, one has

$$g_0(x_1) = p_l(x_1) + p_r(x_1) \qquad g_1(x_1) = c_\infty \left[\partial_{x_1} p_l(x_1) - \partial_{x_1} p_r(x_1)\right]$$

and by integration,

$$\int_0^{x_1} g_1(\xi) d\xi = c_\infty \left[p_l(x_1) - p_r(x_1)\right] + \mathrm{cst}$$

Acoustic wave equation : d'Alembert's solution

Initial value problem (cont'd)

The two functions p_l and p_r can be determined as follows,

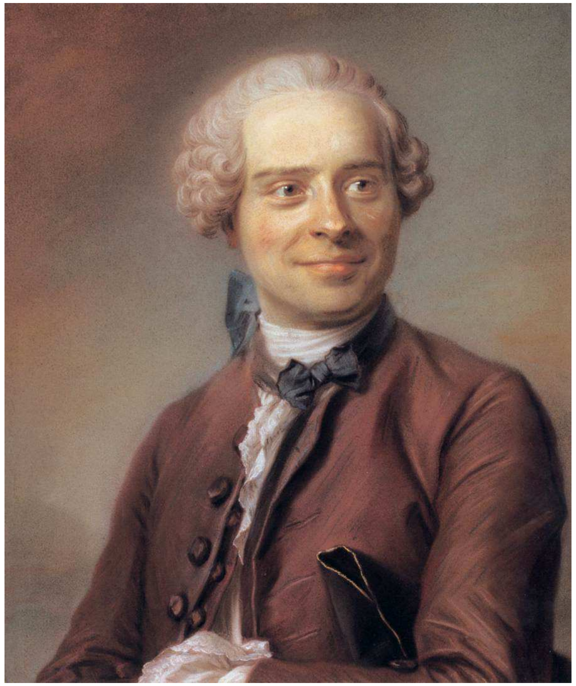
$$\begin{cases} p_l(x_1) = \dfrac{1}{2}\left(g_0(x_1) + \dfrac{1}{c_\infty}\displaystyle\int_0^{x_1} g_1(\xi)d\xi\right) - \dfrac{1}{2c_\infty}\text{cst} \\ p_r(x_1) = \dfrac{1}{2}\left(g_0(x_1) - \dfrac{1}{c_\infty}\displaystyle\int_0^{x_1} g_1(\xi)d\xi\right) + \dfrac{1}{2c_\infty}\text{cst} \end{cases}$$

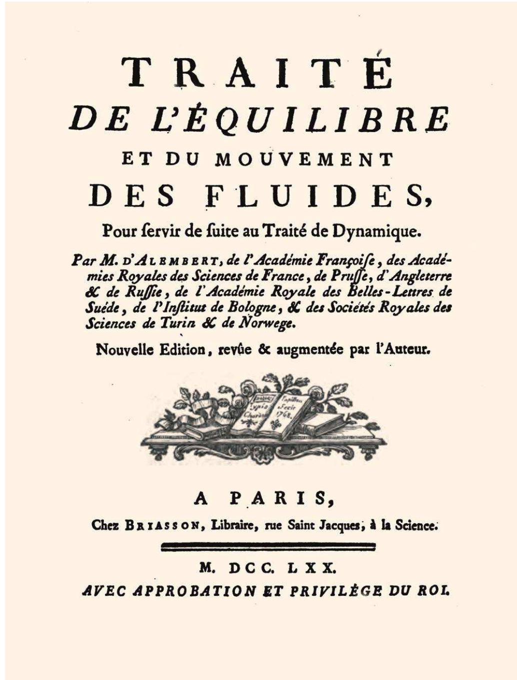
d'Alembert's solution

$$p'(x_1,t) = \frac{1}{2}[g_0(x_1 + c_\infty t) + g_0(x_1 - c_\infty t)] + \frac{1}{2c_\infty}\int_{x_1 - c_\infty t}^{x_1 + c_\infty t} g_1(\xi)d\xi$$

Acoustic wave equation : d'Alembert's solution

Initial value problem (cont'd)





TRAITÉ
DE L'ÉQUILIBRE
ET DU MOUVEMENT
DES FLUIDES,
Pour servir de suite au Traité de Dynamique.
Par M. D'ALEMBERT, de l'Académie Françoise, des Académies Royales des Sciences de France, de Prusse, d'Angleterre & de Russie, de l'Académie Royale des Belles-Lettres de Suède, de l'Institut de Bologne, & des Sociétés Royales des Sciences de Turin & de Norwege.
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A PARIS,
Chez BRIASSON, Libraire, rue Saint Jacques, à la Science.
M. DCC. LXX.
AVEC APPROBATION ET PRIVILÈGE DU ROI.

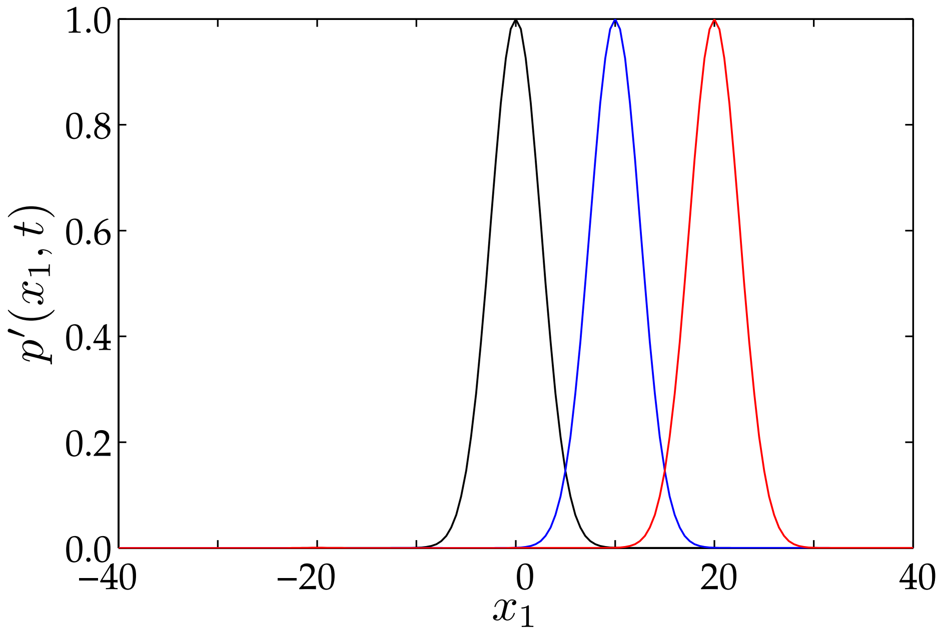
Jean Le Rond d'Alembert (1717–1783)

Acoustic wave equation : d'Alembert's solution

$$g_0(x_1) = e^{-\ln 2\,(x_1/b)^2} \qquad (c_\infty = 1)$$

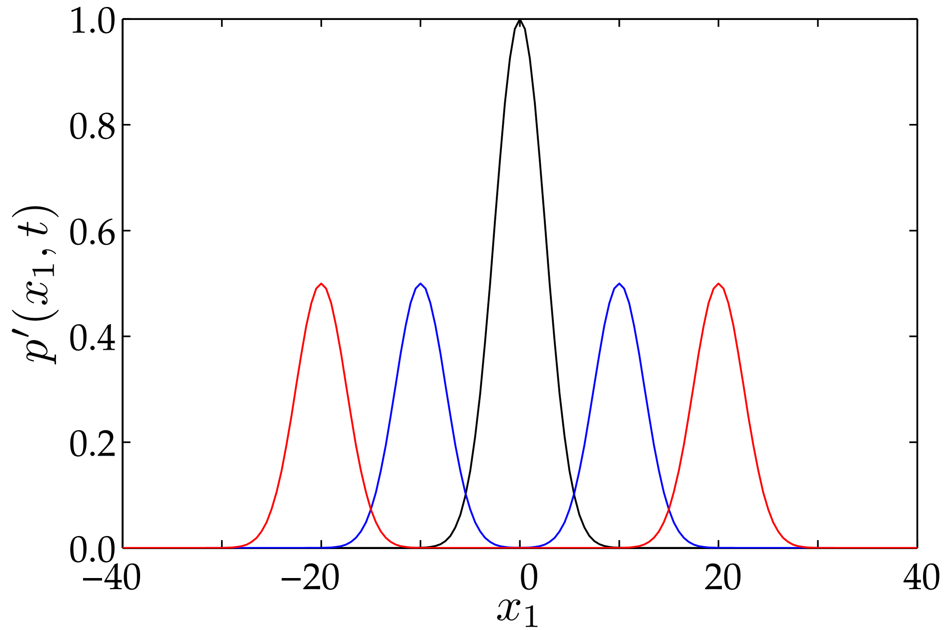
$\boxed{g_1 = -c_\infty g_0'}$

$t = 0 \quad t = 10 \quad t = 20$



$\boxed{g_1 = 0}$

$t = 0 \quad t = 10 \quad t = 20$

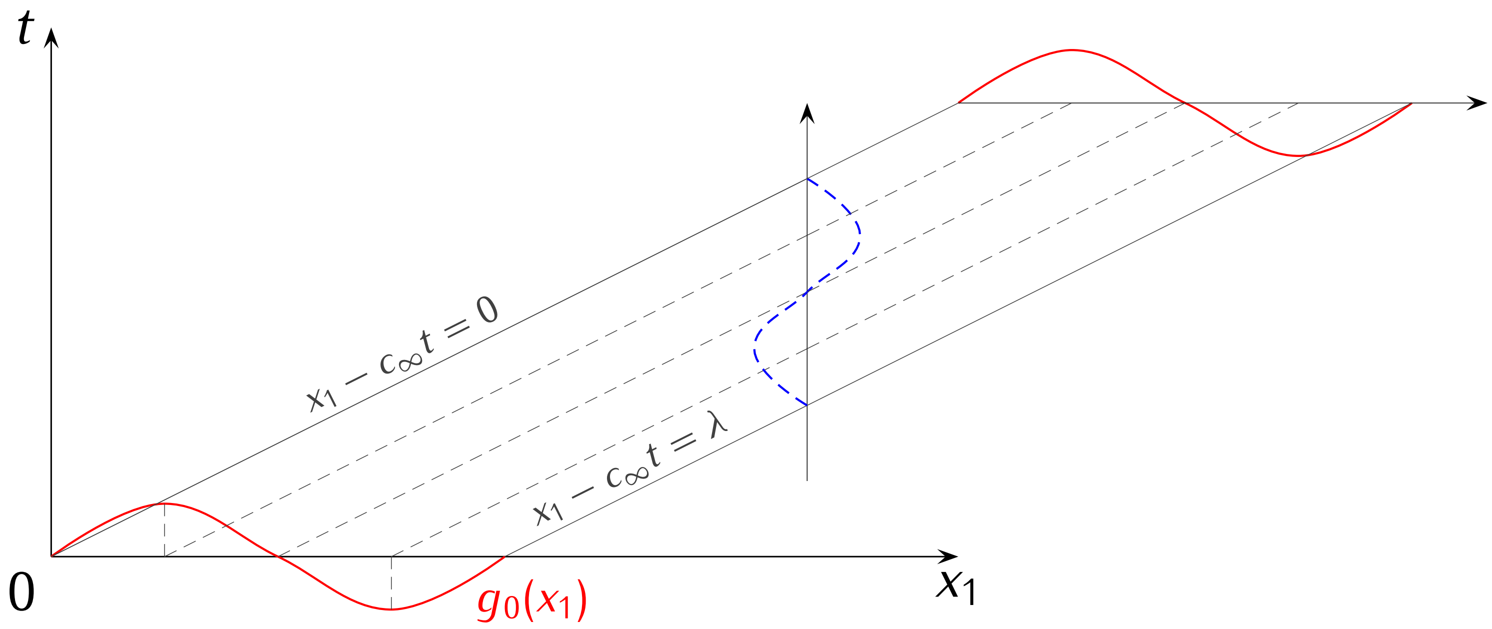


Acoustic wave equation : d'Alembert's solution

Interpretation in terms of characteristic curves

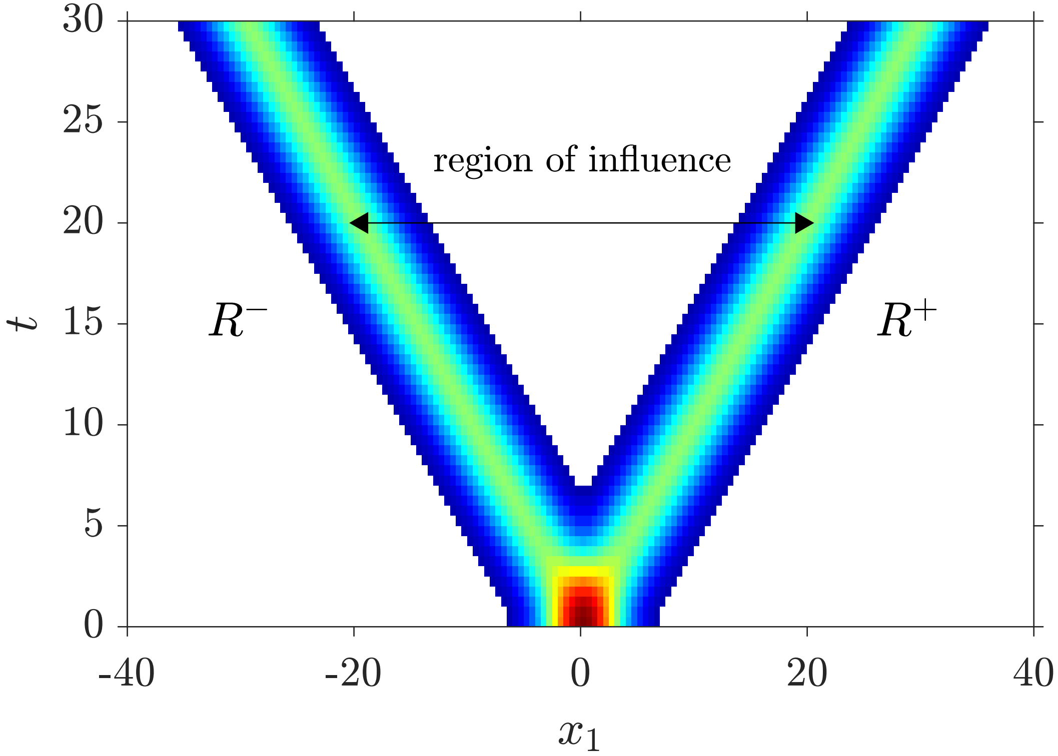
$p'(x_1, t) = g_0(x_1 - c_\infty t)$

means that $g_0(x_1)$ is preserved along the lines $dx_1 = c_\infty dt$



Acoustic wave equation : d'Alembert's solution

Interpretation in terms of characteristic curves



p_r constant along the curve (line here) $dx_1 = +c_\infty dt \quad (R^+)$

p_l constant along the curve (line here) $dx_1 = -c_\infty dt \quad (R^-)$

Acoustic wave equation : solution by Fourier integral

$$\int_{-\infty}^{+\infty} \left\{ \frac{\partial^2 p'}{\partial t^2} - c_\infty^2 \frac{\partial^2 p'}{\partial x_1^2} \right\} \, e^{-ik_1 x_1} dx_1 = 0$$

For p' and $\partial_{x_1} p' \to 0$ as $x_1 \to \infty$,

$$\frac{\partial^2 \hat{p}}{\partial t^2} + c_\infty^2 k_1^2 \hat{p} = 0 \quad \Longrightarrow \quad \hat{p}(k_1, t) = \hat{f}_1(k_1) e^{-ik_1 c_\infty t} + \hat{f}_2(k_1) e^{ik_1 c_\infty t}$$

1-D Fourier transform $\qquad p'(x_1) = \mathcal{F}^{-1}\left[\hat{p}(k_1)\right] \equiv \int_{-\infty}^{+\infty} \hat{p}(k_1) e^{ik_1 x_1} dk_1$

The solution is the sum of two travelling wave packets

$$p'(x_1, t) = \int_{-\infty}^{+\infty} \hat{f}_1(k_1) e^{i(k_1 x_1 - k_1 c_\infty t)} dk_1 + \int_{-\infty}^{+\infty} \hat{f}_2(k_1) e^{i(k_1 x_1 + k_1 c_\infty t)} dk_1$$

$= f_1(x_1 - c_\infty t) + f_2(x_1 + c_\infty t)$, each containing progressive (moving to the right) and retrograde (moving to the left) elementary plane waves wrt to the sign of the wavenumber k_1

- **General solution by Fourier integral**
  $\rightsquigarrow$ sum written over the n modes $\Omega(\boldsymbol{k})$ of the dispersion relation

  e.g. for the acoustic wave equation, $\omega = \pm\Omega(\boldsymbol{k}) = \pm c_\infty k$
  1-D to simplify algebra, for an arbitrary variable ζ

$$\zeta(x_1,t) = \int_{-\infty}^{+\infty} \hat{f}_1(k_1)e^{i(k_1x_1-\Omega(\boldsymbol{k})t)}dk_1 + \int_{-\infty}^{+\infty} \hat{f}_2(k_1)e^{i(k_1x_1+\Omega(\boldsymbol{k})t)}dk_1 \tag{9}$$

  where the functions $\hat{f}_1$ and $\hat{f}_2$ are determined to fit initial or boundary conditions

- Dispersion relation with only one mode $\omega = \Omega(\boldsymbol{k})$

  At $t=0$, $\quad \zeta(x_1,0) = \int_{-\infty}^{+\infty} \hat{f}_1(k_1)e^{ik_1x_1}dk_1 = \mathcal{F}^{-1}\big[\hat{f}_1(k_1)\big] = f_1(x_1)$

  and f_1 is then determined by the initial condition, $f_1(x_1) = g_0(x_1)$

Solution by Fourier integral

- Initial value problem for two modes $\omega = \pm\Omega(\boldsymbol{k})$
  $\zeta = g_0(x_1)$ and $\partial_t\zeta = g_1(x_1)$ at time $t = 0$

$$\begin{cases} g_0(x_1) = \displaystyle\int_{-\infty}^{+\infty} \left[\hat{f}_1(k_1) + \hat{f}_2(k_1)\right] e^{ik_1x_1}\,dk_1 \\ g_1(x_1) = \displaystyle\int_{-\infty}^{+\infty} -i\Omega\left[\hat{f}_1(k_1) - \hat{f}_2(k_1)\right] e^{ik_1x_1}\,dk_1 \end{cases}$$

The inverse Fourier transform provides $\hat{g}_0 = \hat{f}_1 + \hat{f}_2$ and $\hat{g}_1 = -i\Omega(\hat{f}_1 - \hat{f}_2)$
The function f_1 and f_2 are thus determined to be

$$\hat{f}_1(k_1) = \frac{1}{2}\left[\hat{g}_0(k_1) + \frac{i\hat{g}_1(k_1)}{\Omega}\right] \qquad \hat{f}_2(k_1) = \frac{1}{2}\left[\hat{g}_0(k_1) + \frac{i\hat{g}_1(k_1)}{-\Omega}\right]$$

Solution by Fourier integral

- Initial value problem for two modes (cont'd)
  Let us consider the particular case $\zeta = g_0(x_1)$ when g_0 is real
  and $\partial_t \zeta = 0$ at time $t = 0$

$$\zeta(x_1, t) = \frac{1}{2}\int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i(k_1 x_1 - \Omega(k)t)} dk_1 + \frac{1}{2}\int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i(k_1 x_1 + \Omega(k)t)} dk_1$$

It can be shown that (leave it as an exercise)

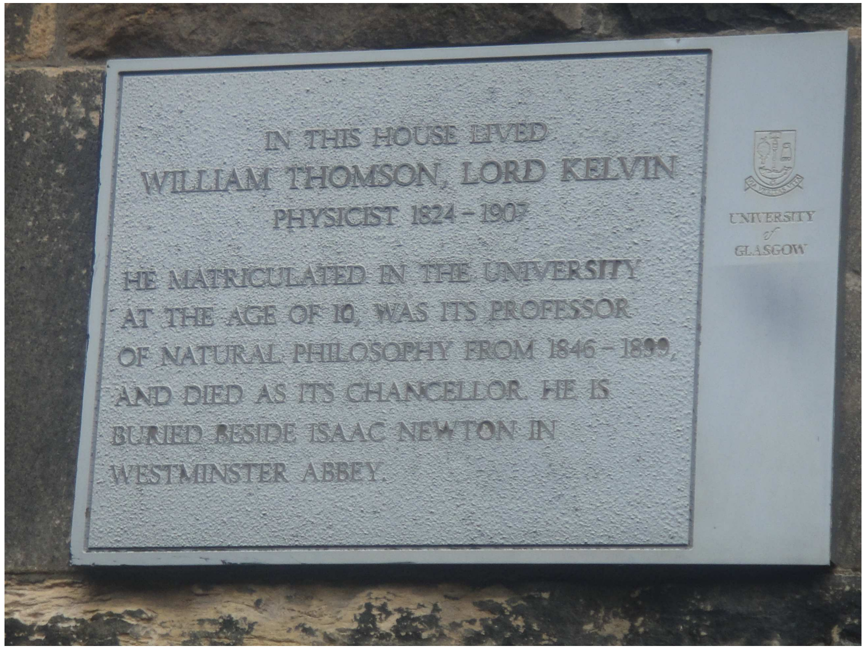
$$\zeta(x_1, t) = \mathcal{R}_e\left\{\int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i(k_1 x_1 - \Omega(k)t)} dk_1\right\} \tag{10}$$

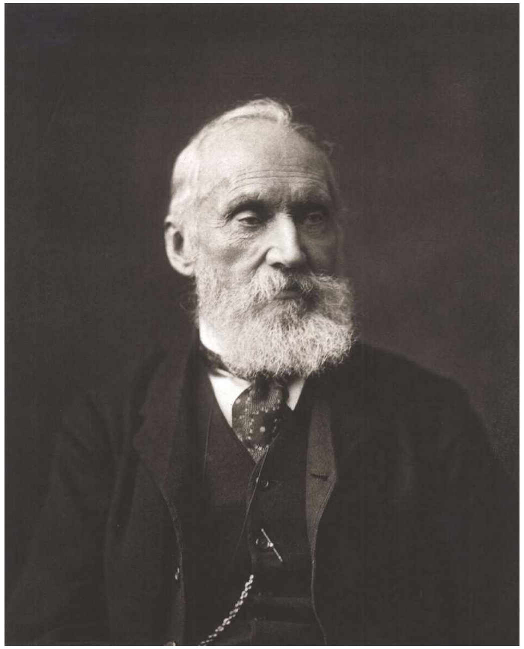
An explicit integration is possible for a very few functions g_0, direct numerical integration often tricky, but the asymptotic beheviour as $x_1, t \to \infty$ can be easily obtained by the stationary phase method

