Laboratoire de Mécanique des Fluides et d'Acoustique
LMFA UMR 5509

ÉCOLE CENTRALE LYON

Fluid mechanics and energy

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Textbooks on fluid dynamics

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National Committee for Fluid Mechanics Films (NCFMF)

Gallery of fluid motion (APS division of fluid mechanics)

Textbooks on turbulent flows

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airfoil	profil
bluff body	corps non profilé
boundary layer	couche limite
bulk velocity	vitesse de débit
buoyancy	flottabilité
curl	rotationnel
chord	corde
conservative force	force qui dérive d'un potentiel (gravité par exemple)
creeping flow	écoulement rampant
Darcy friction coefficient	coefficient de pertes de charge
drag	traînée
density (mass per unit volume)	masse volumique
efficiency	rendement
energy head	charge
friction velocity	vitesse de frottement
head loss	perte de charge
inviscid flow	écoulement non visqueux
leading edge	bord d'attaque (d'un profil)
lift	portance
lift-to-drag ratio	finesse
mass fraction	fraction massique
mixture	mélange
point vortex	tourbillon ponctuel

relative density	densité
shaft work	travail de l'arbre (d'une machine tournante)
skin-friction coefficient	coefficient de frottement
slip boundary condition	condition aux limite glissante
stall	décrochage
strain (deformation) tensor	tenseur des déformations
stream function	fonction de courant
streamlined body	corps profilé
stress tensor	tenseur des contraintes
thrust	poussée
torque (angular momentum)	couple
trailing edge	bord de fuite (d'un profil)
vortex shedding frequency	fréquence du lâcher tourbillonnaire
vortex sheet	nappe (infiniment mince) de vorticité
wake	sillage
wall shear stress	contrainte pariétale

aka	also known as
wrt	with respect to

- **Both indicial and boldface notations are used to indicate vectors**

vector $\boldsymbol{U} \equiv \overrightarrow{U}$, i-th component U_i, norm U, $U^2 = \boldsymbol{U} \cdot \boldsymbol{U}$
gravity $\boldsymbol{g}$, $g_i = -g\delta_{3i}$, $\boldsymbol{g} = (g_1, g_2, g_3) = (0, 0, -g)$, $g = 9.81$ m.s^{-2}

density ρ (kg.m^{-3})
δ_{ij} Kronecker delta

Einstein summation convention
When an index variable appears twice in a single term (dummy index), it implies summation of that term over all the values of the index.

Scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i \quad \text{(dummy index } i \text{ repeated)}$$

Short quiz $\delta_{ij} a_j = ?$ $\delta_{ij} \delta_{ij} = ?$

- **Differential operators (expressed in Cartesian coordinates here)**
  The dot symbol · is never decorative : scalar product

Gradient

$$\boldsymbol{b} = \nabla f \equiv \overrightarrow{\text{grad}}\, f \qquad b_i = \frac{\partial f}{\partial x_i}$$

Divergence

$$\nabla \cdot \boldsymbol{U} = \text{div}(\boldsymbol{U}) = \sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} = \frac{\partial U_i}{\partial x_i}$$

Laplacian

$$\nabla^2 f = \Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_i}$$

Curl

$$\nabla \times \boldsymbol{U} = \overrightarrow{\text{rot}}\, \boldsymbol{U}$$

Differential operators (cont.)

Explicit expression of the velocity gradient tensor $\nabla \boldsymbol{U}$

$$\nabla \boldsymbol{U}\Big|_{ij} = \frac{\partial \boldsymbol{U}}{\partial \boldsymbol{x}}\Big|_{ij} = \begin{pmatrix} \dfrac{\partial U_1}{\partial x_1} & \dfrac{\partial U_1}{\partial x_2} & \dfrac{\partial U_1}{\partial x_3} \\ \dfrac{\partial U_2}{\partial x_1} & \dfrac{\partial U_2}{\partial x_2} & \dfrac{\partial U_2}{\partial x_3} \\ \dfrac{\partial U_3}{\partial x_1} & \dfrac{\partial U_3}{\partial x_2} & \dfrac{\partial U_3}{\partial x_3} \end{pmatrix}$$

Differential operators (cont.)