Propagation of a wave-packet : asymptotic behaviour

$$\zeta(x_1, t) = \int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i[k_1 x_1 - \Omega(k) t]} dk_1 \qquad \text{Asymptotic solution as } t \to \infty\,?$$

Method of the stationary phase (Kelvin, 1887)



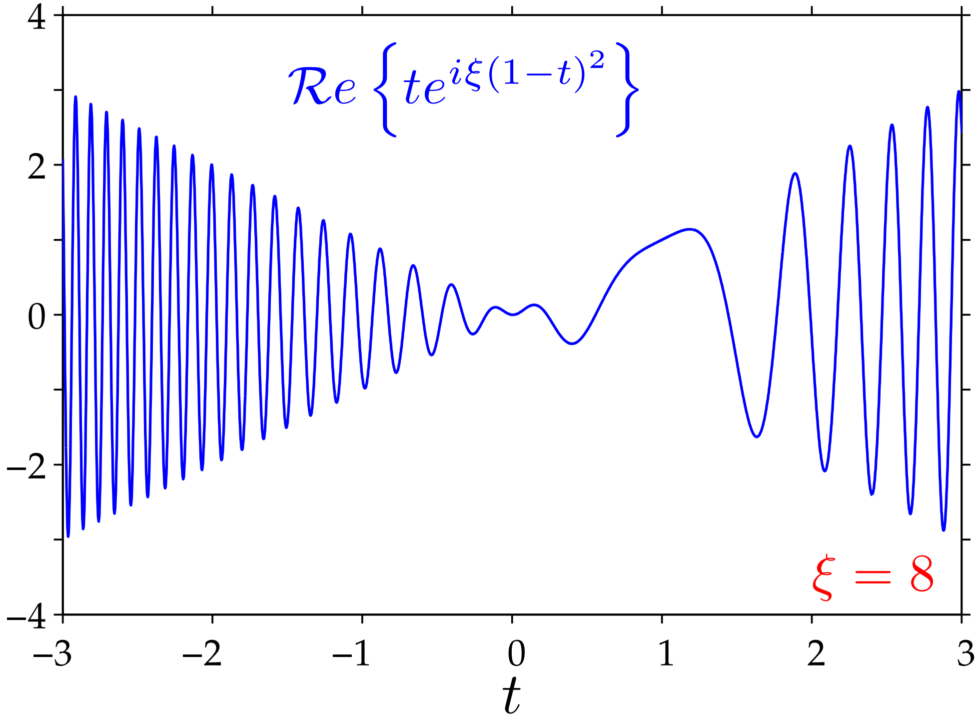


Lord Kelvin (William Thomson), 1824 - 1907
http://www-history.mcs.st-andrews.ac.uk/Biographies/Thomson.html

Asymptotic behaviour of integrals

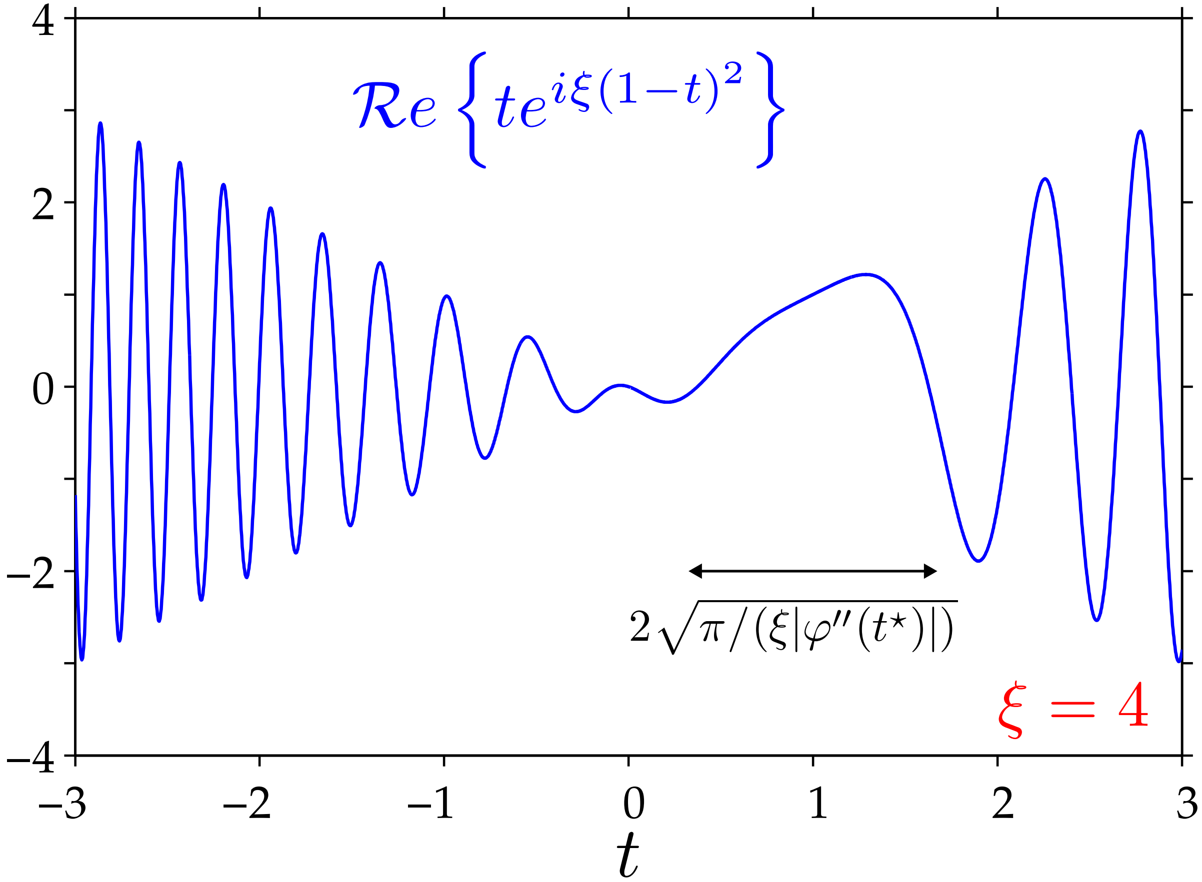
$$I(\xi) = \int_a^b f(t) e^{i\xi\varphi(t)} dt \qquad \text{as } \xi \to \infty \quad (a, b, \varphi) \text{ reals}$$

For large ξ, the function $e^{i\xi\varphi(t)}$ oscillates quickly with almost complete cancellation for $I(\xi)$. The main contribution comes from intervals of t where $\varphi(t)$ varies slowly, that is for which $\varphi'(t^\star) = 0$ ($t^\star$ stationary point)



Asymptotic behaviour of integrals

$$e.g. \quad I(\xi) = \int_{-\infty}^{+\infty} t e^{i\xi(1-t)^2} dt \qquad \varphi(t) = (1-t)^2 \quad t^\star = 1 \quad \varphi'(t^\star) = 0$$



Asymptotic behaviour of integrals

Easiest case, only one stationary point $t^\star$, $\varphi'(t^\star) = 0$, $a < t^\star < b$
Expanding the phase in a Taylor series near $t^\star$

$$\varphi(t) \simeq \varphi(t^\star) + \frac{1}{2}(t-t^\star)^2\varphi''(t^\star) + \mathcal{O}\left[(t-t^\star)^3\right]$$

Method of the stationary phase (Kelvin, 1887)

$$I(\xi) = \int_a^b f(t)e^{i\xi\varphi(t)}dt \sim f(t^\star)e^{i\xi\varphi(t^\star)} \underbrace{\int_a^b e^{i\frac{\xi\varphi''(t^\star)}{2}(t-t^\star)^2}dt}_{\text{can be exactly calculated}} \qquad \text{as } \xi \to \infty$$

$$I(\xi) = \int_a^b f(t)e^{i\xi\varphi(t)}dt \sim f(t^\star)\sqrt{\frac{2\pi}{\xi|\varphi''(t^\star)|}}\; e^{i\varphi(t^\star)\xi \pm i\frac{\pi}{4}} \quad \text{as } \xi \to \infty$$

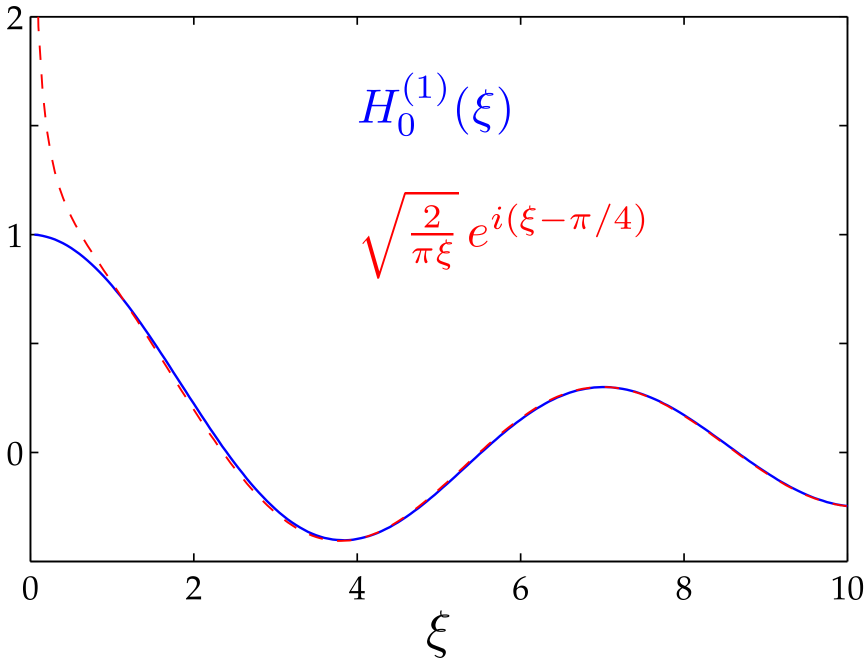
with the sign $\pm$ according as $\varphi''(t^\star) > 0$ or $\varphi''(t^\star) < 0$

- **Asymptotic behaviour of integrals**

An example : Hankel function $H_0^{(1)}(\xi)$

$$H_0^{(1)}(\xi) = \frac{1}{i\pi}\int_{-\infty}^{+\infty} e^{i\xi\cosh t}dt \qquad \begin{cases} \varphi(t) = \cosh(t), \varphi'(t) = \sinh(t), t^\star = 0 \\ \varphi''(t) = \cosh(t), \varphi''(t^\star) = 1 > 0 \end{cases}$$

$$H_0^{(1)}(\xi) \sim \frac{1}{i\pi}\sqrt{\frac{2\pi}{\xi}}\, e^{i\xi + i\pi/4} \sim \sqrt{\frac{2}{\pi\xi}}\, e^{i(\xi-\pi/4)} \quad \text{as } \xi \to \infty$$



$$H_0^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\pi/4)}$$

$$\text{as } \xi = kr \to \infty$$

Asymptotic behaviour of integrals

Additional remarks

- stationary point at an end point, $t^\star = a$ for instance, half contribution $\rightsquigarrow$ 1/2 factor

$$\int_0^\infty \cos(\xi t^2 - t)dt \sim \frac{1}{2}\sqrt{\frac{\pi}{2\xi}} \quad \text{as } \xi \to \infty \qquad (t^\star = 0)$$

- several stationary points : summation of their contributions

- notation : symbol $\sim$ means asymptotic equivalence ; $f \sim g$ as $\xi \to \infty$ means $f/g \to 1$ as $\xi \to \infty$

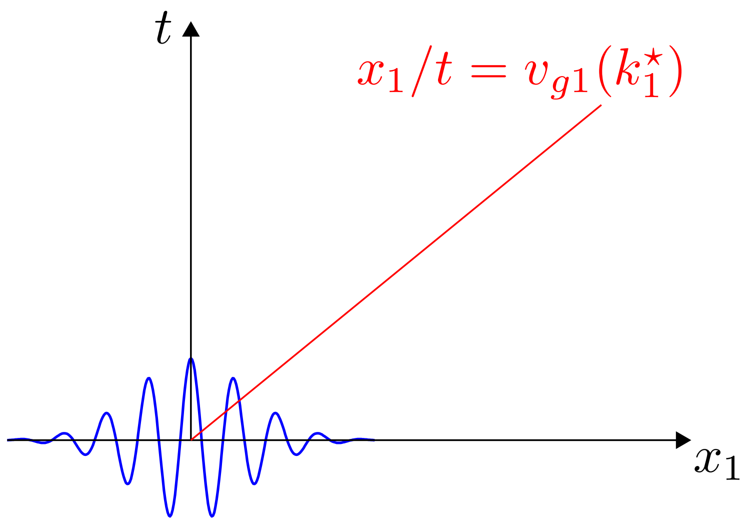
Propagation of a wave-packet : asymptotic behaviour

$$\zeta(x_1, t) = \int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i[k_1 x_1 - \Omega(\boldsymbol{k}) t]} dk_1 = \int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i\varphi(k_1) t} dk_1$$

Method of stationary phase

- phase given by $\varphi(k_1) = k_1 \dfrac{x_1}{t} - \Omega(\boldsymbol{k})$
- turning point $k_1^\star$

$$\varphi'(k_1^\star) = 0 \quad \Longrightarrow \quad \frac{x_1}{t} = \left.\frac{\partial \Omega}{\partial k_1}\right|_{k_1 = k_1^\star} \equiv v_{g1}^\star$$



Asymptotic solution as $t \to \infty$ along the ray $x_1/t = v_{g1}(k_1^\star)$, that is with x_1/t held fixed (parameter)

Propagation of a wave-packet : asymptotic behaviour

In summary

$$\zeta(x_1 = v_{g1}^\star t, t) \sim \frac{\sqrt{2\pi}}{\sqrt{t|\Omega''(k_1^\star)|}} \, \hat{g}_0(k_1^\star) \, e^{i\left\{k_1^\star x_1 - \Omega(k_1^\star)t + i\frac{\pi}{4}\mathrm{sgn}[-\Omega''(k_1^\star)]\right\}}$$

- dominant contribution for a component at wavenumber $k_1^\star$, namely $\hat{g}_0(k_1^\star)$, is observed at $x_1 = v_{g1}(k_1^\star)t$

- amplitude decays like $t^{-1/2}$ as $t \rightarrow \infty$ (and the signal therefore widens to conserve energy)

- formal definition of the group velocity, $\boldsymbol{v}_g = \nabla_{\boldsymbol{k}}\omega$ (reminder : phase velocity $v_\varphi = \omega/k \equiv$ propagation of constant phase lines in the $\boldsymbol{k}$ direction $\boldsymbol{\nu} = \boldsymbol{k}/k$)

Table of Fourier transforms

$$g_0(x_1) = \mathcal{F}^{-1}\left[\hat{g}_0(k_1)\right] = \int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{ik_1x_1} dk_1$$

$g_0(x_1)$	$\hat{g}_0(k_1)$
$e^{-\ln 2 \left(\frac{x_1}{b}\right)^2}$	$\frac{b}{2\sqrt{\pi \ln 2}} e^{-\frac{(bk_1)^2}{4\ln 2}}$
$\delta(x_1)$	$\frac{1}{2\pi}$
$\cos(k_0x_1)$	$\frac{1}{2}\left[\delta(k_1 - k_0) + \delta(k_1 + k_0)\right]$
$e^{-\ln 2\,(x_1/b)^2} \quad \cos(k_w x_1)$	$\frac{1}{4}\frac{b}{\sqrt{\pi \ln 2}}\left\{ e^{-\frac{[b(k_1 - k_w)]^2}{4\ln 2}} + e^{-\frac{[b(k_1 + k_w)]^2}{4\ln 2}} \right\}$
$\int_{-\infty}^{x_1} g(\xi) d\xi$	$\frac{1}{2\pi}\frac{\hat{g}(k_1)}{ik_1} + \frac{1}{2}\hat{g}(0)\delta(k_1)$
$\frac{1}{1+(x_1/l)^2}$	$\frac{l}{2}e^{-l\lvert k_1\rvert}$

Stationary phase applied to surface gravity waves

Dispersion relation for long waves ($kl_c \ll 1$) in deep water ($kh \gg 1$)
$\omega = \pm\sqrt{gk} = \pm\Omega(k)$

Initial value problem for the surface displacement ζ
$\zeta(x_1) = g_0(x_1)$ and $\partial_t \zeta = 0$ at $t = 0$

Since g_0 is a real function - refer to Eq. (10) - it can be shown that

$$\zeta(x_1, t) = \mathcal{R}_e\left\{ \int_{-\infty}^{+\infty} \hat{g}_0(k_1) e^{i(k_1 x_1 - \Omega(k)t)} dk_1 \right\} \qquad \Omega(k) = \sqrt{gk}$$

$$\sim \; ? \quad \text{as } t \to \infty$$

Stationary phase applied to surface gravity waves

$\varphi(k_1) = k_1\, x_1/t - \Omega(k)$, stationary points $\partial_{k_1}\varphi(k_1) = 0$

$$\frac{x_1}{t} = \frac{\partial\Omega}{\partial k_1} = \frac{1}{2}\sqrt{\frac{g}{k}}\frac{k_1}{k} = \frac{1}{2}\sqrt{\frac{g}{k_1}} \quad \text{for} \quad \underline{x_1 > 0} \qquad k = \sqrt{k_1^2} \quad \text{(1-D)}$$

$$\Longrightarrow \qquad k_1^\star = \frac{g\,t^2}{4x_1^2} > 0$$

$$\left.\frac{\partial^2\Omega}{\partial k_1^2}\right|_{k_1^\star} = -\frac{1}{4k_1}\sqrt{\frac{g}{k_1}}\bigg|_{k_1^\star} = -\frac{\sqrt{g}}{4}\left(\frac{4x_1^2}{g\,t^2}\right)^{3/2} < 0$$

$$\sqrt{t|\partial^2_{k_1k_1}\Omega(k_1^\star)|} = \frac{2}{g}\frac{x_1^3}{t^2}$$

$$\zeta \sim \hat{g}_0(k_1)\,\sqrt{\pi g}\,\frac{t}{x_1^{3/2}}\,\cos\left(-\frac{g\,t^2}{4x_1} + \frac{\pi}{4}\right)$$

Application to surface gravity waves

Initial value of the surface elevation ζ

$$g_0(x_1) = \frac{\zeta_0}{1 + (x_1/l)^2} \quad \hat{g}_0(k_1) = \zeta_0 \frac{l}{2} e^{-l|k_1|}$$

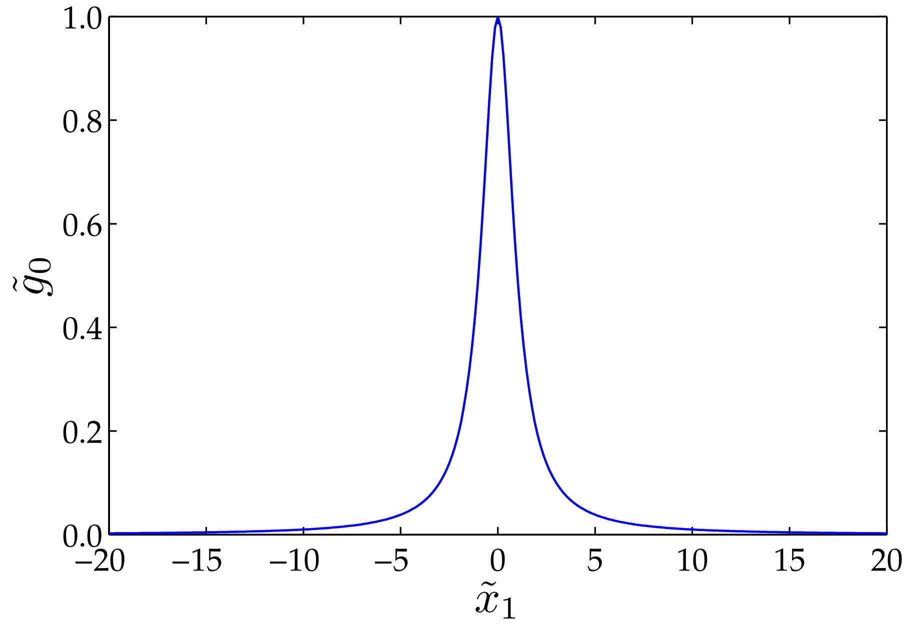
By introducing dimensionless variables
$\tilde{t} = t\sqrt{g/l}$, $\tilde{x}_1 = x_1/l$ and $\tilde{\zeta} = \zeta/\zeta_0$ (linear problem),

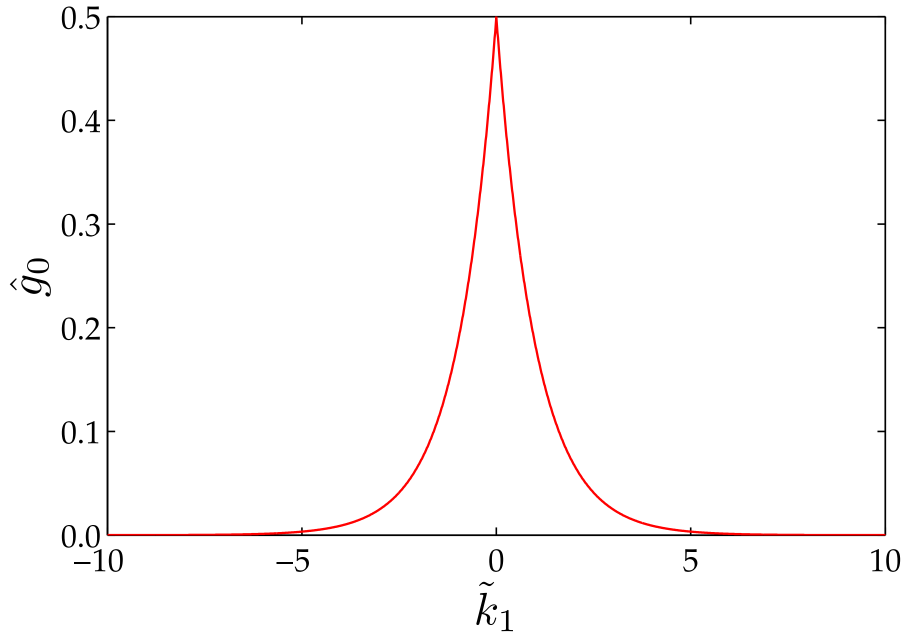
$$\boxed{\tilde{\zeta} \sim \frac{\sqrt{\pi}}{2} \; e^{-\frac{\tilde{t}^2}{4\tilde{x}_1^2}} \; \frac{\tilde{t}}{\tilde{x}_1^{3/2}} \; \cos\left(\frac{\tilde{t}^2}{4\tilde{x}_1} - \frac{\pi}{4}\right)}$$

Application to surface gravity waves

Initial value of the surface elevation

$$\tilde{g}_0(x_1) = \frac{1}{1+\tilde{x}_1^2} \qquad \hat{\tilde{g}}_0(\tilde{k}_1) = \frac{1}{2}e^{-|\tilde{k}_1|}$$





Application to surface gravity waves

Tsunami generated by (submarine) earthquake, landslide, volcanic eruption ...

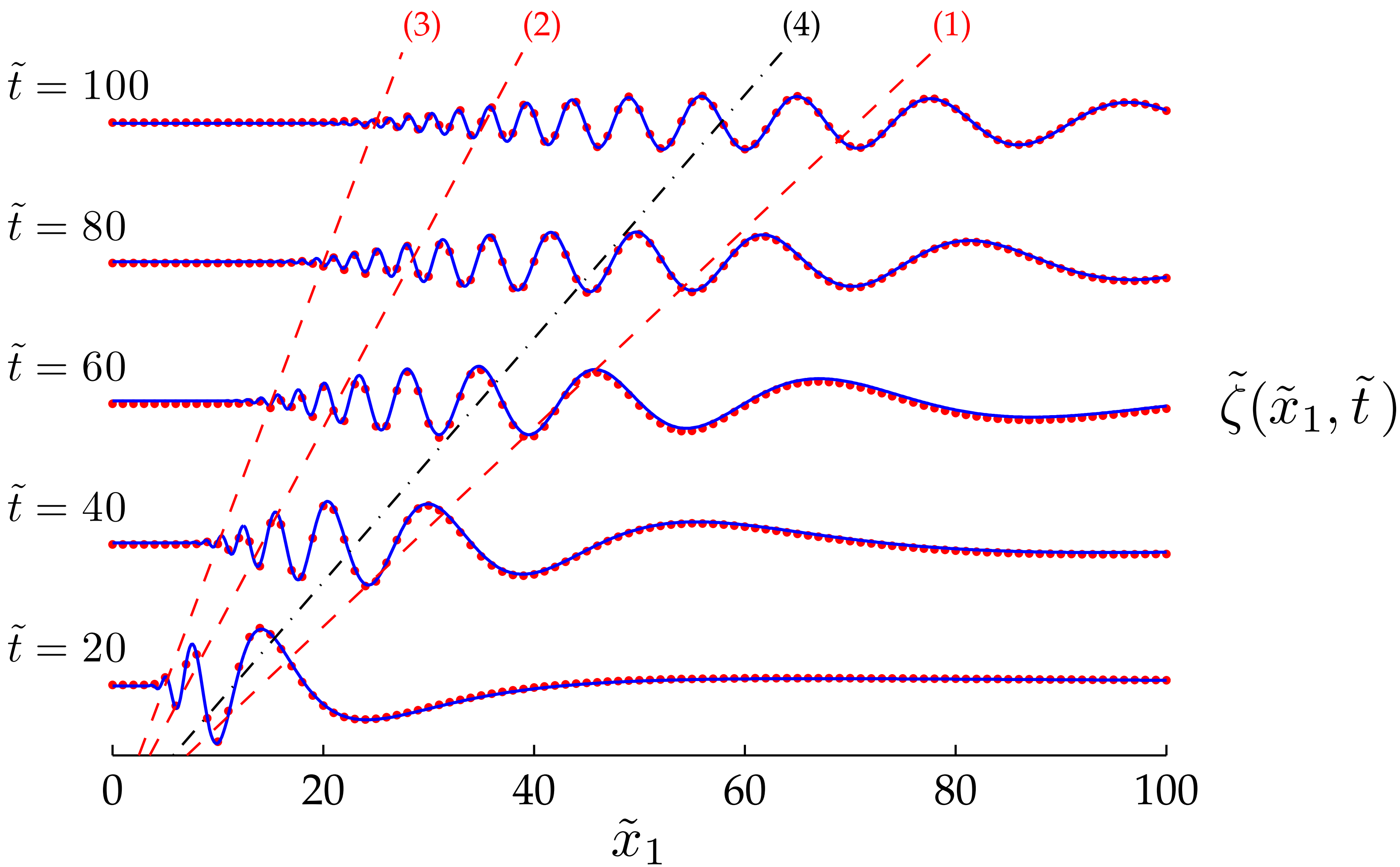
Cape Verde archipelago off western Africa, where a massive flank collapse at Fogo volcano potentially triggered a 'giant tsunami' with devastating effects, reportedly between 65,000 and 124,000 years ago. Fogo is one of the most active and prominent oceanic volcanoes on Earth, presently standing 2829 m above mean sea level and 7 km above the surrounding seafloor.

Ramalho *et al.*, 2015, *Science Advances*

Application to surface gravity waves

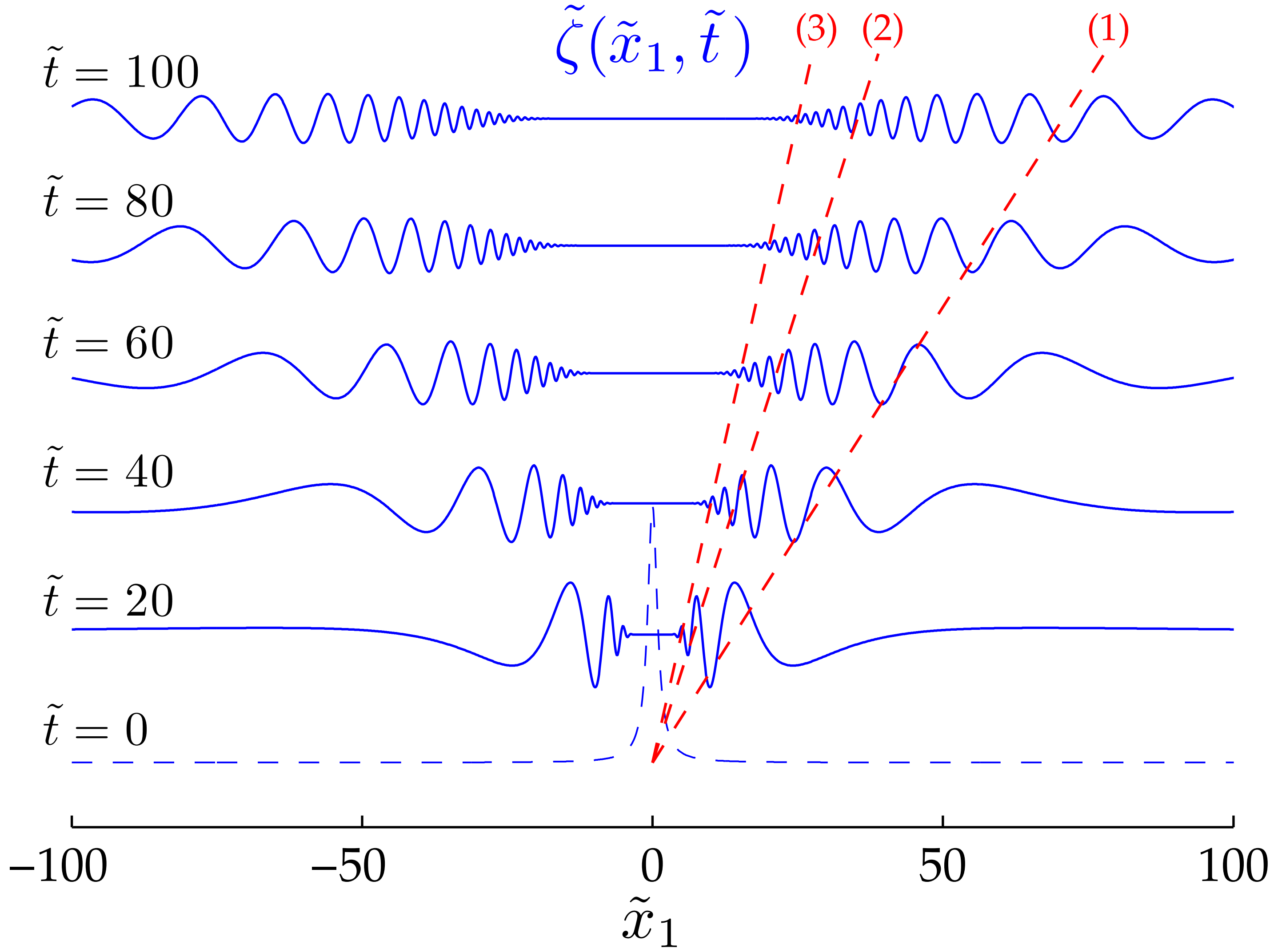
—— numerical solution (Fourier), • stationary phase approximation

$\tilde{x}_1/\tilde{t} = v_g(\tilde{k}_1)$ with (1) $\tilde{k}_1 = 1/2$ (2) $\tilde{k}_1 = 2$ (3) $\tilde{k}_1 = 4$ (4) maximum amplitude $\tilde{x}_1/\tilde{t} = \sqrt{3}$



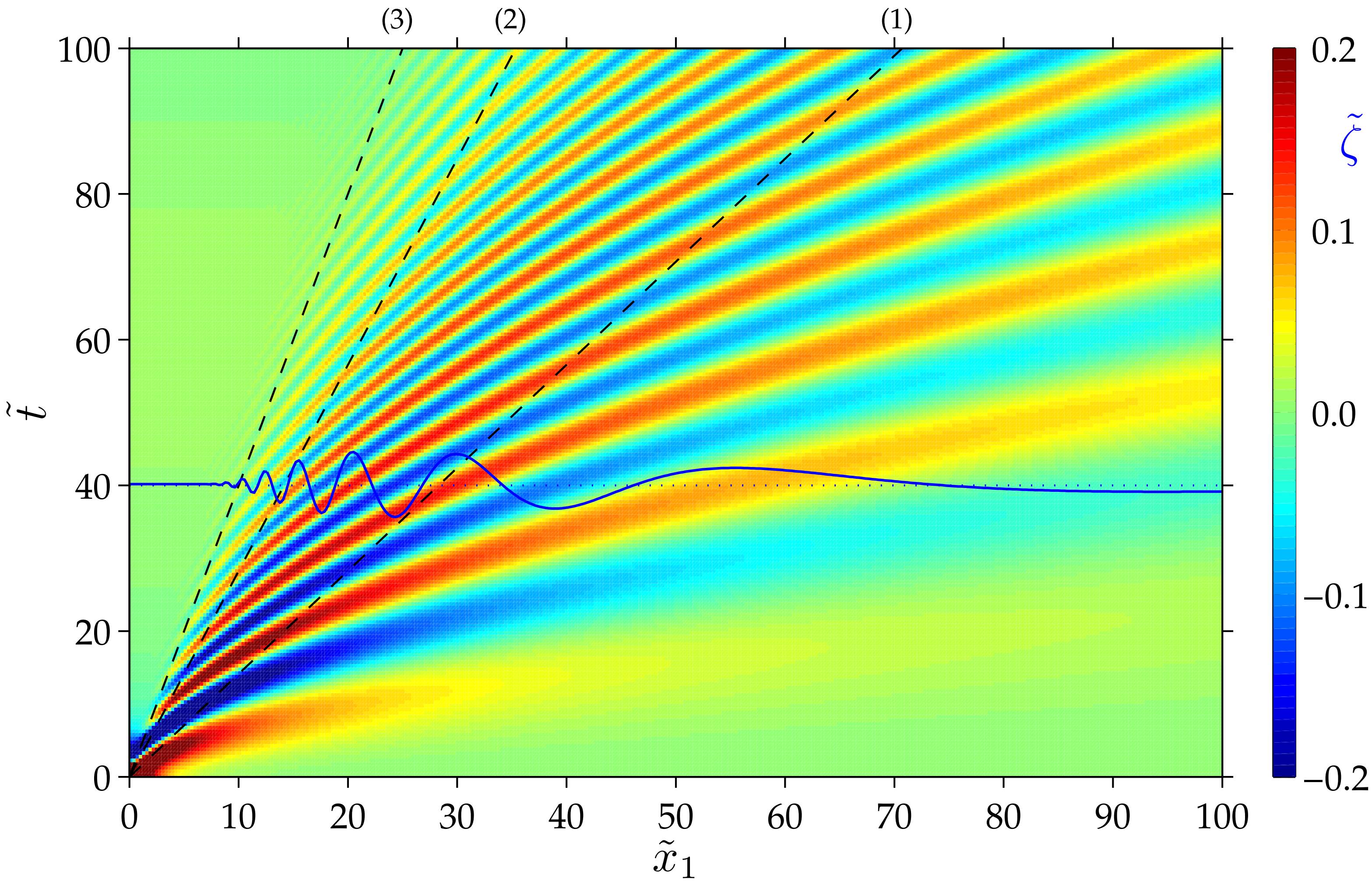
- **Application to surface gravity waves**

—— numerical solution (Fourier)



- **Surface gravity waves**

  $\tilde{x}_1/\tilde{t} = v_g(\tilde{k}_1)$ (1) $\tilde{k}_1 = 1/2$ (2) $\tilde{k}_1 = 2$ (3) $\tilde{k}_1 = 4$



Application to surface gravity waves : additional remarks

– As $k \to 0$, the group velocity becomes infinite, and $\Omega''(k_1) \to 0$ in the stationary phase approximation

$$v_g = \frac{\partial \Omega}{\partial k_1} = \frac{1}{2}\sqrt{\frac{g}{k_1}}$$

The propagation of large wavelength components at an infinite speed is a direct consequence of the incompressibility assumption.

– For a finite depth h, $\Omega(k) = \sqrt{gk \tanh(kh)}$. The group velocity then remains bounded,

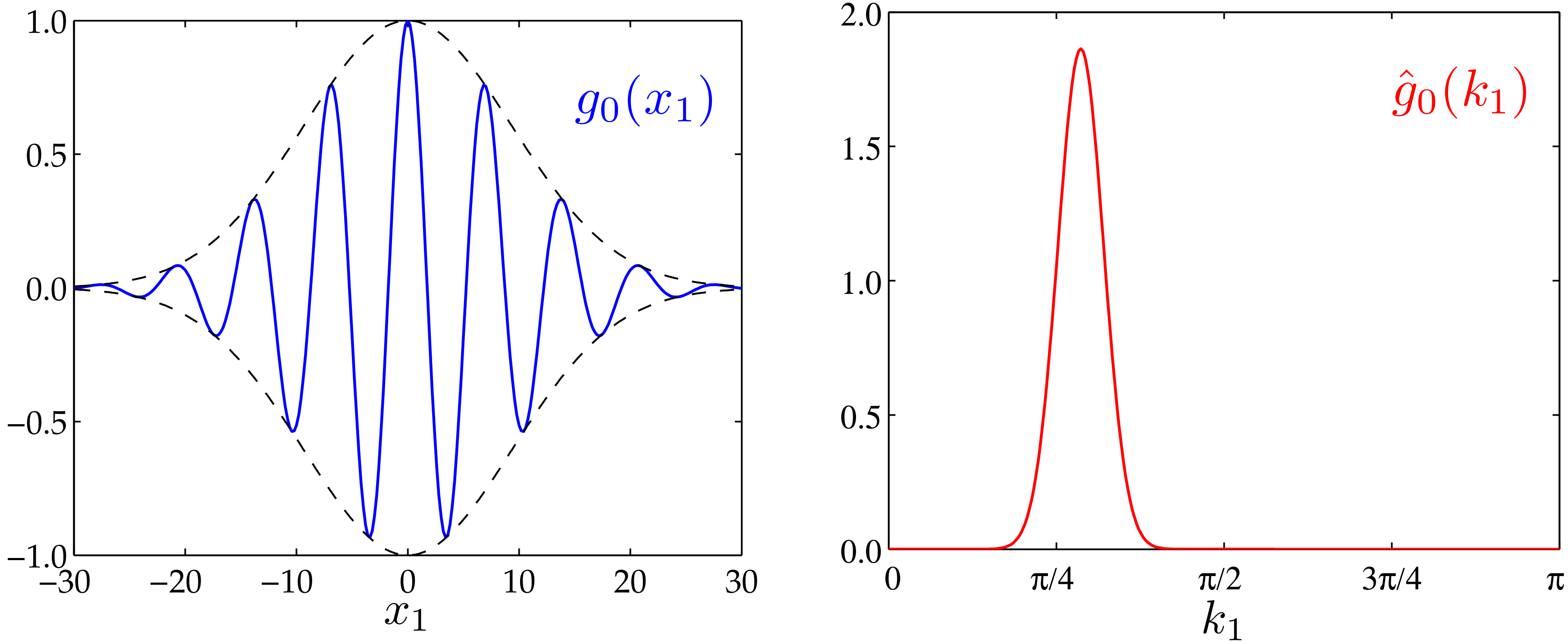
$$v_g = \frac{\partial \Omega}{\partial k_1} \quad \to \quad \sqrt{gh} \ \text{ as } \ k_1 \to 0$$

but $\Omega''(k_1) \to 0$! The treatement of the wavefront requires a little more work.

- A simple wave packet model as initial condition

$$g_0(x_1) = e^{-\ln 2\,(x_1/b)^2}\ \cos(k_w x_1)$$

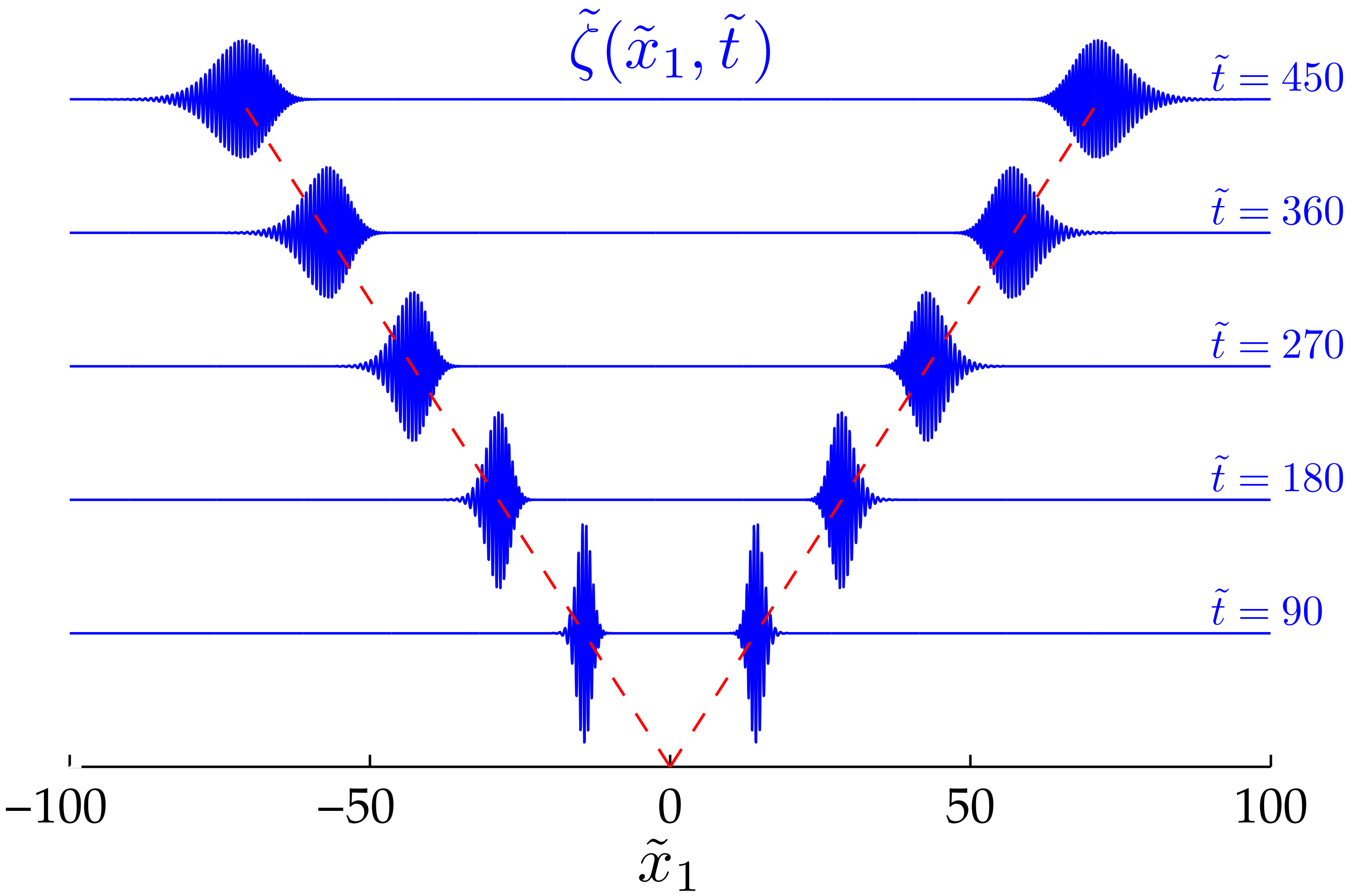
$$\hat{g}_0(k_1) = \frac{1}{4}\frac{b}{\sqrt{\pi \ln 2}}\left\{ e^{-\frac{[b(k_1-k_w)]^2}{4\ln 2}} + e^{-\frac{[b(k_1+k_w)]^2}{4\ln 2}} \right\}$$



$(b = 11,\ k_w = 0.9)$

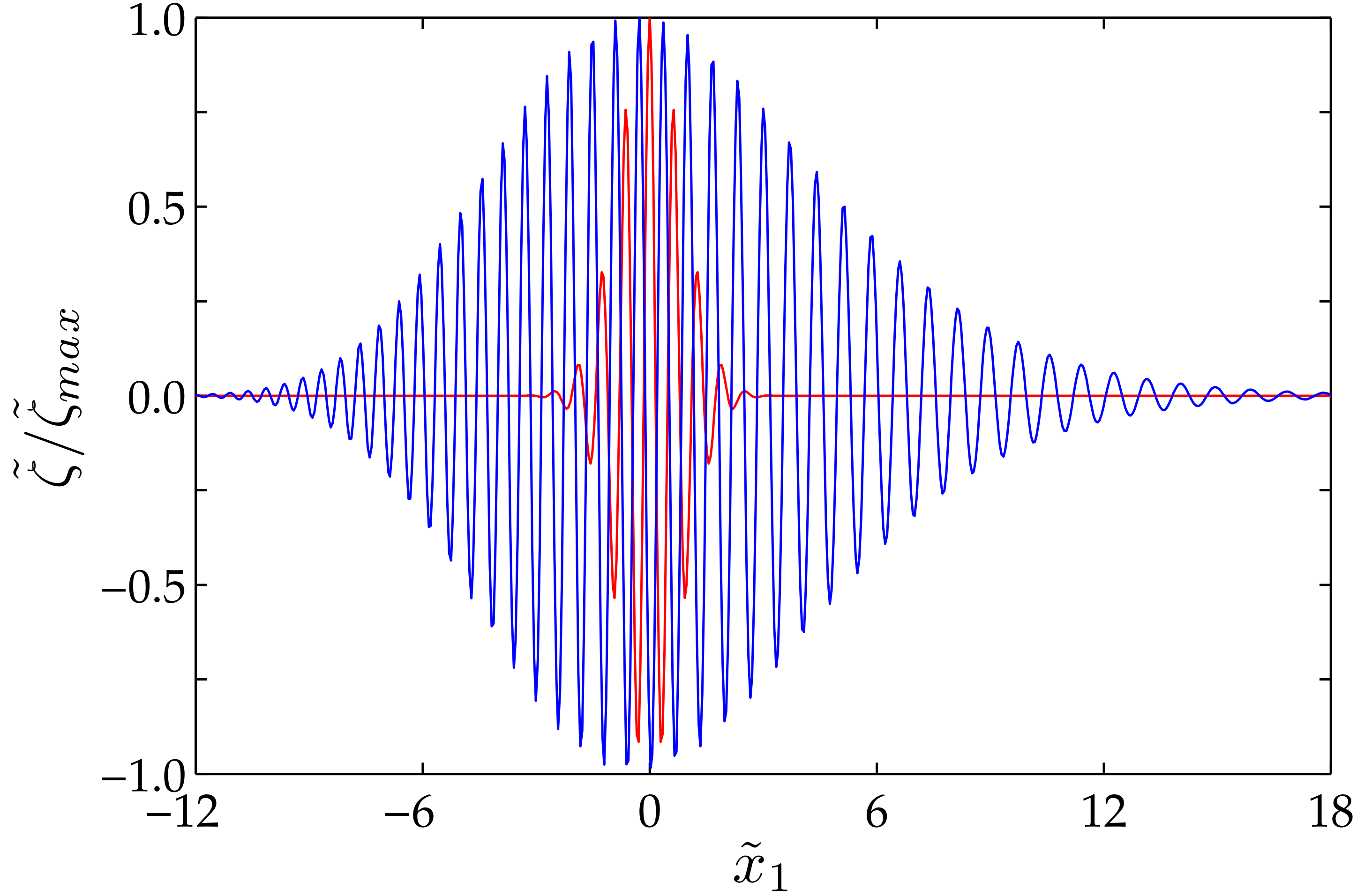
- Application to surface gravity waves

—— numerical solution (Fourier), - - - $\tilde{x}_1/\tilde{t} = v_g(\tilde{k}_w)$



Application to surface gravity waves

numerical solution (Fourier) at $\tilde{t} = 0$ and $\tilde{t} = 450$ (signal translated of $v_g(\tilde{k}_w)\tilde{t}$)



In summary

Linear partial differential equation $\mathcal{L}(\zeta) = 0$
Fourier transform $\sim e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, relation dispersion $\mathcal{D}(\boldsymbol{k},\omega) = 0$

e.g. surface gravity waves $\omega(\boldsymbol{k}) = \pm\Omega(k)$ with $\Omega(k) = \sqrt{gk\tanh(kh)}$

With the initial conditions $\zeta(x_1) = g_0(x_1)$ and $\partial_t\zeta(x_1) = 0$ at $t = 0$, the solution can be recast into a single integral

$$\zeta(x_1,t) = R_e\left\{\int_{-\infty}^{+\infty} \hat{g}_0(k_1)e^{i(k_1x_1-\Omega(\boldsymbol{k})t)}dk_1\right\} \qquad \text{(2 modes)}$$

Asymptotic behavior (stationary phase) as $t \to \infty$

$$\zeta(x_1 = v_{g1}^\star t, t) \sim \frac{\sqrt{2\pi}}{\sqrt{t|\Omega''(k_1)|}}\,\hat{g}_0(k_1)\; e^{i\left\{k_1x_1-\Omega(k_1)t+i\frac{\pi}{4}\mathrm{sgn}[-\Omega''(k_1)]\right\}}$$

For an observer travelling at $x/t = v_{g1}(k_1)$, the amplitude varies as $1/\sqrt{t}$ and is modulated thanks to the phase, crests moving at $v_\varphi = \Omega(k_1)/k_1$.

Waves in Fluids : models for linear wave propagation
$\rightsquigarrow$ theories for linear dispersive waves II

Ray theory

Extention of Fourier's integral solutions for a medium with slowly varying properties with respect to the wavelength : geometrical or high frequency approximation

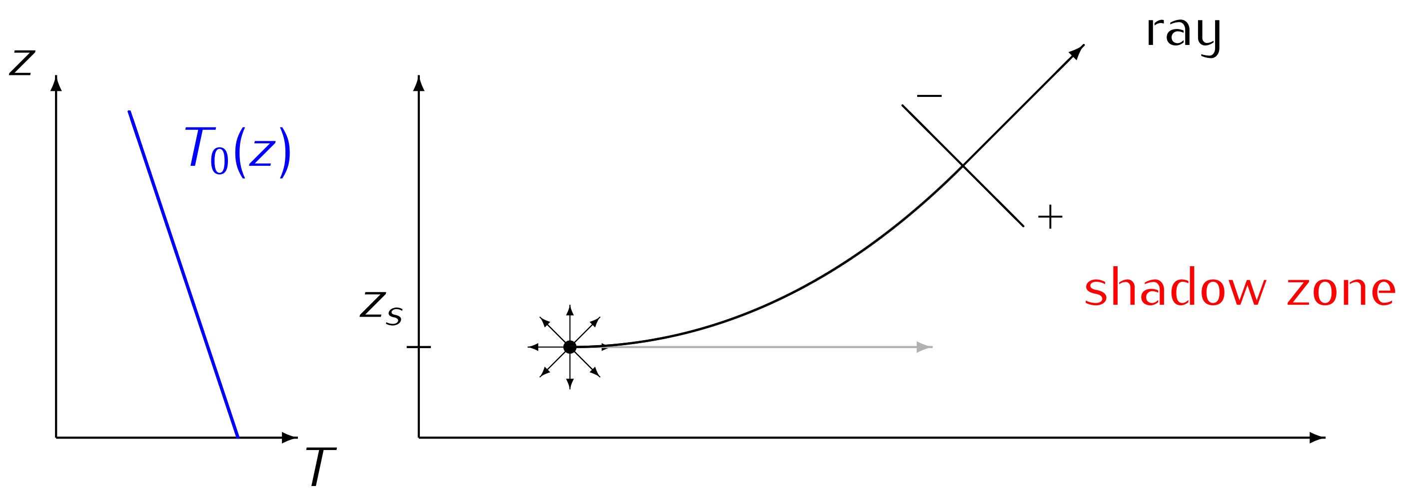
- surface gravity waves with $h = h(\boldsymbol{x})$, $\Omega(\boldsymbol{k}) \pm \sqrt{gk \tanh(kh)}$
- acoustic waves in non homogeneous medium $c_0 = c_0(\boldsymbol{x})$
  or in the presence of a mean flow $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$
  It can be shown that the dispersion relation reads

$$c_0^2 k^2 - (\boldsymbol{k} \cdot \boldsymbol{u}_0 - \omega)^2 = 0 \qquad \text{or} \qquad \omega = \boldsymbol{k} \cdot \boldsymbol{u}_0 \pm c_0 k$$

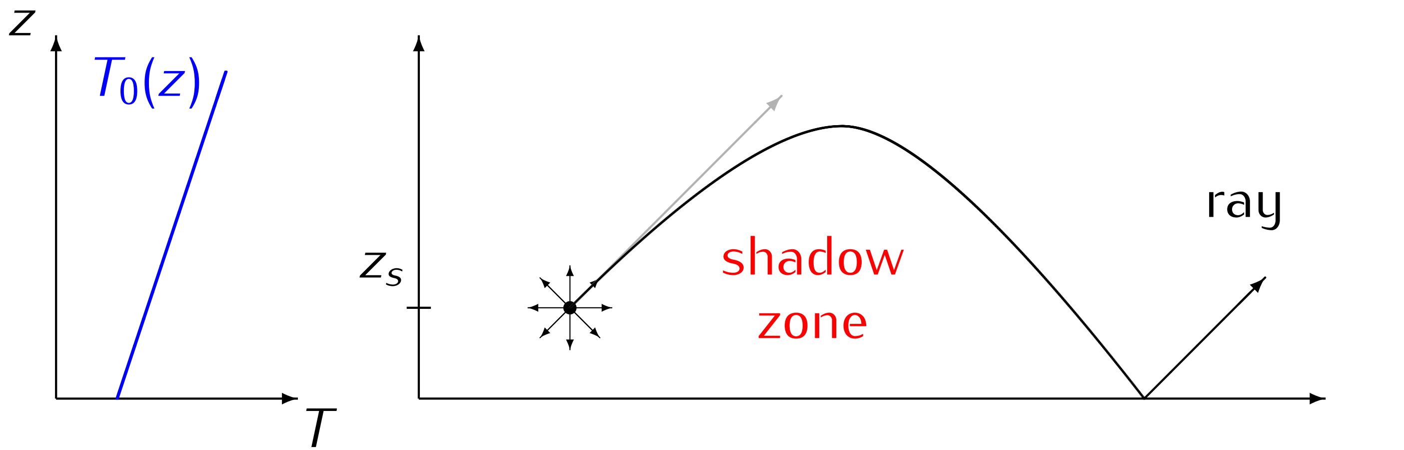
Dispersion relation $\quad \mathcal{D}(\boldsymbol{k}, \omega, \boldsymbol{x}) = 0$

Wave propagation is then governed by partial differential equations with non-constant coefficients, and it is no longer possible to apply a simple Fourier transform.

- Standard temperature profile

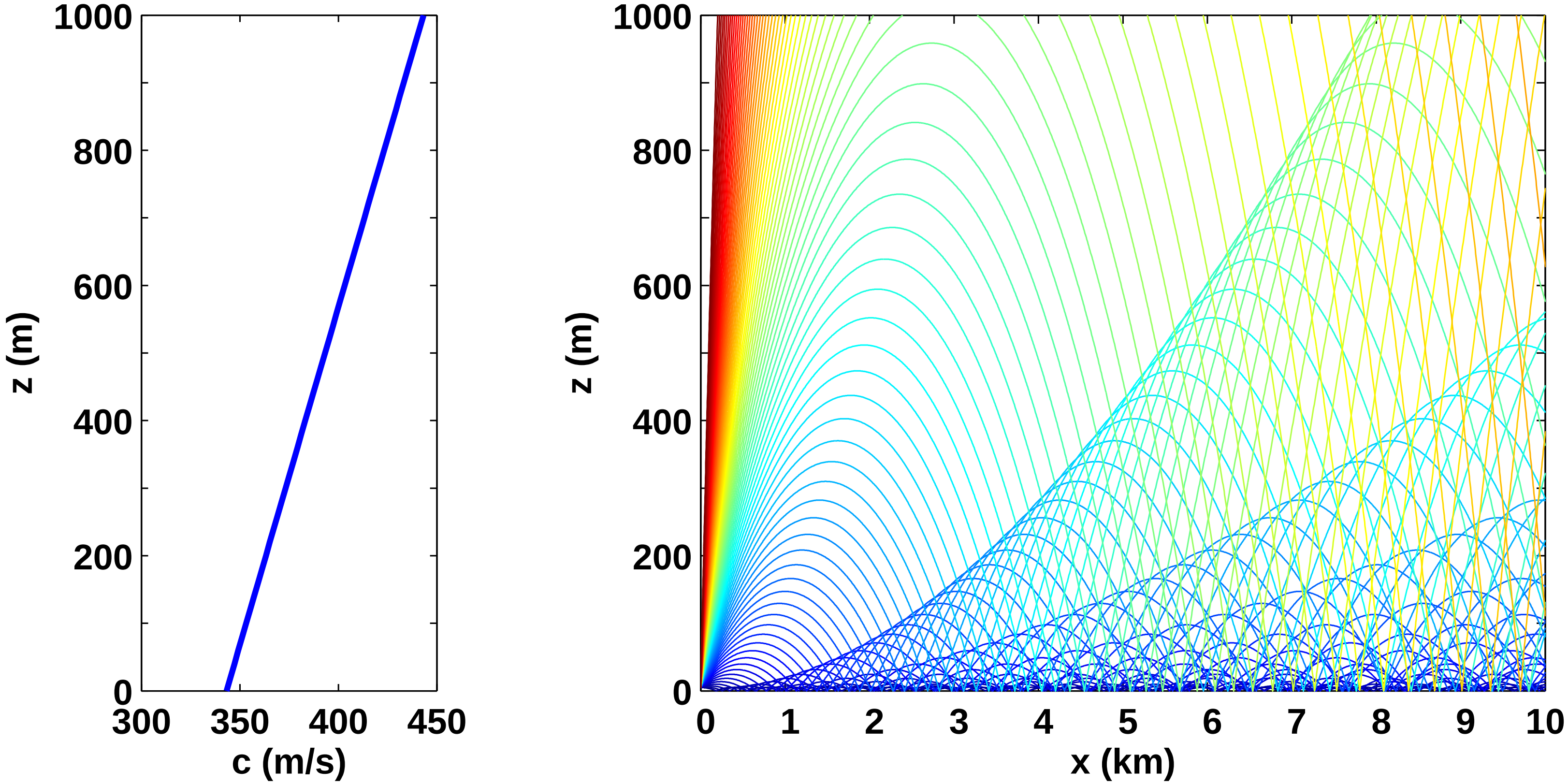


- Temperature profile inversion (pollution)



- **Mean flow effects on sound propagation**

  Ray tracing with strong positive sound speed gradient of 0.1 s^{-1}

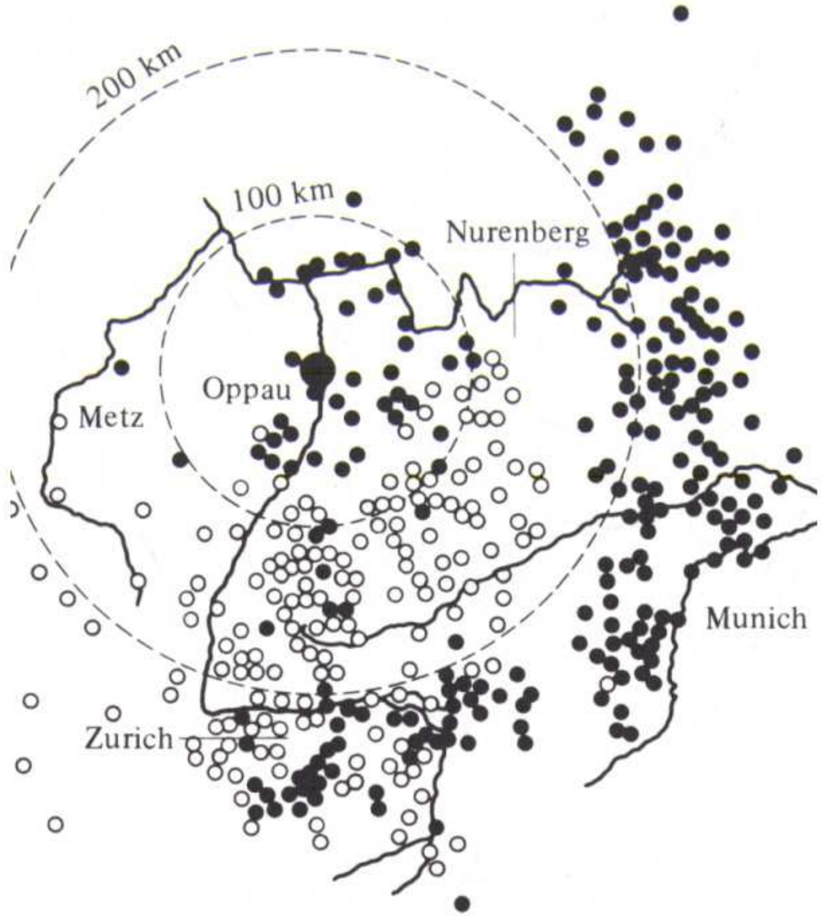


Mean flow effects on sound propagation

Explosion at Oppau, Germany, on sept. 21 1921 (561 deaths)

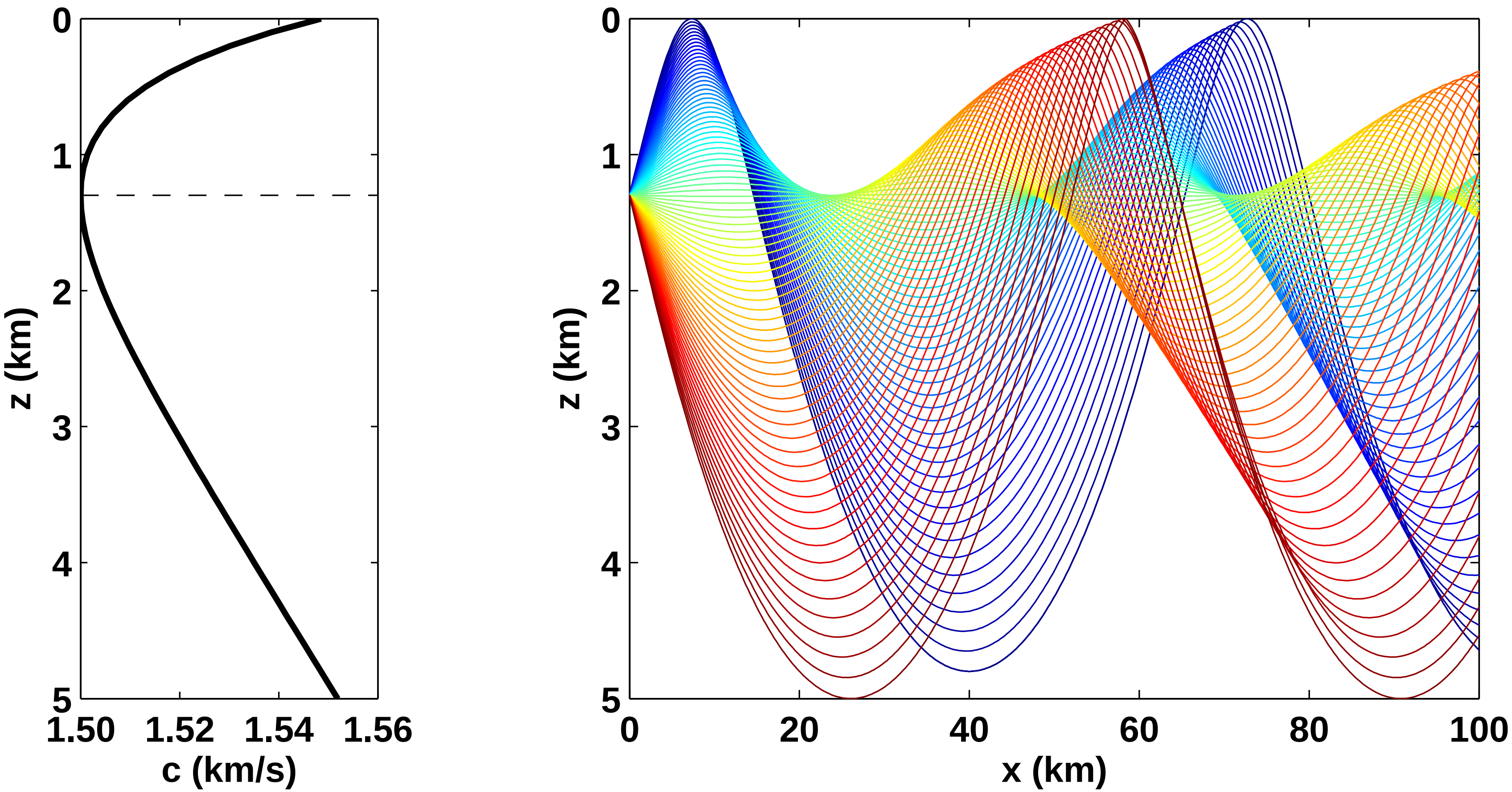
Locations where sound was heard ●
and not heard ○

BASF factory, ammonium nitrate



Cook, R.K., 1962, Strange sounds in the Atmosphere, *Sound*, **1**(2)

- SOFAR (SOund Fixing And Ranging)



Munk, *J. Acoust. Soc. Am.* (1974)

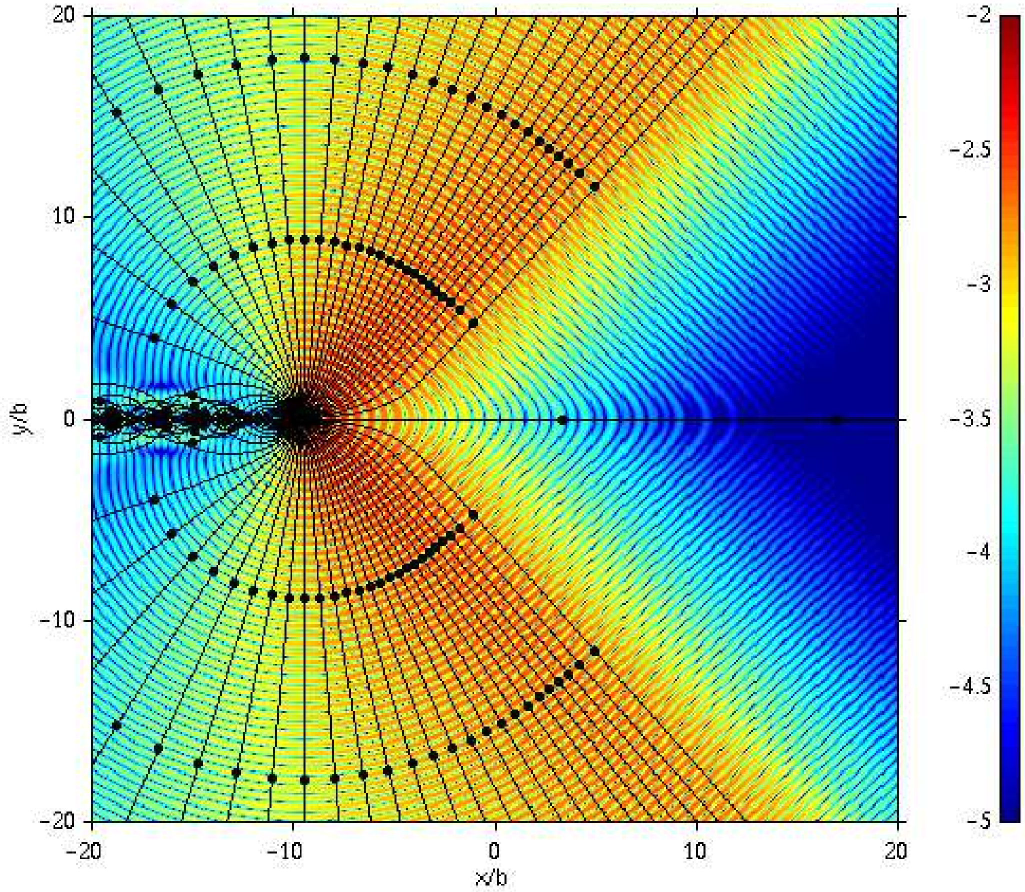
Ghost octopus ‘Casper’

Octopus observed at a depth of 4290 meters by the remotely operated vehicle *Deep Discoverer* (Hawaiian island of Necker ; NOAA, 2016)

Sound propagation in a jet flow

Harmonic source in a Bickley jet $\quad \dfrac{\bar{u}_1}{u_j} = \dfrac{1}{\cosh^2(\beta y/\delta)} \qquad \beta = \ln(1+\sqrt{2})$

$\mathrm{St} = 4.4 \quad \mathrm{M} = 0.5 \quad \lambda \sim \delta$

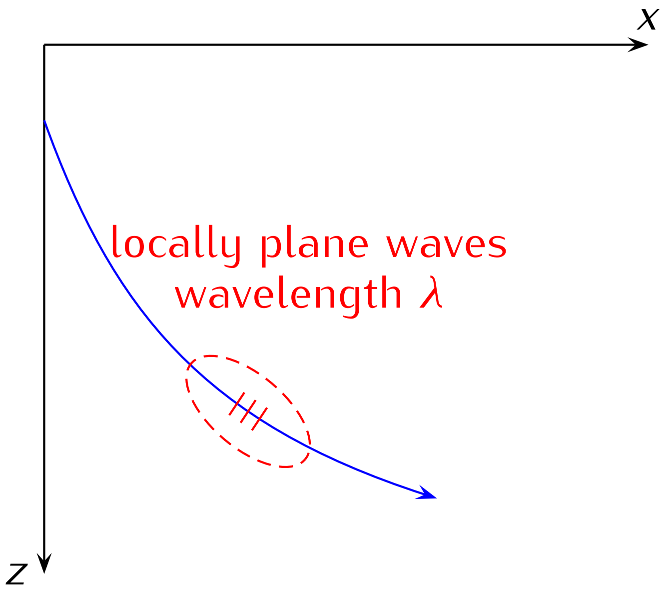


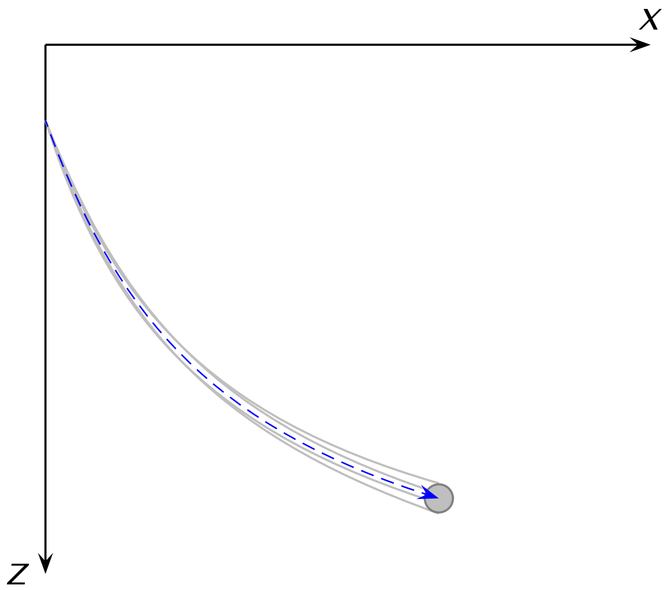
LEE ($\log_{10}(|p'| + \epsilon)$) and ray-tracing

high-frequency noise is diverting away from the jet axis

shadow zone at angles close to the jet axis, $\theta^\star \simeq 48.2^o$ (edge of the silence cone)

Ray theory





slowly varying medium on scale L

ray tube

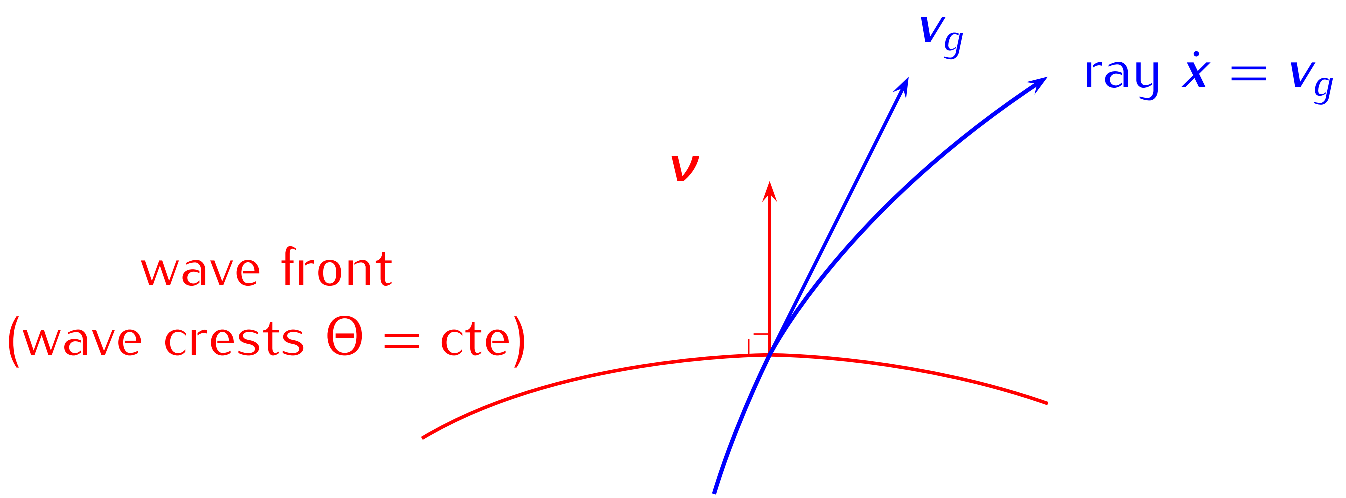
$$\boxed{\text{high frequency approximation} \quad \Longrightarrow \quad \lambda \ll L}$$

Wave kinematics

Dispersion relation $\mathcal{D}(\boldsymbol{k}, \omega, \boldsymbol{x}) = 0$ in an inhomogeneous medium, and by considering one of the modes $\omega = \Omega(\boldsymbol{k}, \boldsymbol{x})$

The solution is now sought as a local plane wave, *e.g.* $\zeta = \tilde{\zeta}(\boldsymbol{x})e^{i\Theta}$, where the amplitude $\tilde{\zeta}(\boldsymbol{x})$ and the wavenumber $\boldsymbol{k}(\boldsymbol{x})$ slowly vary with position $\boldsymbol{x}$ on scale $\lambda = 2\pi/k$, or equivalently $\lambda/L \ll 1$

From the phase Θ of the wave, we can define
a wavenumber vector $\boldsymbol{k}(\boldsymbol{x}, t) = \nabla\Theta$
an angular frequency $\omega(\boldsymbol{x}, t) = -\partial_t\Theta$



Wave kinematics (Whitham, 1960)

The orientation of the normal vector $\boldsymbol{\nu} = \boldsymbol{k}/k$ to the wavefront must be determined along the ray path, through the evolution of $\boldsymbol{k}(\boldsymbol{x}, t)$ along this ray, that is $\partial_t \boldsymbol{k} + \boldsymbol{v}_g \cdot \nabla \boldsymbol{k} = \ ?$

$$\omega = \Omega\left(\boldsymbol{k}, \boldsymbol{x}\right)$$

$$\begin{cases} \dfrac{\partial \omega}{\partial t} = \left.\dfrac{\partial \Omega}{\partial t}\right|_{k,x} + \nabla_k \Omega \cdot \dfrac{\partial \boldsymbol{k}}{\partial t} = \nabla_k \Omega \cdot \dfrac{\partial \boldsymbol{k}}{\partial t} \\ \boldsymbol{v}_g \equiv \nabla_k \Omega \qquad \dfrac{\partial \boldsymbol{k}}{\partial t} = \dfrac{\partial}{\partial t} \nabla \Theta = \nabla \dfrac{\partial \Theta}{\partial t} = -\nabla \omega \qquad \Longrightarrow \qquad \dfrac{\partial \omega}{\partial t} + \boldsymbol{v}_g \cdot \nabla \omega = 0 \end{cases}$$

The angular frequency is convected along rays if the medium is independent of time

Wave kinematics

In a similar way, one has for the wavevector $\boldsymbol{k}$

$$\frac{\partial k_i}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \Theta}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial \Theta}{\partial t} = -\frac{\partial \omega}{\partial x_i} = -\left.\frac{\partial \Omega}{\partial x_i}\right|_k - \frac{\partial \Omega}{\partial \boldsymbol{k}}\frac{\partial \boldsymbol{k}}{\partial x_i} = -\left.\frac{\partial \Omega}{\partial x_i}\right|_k - \underline{v_{gj}\frac{\partial k_j}{\partial x_i}}$$

In order to form the material derivative with the last term, it should be noted that $\nabla \times \boldsymbol{k} = 0$ by construction, since $\boldsymbol{k} = \nabla\Theta$. It yields

$$\frac{\partial k_j}{\partial x_i} - \frac{\partial k_i}{\partial x_j} = 0 \quad \Longrightarrow \quad \underline{v_{gj}\frac{\partial k_j}{\partial x_i}} = v_{gj}\frac{\partial k_i}{\partial x_j} = \boldsymbol{v}_g \cdot \nabla k_i$$

and the transport equation can be rewritten

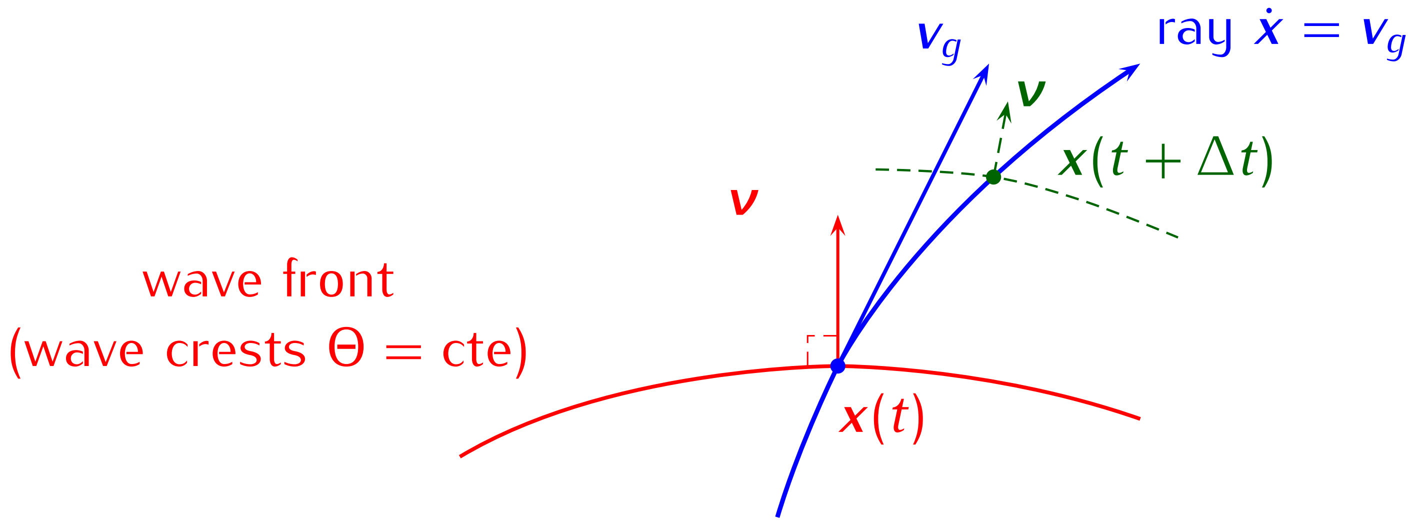
$$\frac{\partial \boldsymbol{k}}{\partial t} + \boldsymbol{v}_g \cdot \nabla \boldsymbol{k} = -\nabla\Omega|_k$$

where the term $\nabla\Omega|_k$ is linked to the explicit dependence on space of the medium.

- **Ray tracing equations** (for a medium independent of time)

$$\begin{cases} \dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{v}_g & (11) \\[2ex] \dfrac{d\boldsymbol{k}}{dt} = -\nabla\Omega|_{\boldsymbol{k}} & (12) \end{cases}$$

Eq. (11) provides rays, Eq. (12) provides the orientation of wave fronts along the rays, and refraction effects are included in the term $-\nabla\Omega$ (frequency remains constant along these rays)



Ray tracing equations in acoustics

$\omega = \Omega(\boldsymbol{k}, \boldsymbol{x}) = \boldsymbol{k} \cdot \boldsymbol{u}_0 + c_0 k$

system of differential equations
to (numerically) solve

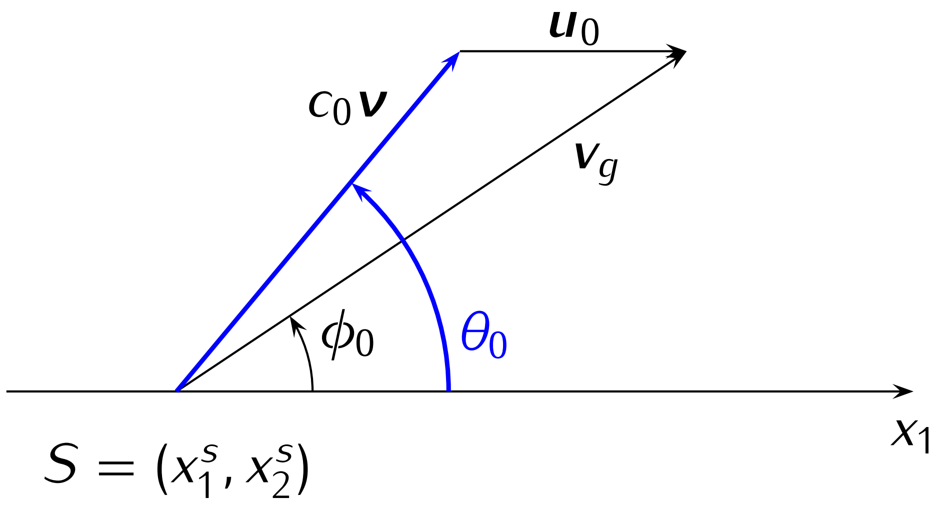
$$\begin{cases} \dfrac{dx_i}{dt} = c_0 \dfrac{k_i}{k} + u_{0i} \\ \dfrac{dk_i}{dt} = -k \dfrac{\partial c_0}{\partial x_i} - k_j \dfrac{\partial u_{0j}}{\partial x_i} \end{cases}$$

The system requires initial conditions. In 2-D,

- Source position S
- Orientation of the wavefront,
  with shooting angle θ_0

$$\cos\phi_0 = \frac{M_0 + \cos\theta_0}{\sqrt{(M_0 + \cos\theta_0)^2 + \sin^2\theta_0}}$$

$M_0 = u_0/c_0$



Additional remarks

- Ray equations are also called characteristic equations
  and they are intensively used in fluid dynamics (hyperbolic systems)

- General framework : WKB (Wentzel, Kramer, Brillouin) expansion method

  small parameter $\epsilon \sim \dfrac{\lambda}{L} \sim \dfrac{\text{acoustic wavelength}}{\text{medium length scale}}$

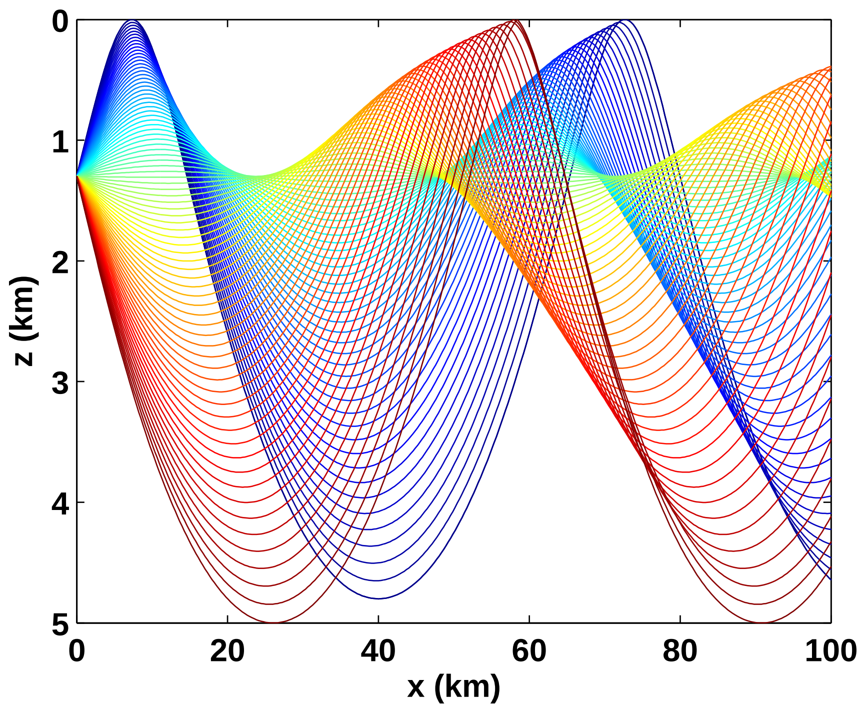
  $$\zeta = \tilde{\zeta}(\boldsymbol{X}) e^{i\Theta(\boldsymbol{X},T)/\epsilon} \text{ with } \boldsymbol{x} = \boldsymbol{X}/\epsilon \text{ and } t = T/\epsilon \qquad \tilde{\zeta}(\boldsymbol{X}) = \sum_{n=0}^{\infty} \epsilon^n \tilde{\zeta}^{(n)}$$

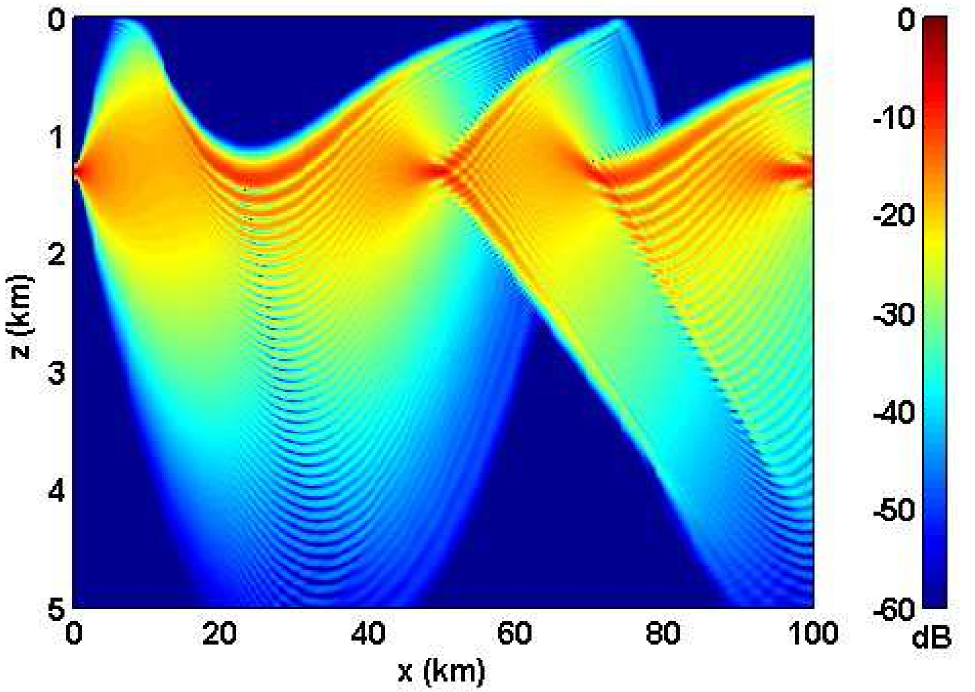
- Propagation of energy along rays

  $$\frac{\partial E}{\partial t} + \nabla \cdot \left(E \boldsymbol{v}_g \right) = 0$$

Underwater acoustics : ray-tracing versus parabolic approximation

(Munk's profile for the speed of sound)





Surface wave energy

Conservation of kinetic energy for an inviscid fluid

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) + \nabla \cdot \left(\frac{\rho u^2}{2}\boldsymbol{u}\right) + \boldsymbol{u} \cdot \nabla p = \rho \boldsymbol{f}_v \cdot \boldsymbol{u}$$

Incompressible flow, $\boldsymbol{u} \cdot \nabla p = \nabla \cdot (p\boldsymbol{u})$

$\rho \boldsymbol{f}_v = \rho \boldsymbol{g} = \nabla \phi_g$ with $\phi_g = -\rho g x_3$ and $\rho = \text{cst}$ (water)

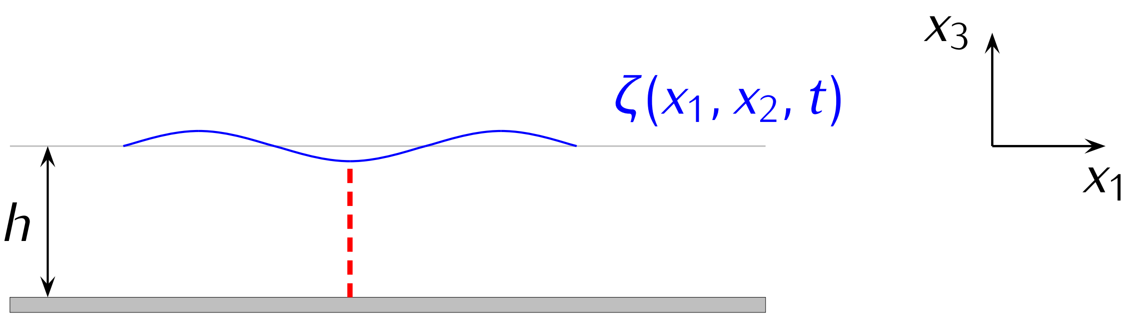
$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) + \nabla \cdot \left[\left(\frac{\rho u^2}{2} + p + \rho g x_3\right)\boldsymbol{u}\right] = 0$$

Equation of energy for a linear flow : quantities of third order (and higher) are discarded,

$$\frac{\partial}{\partial t}\left(\frac{\rho u'^2}{2}\right) + \nabla \cdot \left[\left(p' + \rho g x_3\right)\boldsymbol{u}'\right] = 0$$

Surface wave energy

Integration along $-h \leq x_3 \leq \zeta$



$$\int_{-h}^{\zeta} \frac{\partial}{\partial t}\left(\frac{\rho u'^2}{2}\right) dx_3 + \int_{-h}^{\zeta} \nabla \cdot \left[\left(p' + \rho g x_3\right) \boldsymbol{u}'\right] dx_3 = 0 \tag{13}$$

Leibniz integral rule : integral whose limits are functions of the differential variable

$$I(x) = \frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,y)dy = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x} dy + f(x,b)\frac{\partial b}{\partial x} - f(x,a)\frac{\partial a}{\partial x}$$

Surface wave energy

First term of Eq. (13)

$$\int_{-h}^{\zeta} \frac{\partial}{\partial t}\left(\frac{\rho u'^2}{2}\right) dx_3 = \frac{\partial}{\partial t}\int_{-h}^{\zeta} \frac{\rho u'^2}{2} dx_3 - \left.\frac{\rho u'^2}{2}\right|_{\zeta} \frac{\partial \zeta}{\partial t}$$

$$= \frac{\partial}{\partial t}\int_{-h}^{0} \frac{\rho u'^2}{2} dx_3 + \text{high order terms}$$

Surface wave energy

Second term of Eq. (13)

by introducing $\nabla \cdot \equiv \nabla_h \cdot + \partial_{x_3}$ where $\boldsymbol{x}_h \equiv (x_1, x_2)$

$$\int_{-h}^{\zeta} \nabla \cdot [(p' + \rho g x_3)\boldsymbol{u}']\, dx_3 = \int_{-h}^{\zeta} \nabla_h \cdot [(p' + \rho g x_3)\boldsymbol{u}'_h]\, dx_3 + \big[(p' + \rho g x_3)u'_3\big]_{-h}^{\zeta}$$

$$\begin{cases} \displaystyle\int_{-h}^{\zeta} \nabla_h \cdot [(p' + \rho g x_3)\boldsymbol{u}'_h]\, dx_3 = \nabla_h \cdot \int_{-h}^{0} (p' + \rho g x_3)\boldsymbol{u}'_h dx_3 + \begin{array}{c}\text{high order}\\ \text{terms}\end{array} \\ \big[\left(p' + \rho g x_3\right) u'_3\big]_{-h}^{\zeta} = \rho g \zeta \dfrac{\partial \zeta}{\partial t} = \dfrac{\partial}{\partial t}\left(\dfrac{1}{2}\rho g \zeta^2\right) \end{cases}$$

Kinematic condition for the free surface deformation $u'_3 = \partial \zeta / \partial t$

Notation $\breve{p}' \equiv p' + \rho g x_3$ (so-called dynamic pressure)

Surface wave energy

$$\underbrace{\frac{\partial}{\partial t}\left(\int_{-h}^{0}\frac{\rho u'^2}{2}dx_3 + \frac{1}{2}\rho g \zeta^2\right)}_{\text{(a)}} + \nabla_h \cdot \underbrace{\int_{-h}^{0}\breve{p}'\boldsymbol{u}'_h dx_3}_{\text{(b)}} = 0$$

(a) E = kinetic + potential energy (per unit surface area)

(b) $\boldsymbol{I}$ = energy flux in the plane $\boldsymbol{x}_h = (x_1, x_2)$

conservation of energy $$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{I} = 0$$

Illustration taken from acoustics

Linearized Euler equations around a uniform mean flow $\boldsymbol{u}_0 = u_0 \boldsymbol{x}_1$

$$\begin{cases} \dfrac{D\rho'}{Dt} + \rho_0 \nabla \cdot \boldsymbol{u}' = 0 & (14) \\[2ex] \rho_0 \dfrac{D\boldsymbol{u}'}{Dt} = -\nabla p' & (15) \end{cases}$$

where $D/Dt = \partial/\partial t + \boldsymbol{u}_0 \cdot \nabla = \partial/\partial t + u_0 \partial/\partial x_1$,
and by assuming that $p' = c_0^2 \rho'$

By taking the time derivative of Eq. (14) and the divergence of Eq. (15), and by substraction to eliminate the velocity fluctuations,

$$\frac{D^2 p'}{Dt^2} - c_0^2 \nabla^2 p' = 0$$

Dispersion relation, $(-i\omega + i\boldsymbol{k} \cdot \boldsymbol{u}_0)^2 - c_0^2 (ik)^2 = 0$
that is $\mathcal{D}(\boldsymbol{k}, \omega) = c_0^2 k^2 - (\boldsymbol{k} \cdot \boldsymbol{u}_0 - \omega)^2$

Illustration taken from acoustics

Conservation of energy

By multiplying Eq. (14) by ρ' and Eq. (15) by $\boldsymbol{u}'$, it yields

$$\frac{D}{Dt}\left(\frac{\rho'^2}{2}\right) + \rho_0\rho'\nabla\cdot\boldsymbol{u}' = 0 \qquad \text{and} \qquad \rho_0\frac{D}{Dt}\left(\frac{\boldsymbol{u}'^2}{2}\right) + \boldsymbol{u}'\cdot\nabla p' = 0$$

Using $p' = c_0^2\rho'$, the following energy budget equation can be derived

$$\frac{D}{Dt}\left(\frac{p'^2}{2\rho_0 c_0^2} + \frac{\rho_0\boldsymbol{u}'^2}{2}\right) + \nabla\cdot\left(p'\boldsymbol{u}'\right) = 0$$

that is,

$$\frac{\partial E}{\partial t} + \nabla\cdot(E\,\boldsymbol{u}_0 + \boldsymbol{I}) = 0 \quad \text{with} \quad E = \frac{p'^2}{2\rho_0 c_0^2} + \frac{\rho_0\boldsymbol{u}'^2}{2} \quad \text{and} \quad \boldsymbol{I} = p'\boldsymbol{u}'$$

$E \sim \mathrm{J.m}^{-3}$ sound energy density

Illustration taken from acoustics

Conservation of energy

For the case of a plane wave, $p' = \rho_0 c_0 u'$,

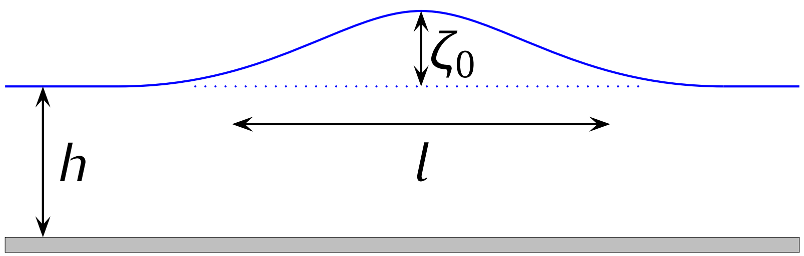
$$E \simeq \frac{p'^2}{\rho_0 c_0^2} \qquad \text{and} \qquad \boldsymbol{I} \simeq \frac{p'^2}{\rho_0 c_0} \boldsymbol{\nu} = E c_0 \boldsymbol{\nu}$$

and thus,

$$\frac{\partial E}{\partial t} + \nabla \cdot \left(E \boldsymbol{v}_g \right) = 0 \qquad \boldsymbol{v}_g = c_0 \boldsymbol{\nu} + \boldsymbol{u}_0 \quad \text{(group velocity)}$$

Introduction to nonlinear wave propagation

The linear Korteweg-de Vries (KdV) equation



(surface tension neglected, $l_c/\lambda \ll 1$)

Linearized equations were derived under the assumption that $\epsilon_\zeta = \zeta_0/h \ll 1$

$$\begin{cases} \nabla^2\phi' = 0 \\ \partial_{tt}\phi' + g\partial_{x_3}\phi' = 0 \text{ on } x_3 = 0 \\ \partial_{x_3}\phi' = 0 \text{ on } x_3 = -h \end{cases}$$

- Many ways to construct approximations, *e.g.* in deep water $h/l \gg 1$, or in shallow water $\epsilon_h = h/l \ll 1$
- Long-time evolution of tidal waves, $\epsilon_\zeta \ll 1$ and $\epsilon_h \sim kh \ll 1$
  $\omega = \pm\Omega(k) \qquad \Omega(k) = \sqrt{gk\tanh(kh)} \simeq \alpha k - \beta k^3 + \mathcal{O}\left(k^5\right)$

associated wave equation $\qquad \dfrac{\partial f}{\partial t} + \alpha\dfrac{\partial f}{\partial x_1} + \beta\dfrac{\partial^3 f}{\partial x_1^3} = 0 \qquad \alpha = \sqrt{gh} \quad \beta = \sqrt{gh}\,\dfrac{h^2}{6}$

The Korteweg-de Vries equation

Fully nonlinear model for surface waves. The derivation of the Korteweg-de Vries equation is rather tedious, and this step is skipped here. Using dimensionless variables, the KdV equation can be recast as

$$\frac{\partial \eta}{\partial \tau} + 6\eta \frac{\partial \eta}{\partial \xi} + \frac{\partial^3 \eta}{\partial^3 \xi} = 0$$

Korteweg & de Vries (1895)

$\eta = \zeta / h \quad \xi = (x_1 - \sqrt{gh}\, t)/l_{\mathrm{ref}} \quad \tau = t / t_{\mathrm{ref}}$

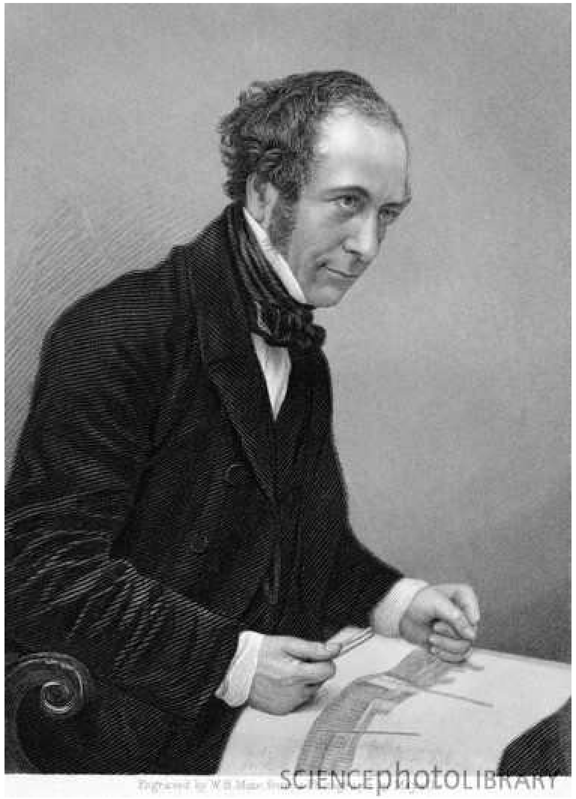
Historically : solitary waves or solitons (unchanging form during propagation, cancellation of nonlinear and dispersive effects)

John Scott Russell (1834, ..., 1885)

Joseph Valentin Boussinesq (1871, 1872)

Diederik Korteweg & Gustav de Vries (1895)

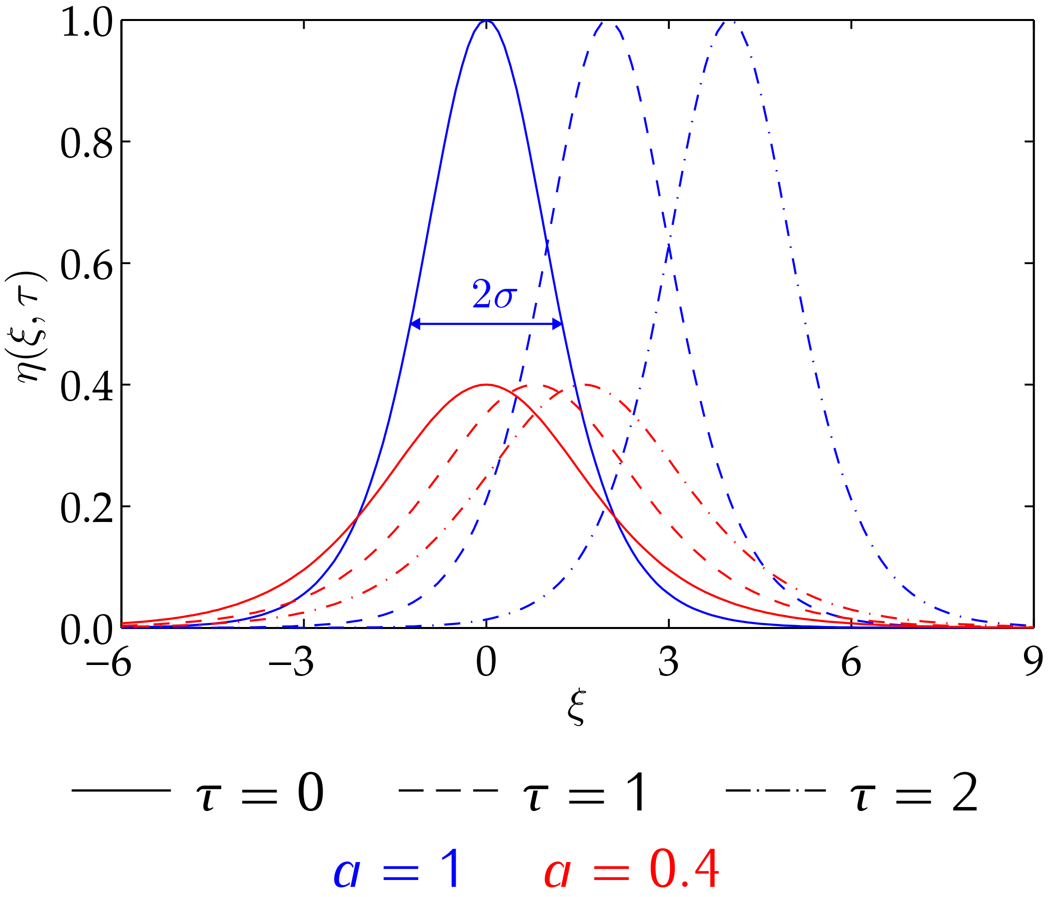
Solitons



John Scott Russell (1808-1882)

Collision of two solitons (Oregon Coast, USA, 2004, Terry Toedtemeier)

Solitary-wave solution



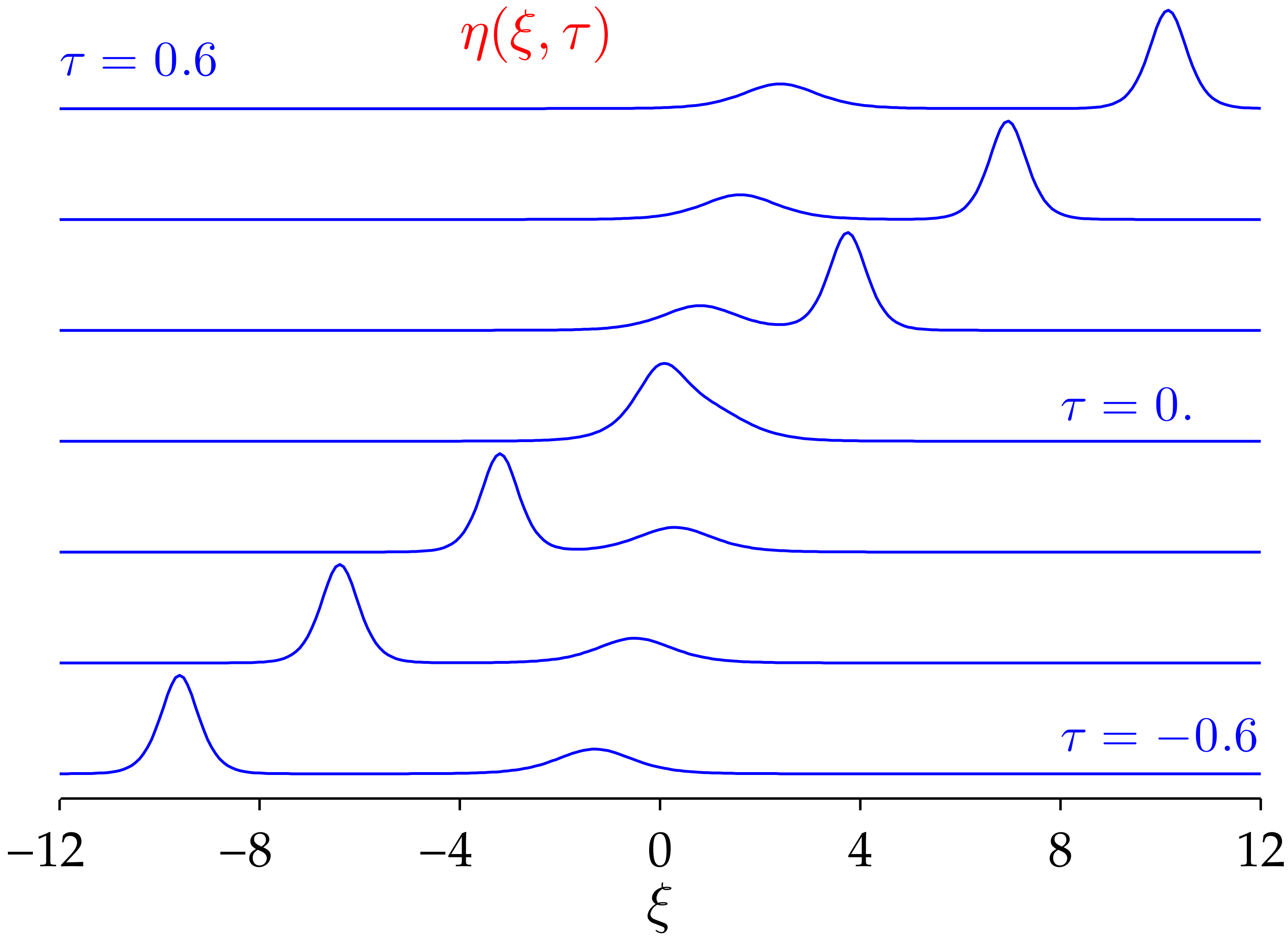
$$\eta = \frac{a}{\cosh^2\left[\sqrt{a/2}(\xi - 2a\tau)\right]}$$

Soliton of amplitude a, and of half-width σ, moves with velocity $2a > 0$

$$\sigma = \ln(1+\sqrt{2})\,\sqrt{2/a}$$

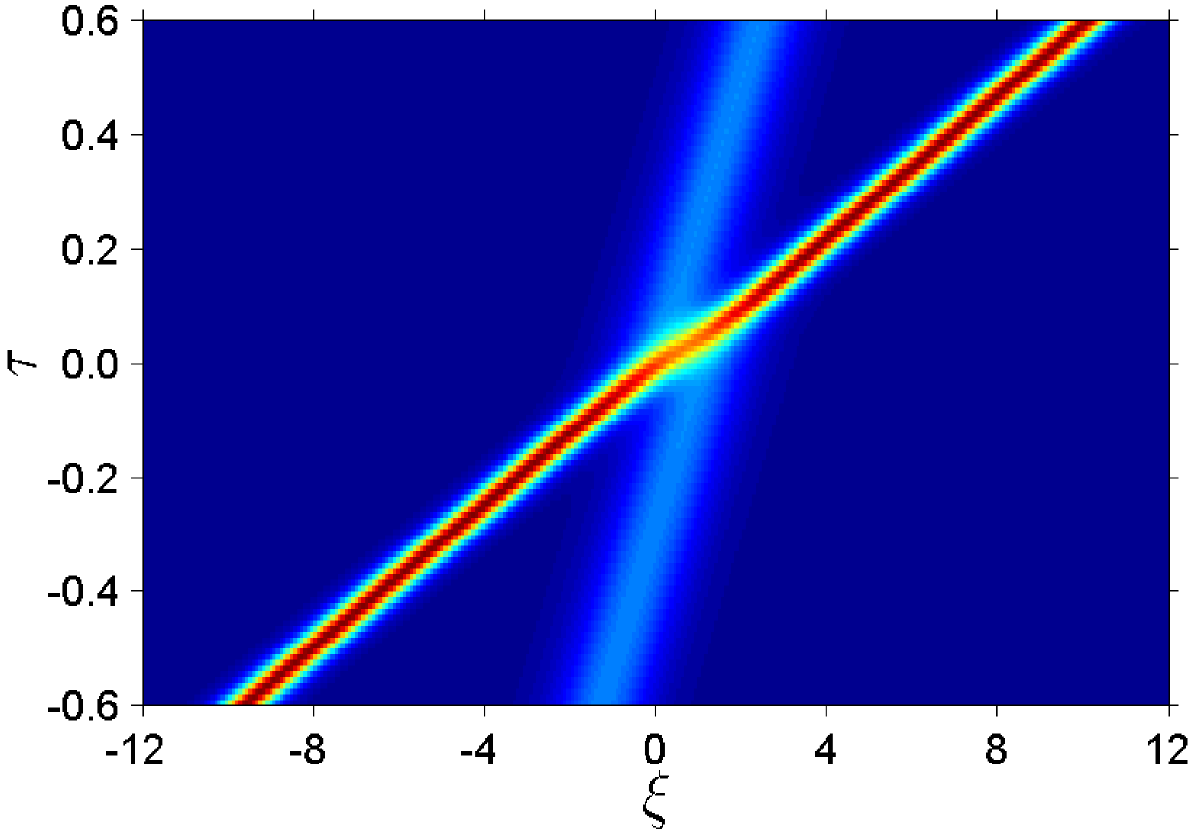
Zabusky & Kruskal (1965)
Gardner *et al.* (1967)

- Interacting solitary waves !



Interacting solitary waves

Elastic collision, and the nonlinear interaction produces a phase shift (taller wave moved forward, smaller one backward)

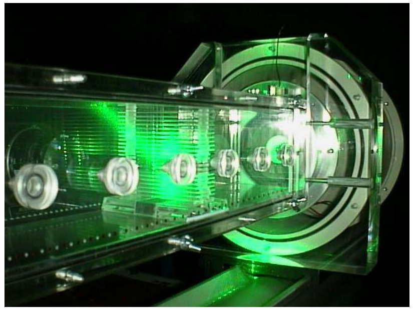


Motivations

Supersonic flying object : aircraft, missile, rocket, meteorite, ...

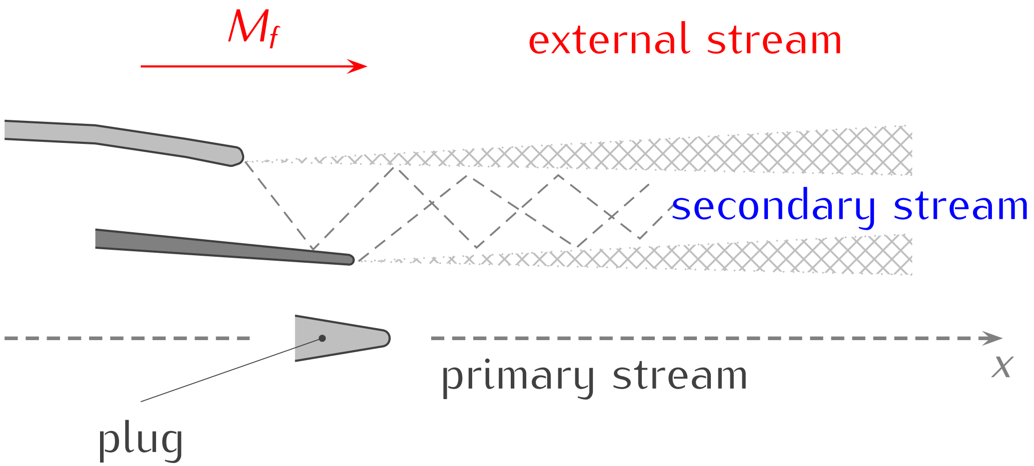
High-speed jet noise, cavity noise, ...

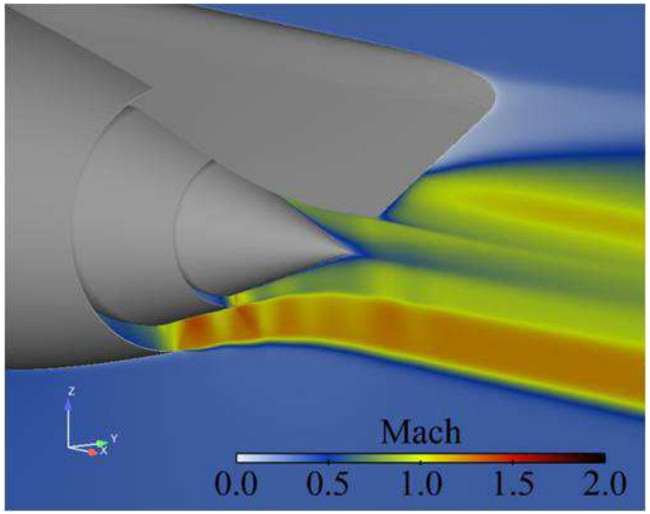
Propagation in resonant systems : thermoacoustics, musical instruments, ...



In aeronautical applications

Secondary flow of a commercial civil engine during the climb and cruise phases

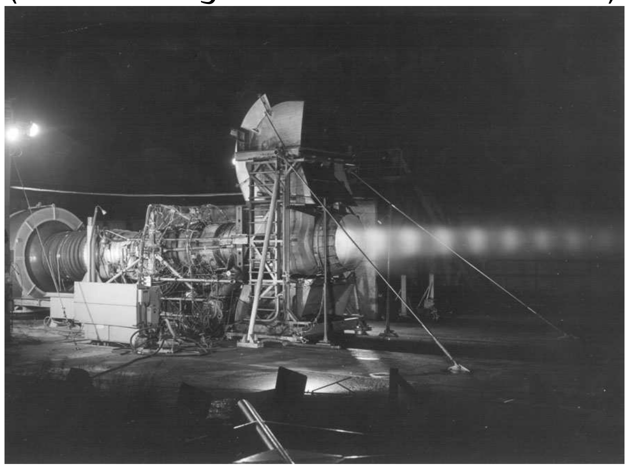




C. Henry (SNECMA)

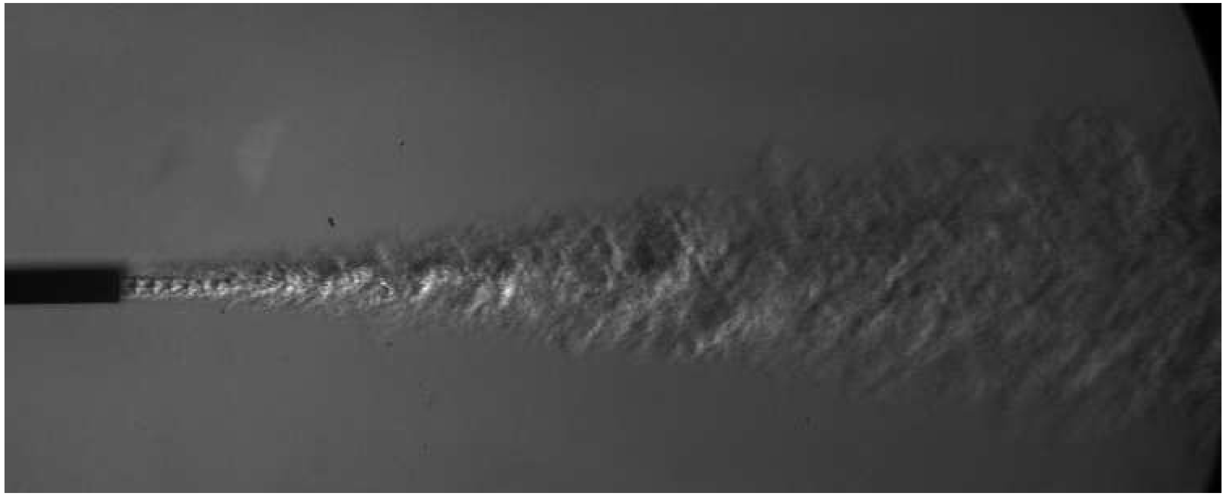
Bell X-1 (1947) flying at Mach 1.07

Olympus 593 Mark 610 (Rolls-Royce & Snecma, 1966)



... but also in domestic life !

Compressed air canister for cleaning your computer
($\mathrm{Re}_D \simeq 5 \times 10^4$)

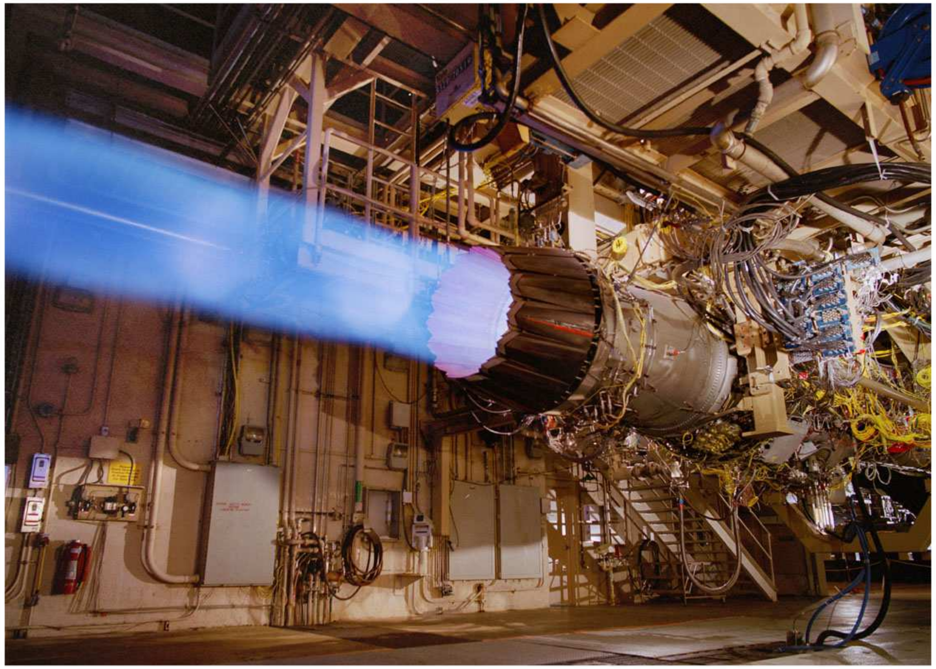


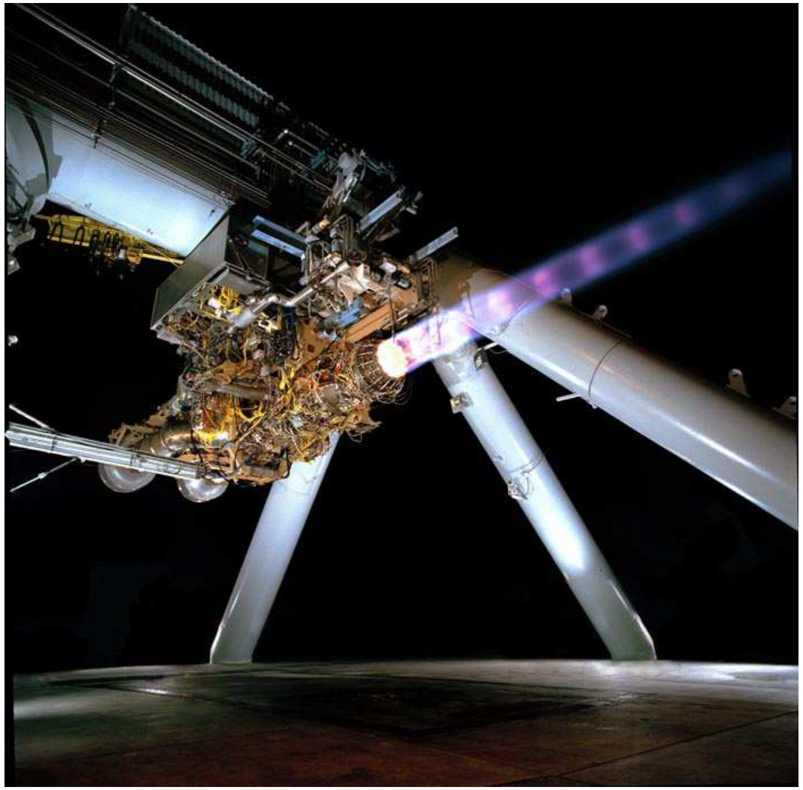
(E. Salze, LMFA)

- **Military and supersonic transport aircrafts**

  Pratt & Whitney FX631 jet engine (F-35 Joint Strike Fighter)

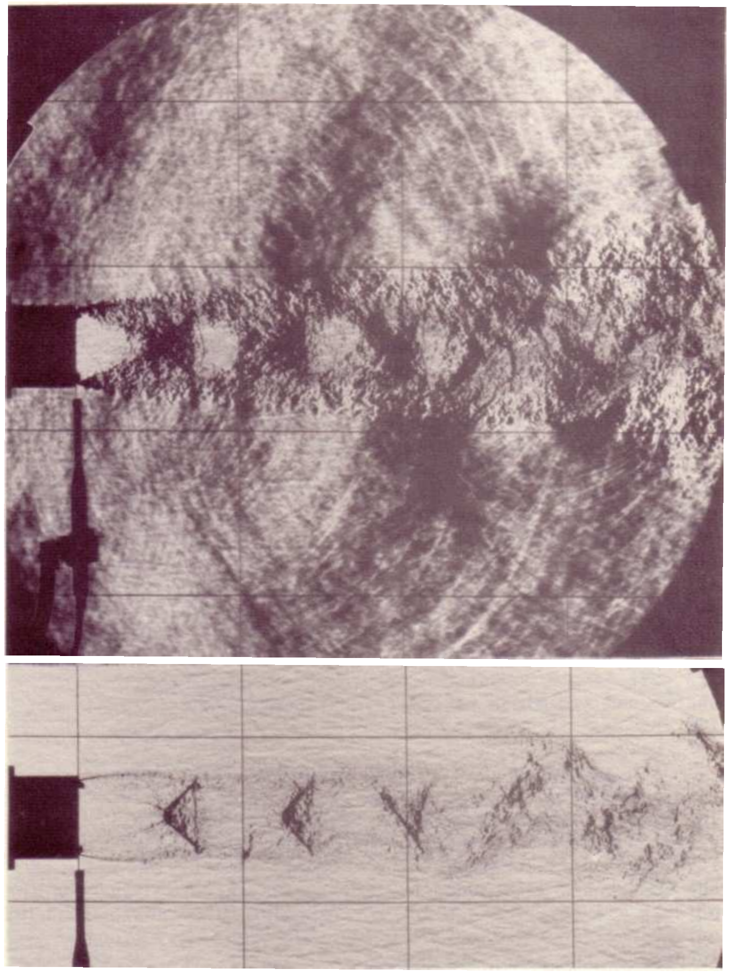
  Kleine & Settles, *Shock Waves* (2008)





http://www.jsf.mil

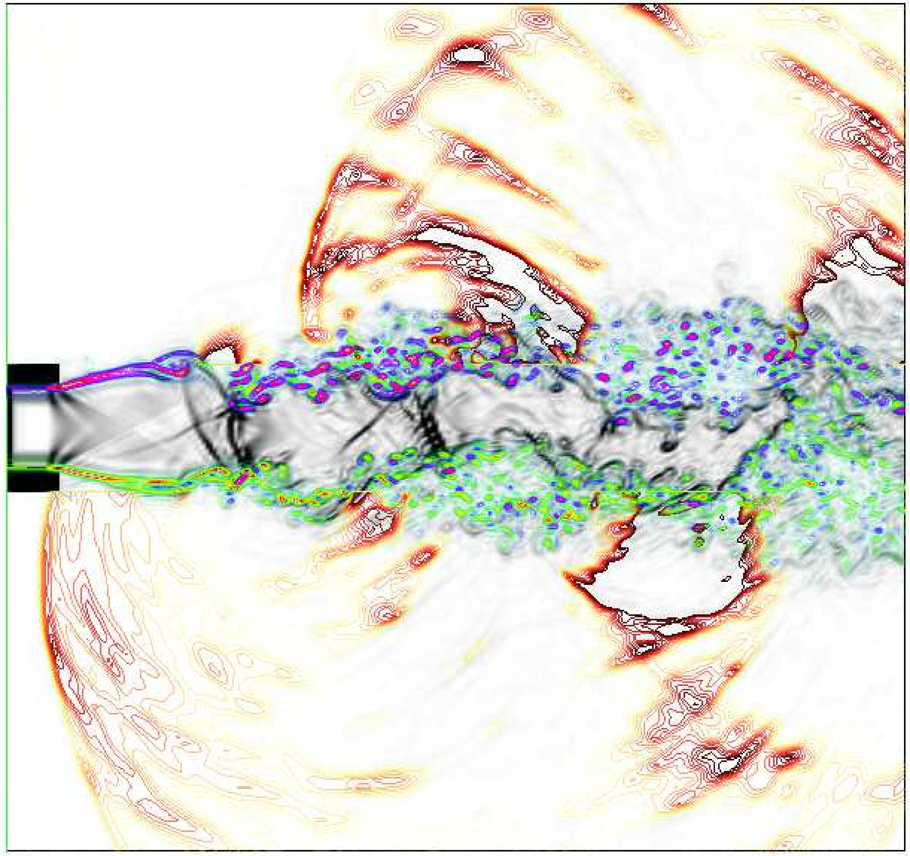
Military and supersonic transport aircrafts



$p_R/p_\infty = 2.48$, $D = 5.76$ cm

$p_e/p_\infty = 2.48$, $\mathrm{M}_j = 1.67$

Westley & Wooley, Prog. Astro. Aero., 43, 1976



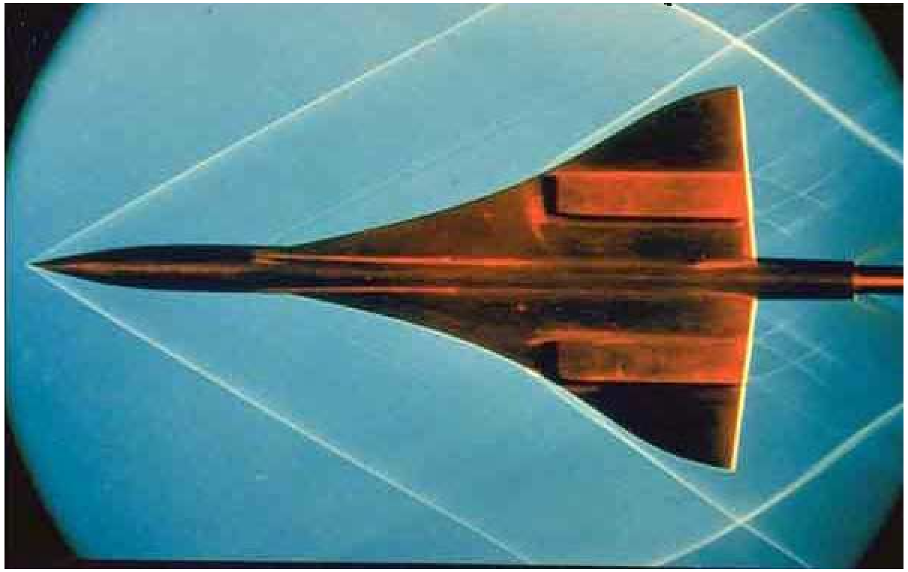
$M_j = 1.55$ & $Re_h = 6 \times 10^4$

$p_e/p_\infty = 2.09$

Berland, Bogey & Bailly, *Phys. Fluids*, 19, 2007

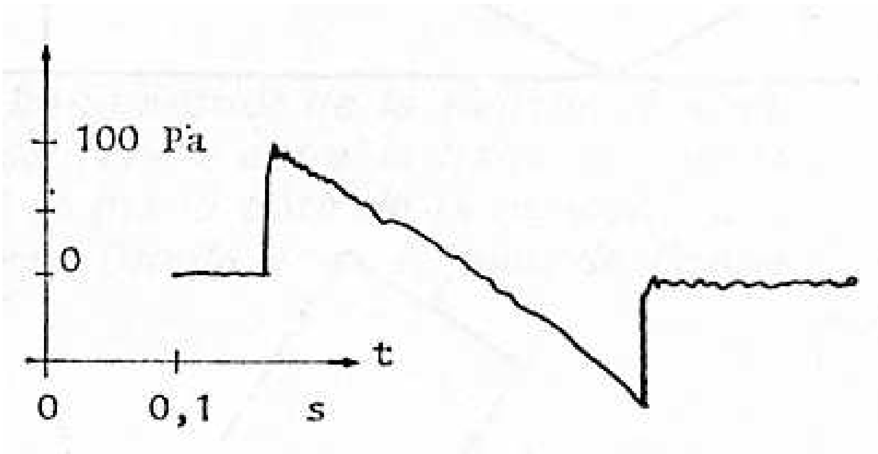
- **Sonic boom**

F/A-18 Hornet
passing through the sound barrier
(Navy Ensign John Gay, July 7, 1999)

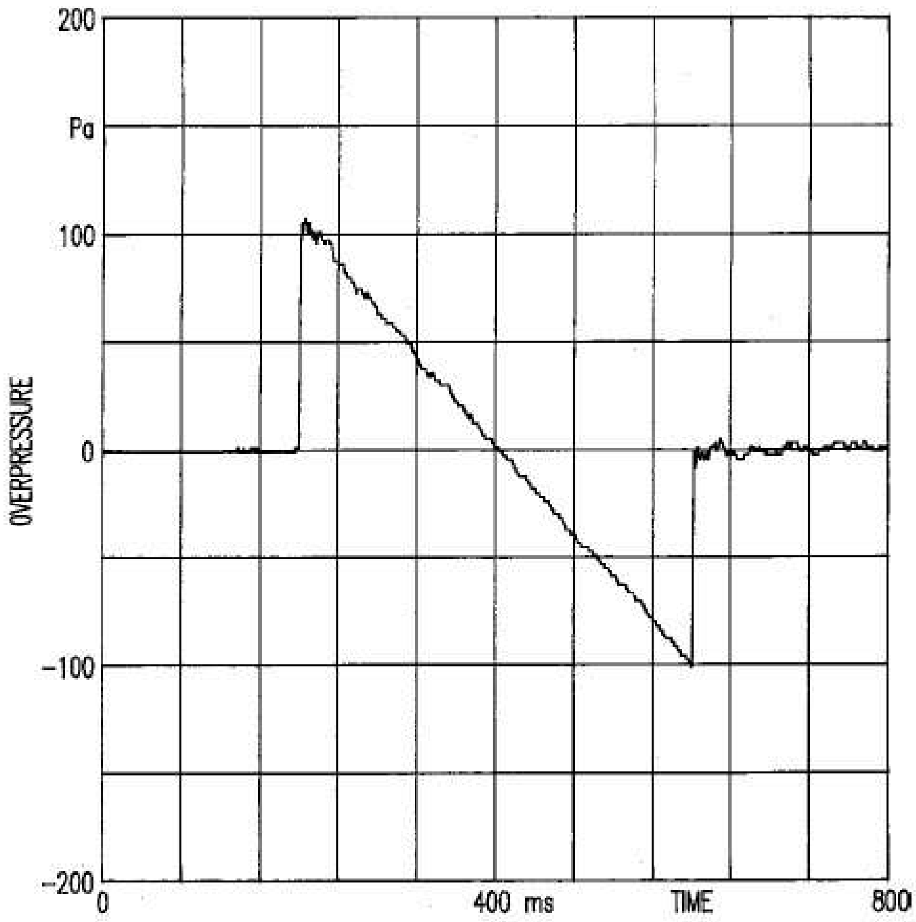


Concorde - Shock waves at Mach 2.2 in wind tunnel (ONERA)

N-wave pattern measured close to the ground from Concorde



- **Space shuttle Columbia – 10 December 1990**
  N-duration 400 ms, overpressure 104 Pa ($z \simeq 18$km, $M \simeq 1.5$)



Young, *J. Acoust. Soc. Am.*, 2002

1-D Euler equations for a homentropic flow

in conservative form

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x_1} = 0 \qquad \boldsymbol{U} = \begin{pmatrix} \rho \\ \rho u_1 \\ \rho e_t \end{pmatrix} \qquad \boldsymbol{E} = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p \\ u_1 \left(\rho e_t + p\right) \end{pmatrix}$$

with $\rho e_t = \rho e + \dfrac{\rho u_1^2}{2} = \dfrac{p}{\gamma - 1} + \dfrac{\rho u_1^2}{2}$ for an ideal gas

In order to highlight nonlinear effects (*e.g.* the formation of a N-wave) while keeping algebra as simple as possible, a more basic flow model is considered here to derive characteristic equations. Namely, the flow is assumed homentropic, $s =$ cst. Hence, $dp = c^2 d\rho$ and

$$\frac{\partial \rho}{\partial t} = \left.\frac{\partial \rho}{\partial p}\right|_s \frac{\partial p}{\partial t} = \frac{1}{c^2}\frac{\partial p}{\partial t}$$

1-D Euler equations for a homentropic flow

$$\frac{\partial \rho}{\partial t} + u_1 \frac{\partial \rho}{\partial x_1} + \rho \frac{\partial u_1}{\partial x_1} = 0 \quad \Longrightarrow \quad \frac{\partial p}{\partial t} + u_1 \frac{\partial p}{\partial x_1} + \rho c^2 \frac{\partial u_1}{\partial x_1} = 0 \tag{16}$$

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + \frac{1}{\rho}\frac{\partial p}{\partial x_1} = 0 \tag{17}$$

By taking Eq. (17) $\pm$ Eq. (16)/(ρc), characteristic equations are obtained,

$$\frac{\partial u_1}{\partial t} \pm \frac{1}{\rho c}\frac{\partial p}{\partial t} + (u_1 \pm c)\left(\frac{\partial u_1}{\partial x_1} \pm \frac{1}{\rho c}\frac{\partial p}{\partial x_1}\right) = 0$$

Let us introduce the Riemann invariants (1860)

$$R_{\pm} = u_1 \pm \int \frac{dp}{\rho c} = u_1 \pm \frac{2}{\gamma - 1} c$$

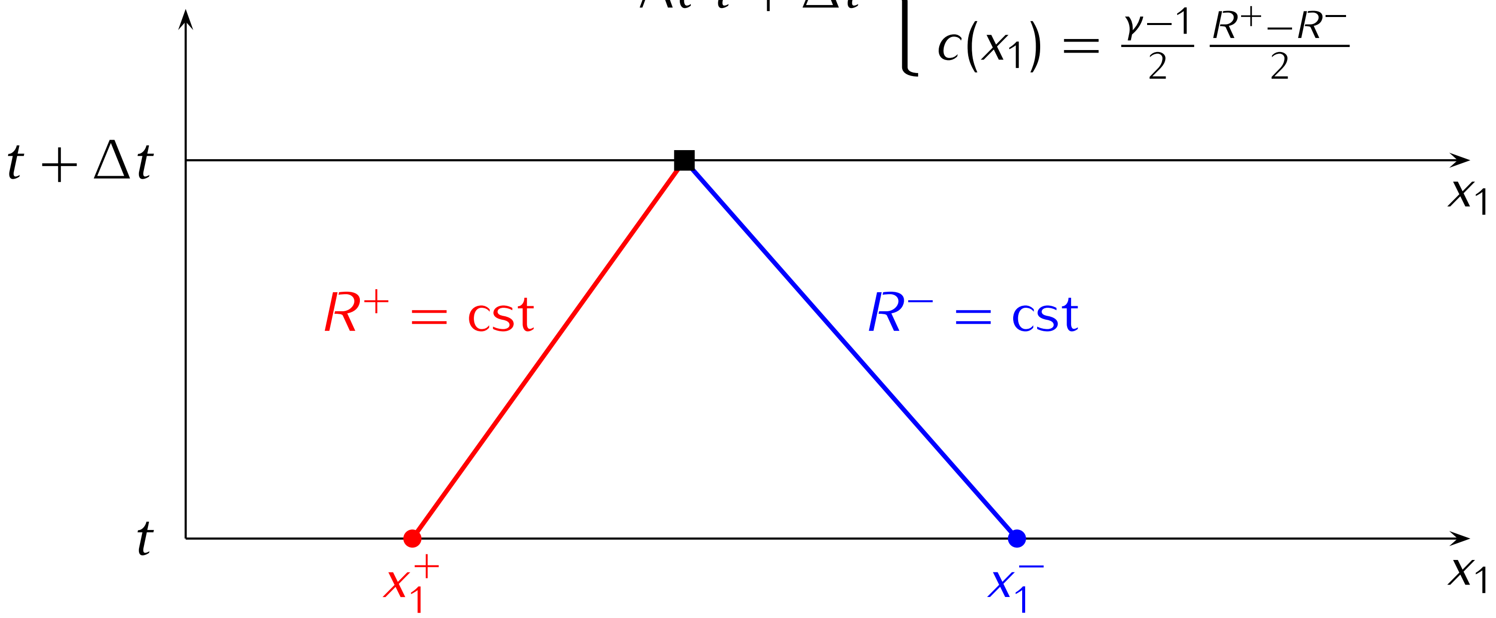
$$c^2 = \frac{\gamma p}{\rho} \quad \Longrightarrow \quad 2\frac{dc}{c} = \frac{dp}{p} - \frac{d\rho}{\rho} \quad \Longrightarrow \quad 2dc = \frac{c}{p}dp - \frac{c}{\rho}\frac{dp}{c^2} = (\gamma - 1)\frac{dp}{\rho c}$$

- ### 1-D Euler equations for a homentropic flow

  Along the curves defined by $dx_1 = (u_1 \pm c)\, dt$, the two Riemann invariants R_+ and R_- are respectively conserved,

$$R_+ = u_1 + \frac{2}{\gamma - 1} c = \text{cst} \qquad\qquad R_- = u_1 - \frac{2}{\gamma - 1} c = \text{cst}$$

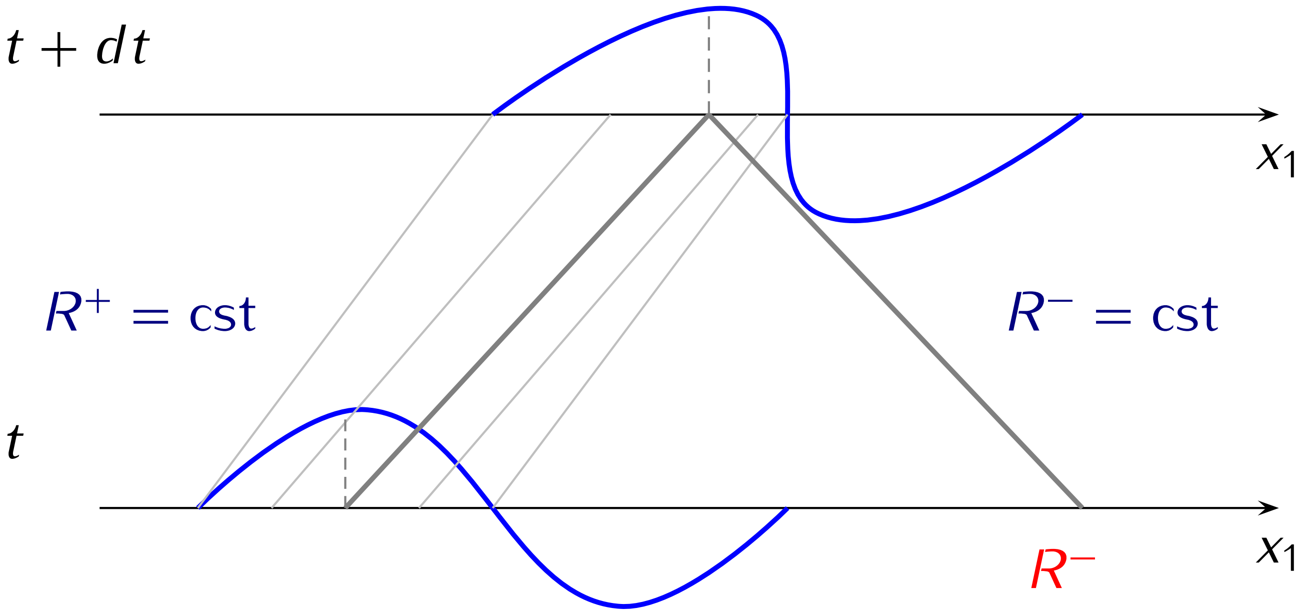
$$\text{At } t + \Delta t \begin{cases} u_1(x_1) = \frac{R^+ + R^-}{2} \\ c(x_1) = \frac{\gamma - 1}{2} \frac{R^+ - R^-}{2} \end{cases}$$



solution known at time $t \Longrightarrow R^+, R^-$ also known

1-D Euler equations for a homentropic flow

Construction of a solution using characteristics

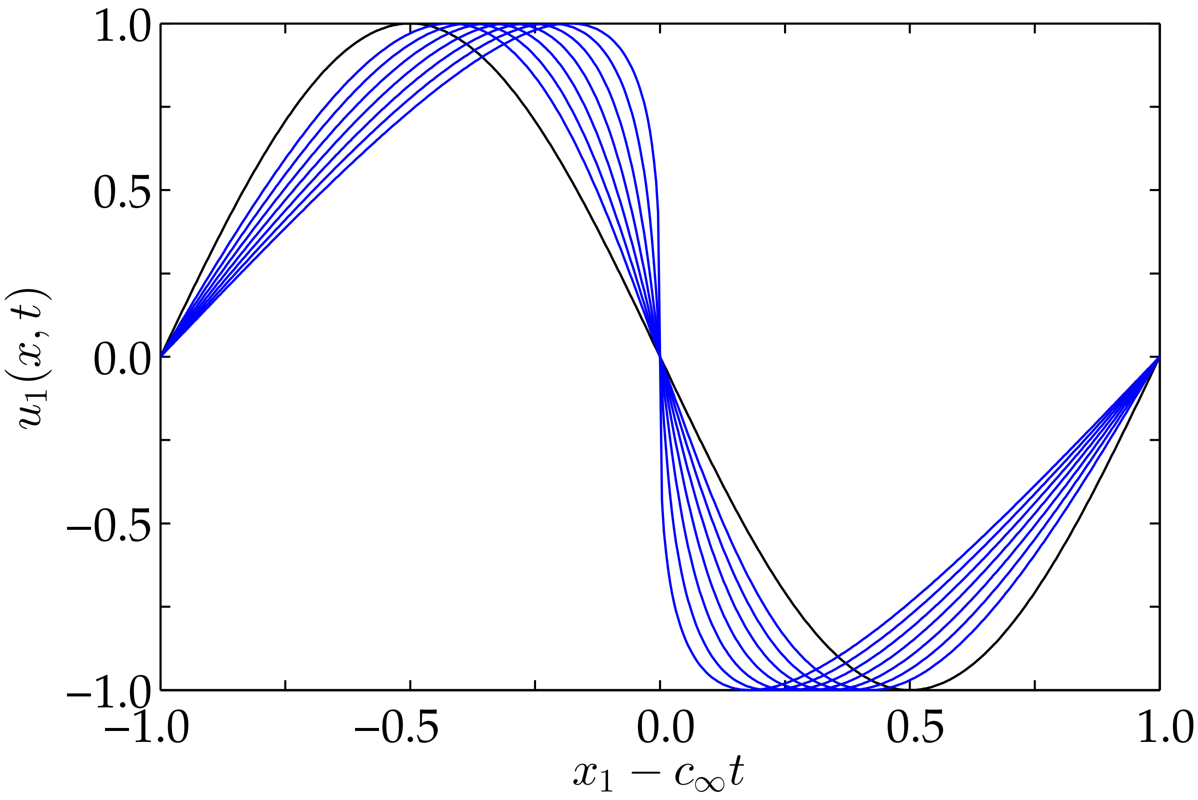


$$R^- = -\frac{2}{\gamma-1}c_\infty\bigg|_t = u_1 - \frac{2}{\gamma-1}c\bigg|_{t+dt} \quad \Longrightarrow \quad c = c_\infty + \frac{\gamma-1}{2}u_1$$

The local speed of sound c is affected by the perturbation amplitude u_1

Formation of a N-wave

The local speed of sound is modified by the velocity amplitude u_1 of the perturbation. The part of the signal corresponding to $u_1 < 0$ travels slower than the part corresponding to $u_1 > 0$. The initial signal is thus distorted, with a stiffening of the front wave and the formation of a weak shock, namely a N-wave.



1-D Euler equations for a homentropic flow

Interpretation : from Euler's equation (17) and the Riemann invariant R^-

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + \frac{2}{\gamma - 1} c \frac{\partial c}{\partial x_1} = 0 \qquad c = c_\infty + \frac{\gamma - 1}{2} u_1$$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} + \left(c_\infty + \frac{\gamma - 1}{2} u_1\right)\frac{\partial u_1}{\partial x_1} = 0$$

which leads to,

$$\frac{\partial u_1}{\partial t} + \left(c_\infty + \frac{\gamma + 1}{2} u_1\right)\frac{\partial u_1}{\partial x_1} = 0$$

Two contributions to nonlinear effects can be identified, which are associated with

– thermodynamics, with the modification of the speed of sound

– the convection itself

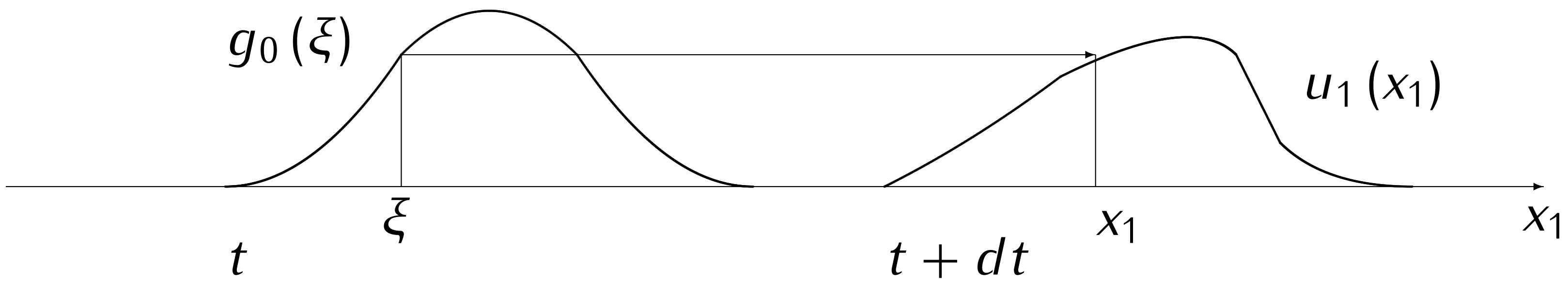
1-D Euler equations for a homentropic flow

Parametric solution – Initial value $u_1 = g_0(x_1)$ at $t = 0$

$$u_1(x_1, t) = g_0\left[x_1 - \left(c_\infty + \frac{\gamma+1}{2}u_1\right)t\right]$$

Time evolution provided by following the characteristic line,

$$x_1 = \xi + \left[c_\infty + \frac{\gamma+1}{2}g_0(\xi)\right]dt$$

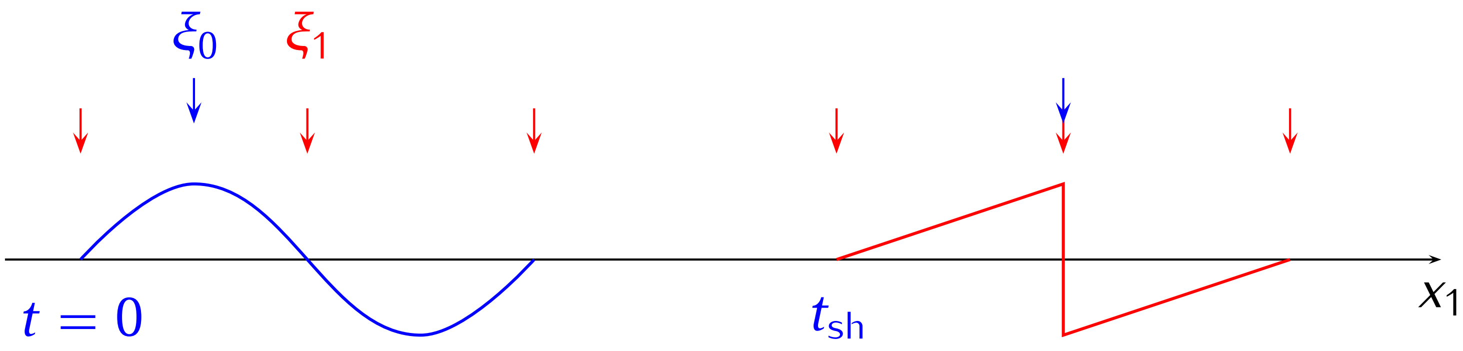


1-D Euler equations for a homentropic flow

Parametric solution - estimation of the shock formation time

As illustration, initial sinusoidal perturbation

$g_0(x_1) = a \sin(kx_1) \qquad 0 \leq x_1 \leq 1 \qquad t = 0 \qquad \lambda = 2\pi/k$



The shock formation time t_{sh} is given by the time needed by the velocity peak a (initially at ξ_0) to reach the next neutral point (initially at ξ_1 with $\xi_1 - \xi_0 = \lambda/4$),

$$x_{sh} = \xi_0 + \left(c_\infty + \frac{\gamma + 1}{2} a \right) t_{sh} \qquad x_{sh} = \xi_1 + c_\infty t_{sh}$$

1-D Euler equations for a homentropic flow

Parametric solution - estimation of the shock formation time t_{sh}

$$\left(c_\infty + \frac{\gamma+1}{2}\,a\right)t_{sh} - c_\infty t_{sh} = \frac{\lambda}{4} \qquad \Longrightarrow \qquad t_{sh} = \frac{2}{\gamma+1}\,\frac{\lambda}{4a}$$

1-D Euler equations for a homentropic flow

General approach to derive characteristic equations associated with a hyperbolic system

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{V}}{\partial x_1} = 0 \qquad \boldsymbol{V} = \begin{pmatrix} \rho \\ u_1 \end{pmatrix} \qquad \boldsymbol{A} = \begin{pmatrix} u_1 & \rho \\ c^2/\rho & u_1 \end{pmatrix}$$

- Eigenvalues $\lambda = u_1 \pm c$ and eigen vectors $\boldsymbol{V}_\lambda$ of matrix $\boldsymbol{A}$

$$\boldsymbol{V}_\lambda = \begin{pmatrix} 1 \\ \pm c/\rho \end{pmatrix} \qquad \boldsymbol{S} = (\boldsymbol{V}_\lambda) = \begin{pmatrix} 1 & 1 \\ c/\rho & -c/\rho \end{pmatrix} \qquad \boldsymbol{S}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \rho/c \\ 1 & -\rho/c \end{pmatrix}$$

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1} \qquad \boldsymbol{\Lambda} = \begin{pmatrix} u_1 + c & 0 \\ 0 & u_1 - c \end{pmatrix}$$

- Characteristic equations

$$\boldsymbol{S}^{-1}\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{S}^{-1}\boldsymbol{A}\frac{\partial \boldsymbol{V}}{\partial x_1} = 0 \quad \Longrightarrow \quad \boldsymbol{S}^{-1}\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{\Lambda}\boldsymbol{S}^{-1}\frac{\partial \boldsymbol{V}}{\partial x_1} = 0$$

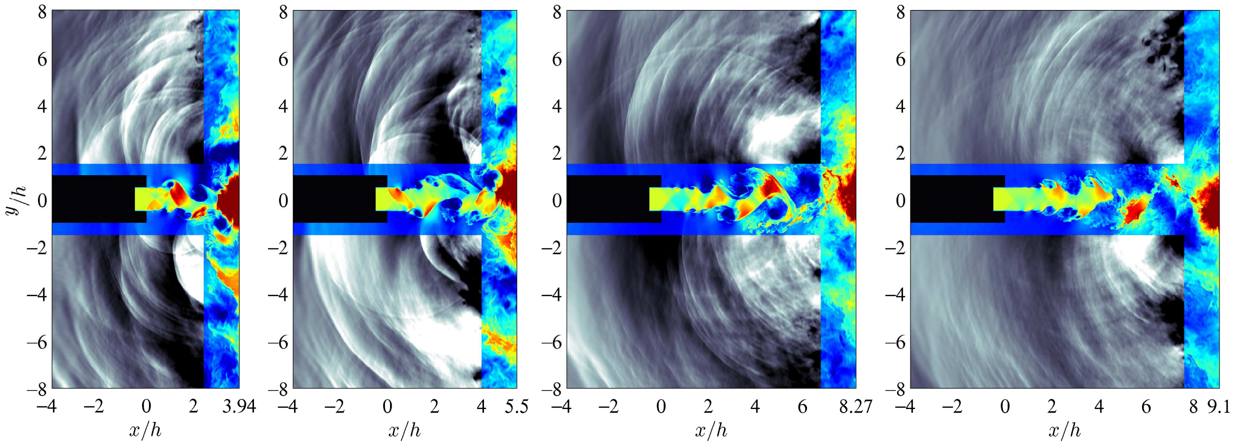
Turbulence and Aeroacoustics

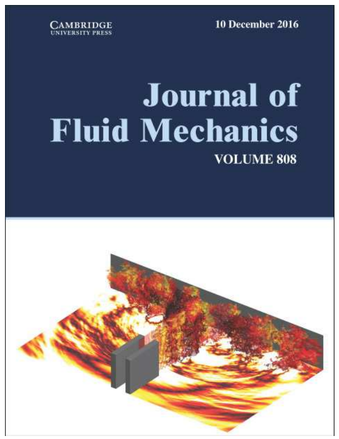
Highly qualified candidates are encouraged

to apply at any time !

http://acoustique.ec-lyon.fr

Investigation of tone generation in ideally expanded supersonic planar impinging jets (Gojon, Bogey & Marsden, *J. Fluid Mech.*, 2016)





Density and pressure fields, $L/h = 3.94, 5.5, 8.27, 9.1$

$M_j = 1.28$, $\mathrm{Re}_h = 5 \times 10^4$