Divergence theorem : the involved surface is a closed surface
(domain $\mathcal{D}$ bounded by the surface $\mathcal{S}$ and $\boldsymbol{n}$ unit outward normal vector)

$$\int_{\mathcal{D}} \nabla \cdot \overline{\overline{\boldsymbol{A}}}\, dv = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{A}}} \cdot \boldsymbol{n}\, ds$$

for any given tensor $\overline{\overline{\boldsymbol{A}}}$.
See for instance application pages 41 and 235. As an illustration, one has for the pressure term :

$$\int_{\mathcal{D}} \nabla p\, dv = \int_{\mathcal{D}} \nabla \cdot (p\overline{\overline{\boldsymbol{I}}})\, dv = \int_{\mathcal{S}} p\overline{\overline{\boldsymbol{I}}} \cdot \boldsymbol{n}\, ds = \int_{\mathcal{S}} p\boldsymbol{n}\, ds$$

Differential operators (cont.)

Useful vectorial identities

$$\nabla \cdot (\alpha \boldsymbol{a}) = \boldsymbol{\alpha} \nabla \cdot \boldsymbol{a} + \nabla \alpha \cdot \boldsymbol{a}$$

$$\nabla \times (\alpha \boldsymbol{a}) = \nabla \alpha \times \boldsymbol{a} + \alpha \nabla \times \boldsymbol{a}$$

Used to compute the torque $\boldsymbol{C}$ (but not so useful)

$$\int_{\mathcal{S}} \boldsymbol{x} \times (-p\boldsymbol{n}\, ds) = -\int_{\mathcal{V}} \nabla \times (-p\boldsymbol{x})\, dv = -\int_{\mathcal{D}} \boldsymbol{x} \times \nabla p\, dv$$

Frequently Asked Questions

Tensor : geometric object ;
must be independent of a particular choice of coordinate system

1st order tensor : vector

2nd-order tensor : the Kronecker delta δ_{ij}, the stress tensor σ_{ij}, or the gradient velocity tensor $(\nabla \boldsymbol{U})_{ij} = \partial U_i / \partial x_j$ among others.

An essential reference for mathematics enthusiasts :

Aris, R., 1962, *Vectors, tensors and the basic equations of fluid mechanics,* Dover Publications, Inc., New York.

Main notations

a	thermal diffusivity (see slide 250)
Bi	Biot number (see slide 270)
$\mathcal{D}$	control volume (material or not)
D/Dt	material derivative (see slide 25)
$\overline{\overline{\boldsymbol{D}}}$	deformation tensor
e	specific internal energy (see slide 215)
$\boldsymbol{g}$	gravity vector ($\boldsymbol{g} = -\nabla\Psi$)
$\Gamma_{\mathcal{C}}$	circulation (see slide 171)
Gr	Grashof number (see slide 277)
h	specific enthalpy (see slide 222)
$\tilde{h}$	global heat transfer coefficient (see slide 266)
h	local heat transfer coefficient (see slide 271)
$\mathcal{H}$	local energy per unit volume (see slide 91)
k	thermal conductivity (see slide 228)
$\overline{\overline{\boldsymbol{I}}}$	identity (unit) tensor
λ	second viscosity (see slide 214)
m	mass of a fluid particle (see slide 22)
M_a	Mach number (see slide 33)
q_∞	dynamic pressure (see slide 57)
U_d	bulk velocity (see slide 121)
U_n	Velocity scale in natural convection (see slide 276)
μ	dynamic (shear) viscosity ($\mu = \rho\nu$)
$\boldsymbol{n}$	unit normal vector pointing outward from $\mathcal{S}$

Main notations (cont.)

Nu	Nusselt number (see slide 267)
Pe	Péclet number (see slides 260 and 295)
Pr	Prandtl number (see slide 252)
Ψ	potential function of gravity ($\Psi = \boldsymbol{g} \cdot \boldsymbol{x}$, see slide 53)
ψ	stream function (see slide 335)
$\boldsymbol{q}$	heat flux vector (see slide 228)
ρ	density (masse volumique)
Sc	Schmidt number (see slide 295)
T	temperature
t	time
$\overline{\overline{\boldsymbol{\sigma}}}$	stress tensor
$\boldsymbol{T}$	stress vector
$\overline{\overline{\boldsymbol{\tau}}}$	viscous stress tensor (see slides 65 and 214)
U_d	bulk velocity (see slide 36)
$\boldsymbol{U}$	velocity vector
$\boldsymbol{U}_{\mathcal{S}}$	velocity of the control surface $\mathcal{S}$ (see slide 27)
$\mathcal{V}$	volume of a fluid particle (see slide 22)
$\boldsymbol{x}$	position vector
$\boldsymbol{x}_P$	material point
$\equiv$	means by definition
$-_f$	subscript for fluid
$-_s$	subscript for solid
$-_w$	subscript for wall

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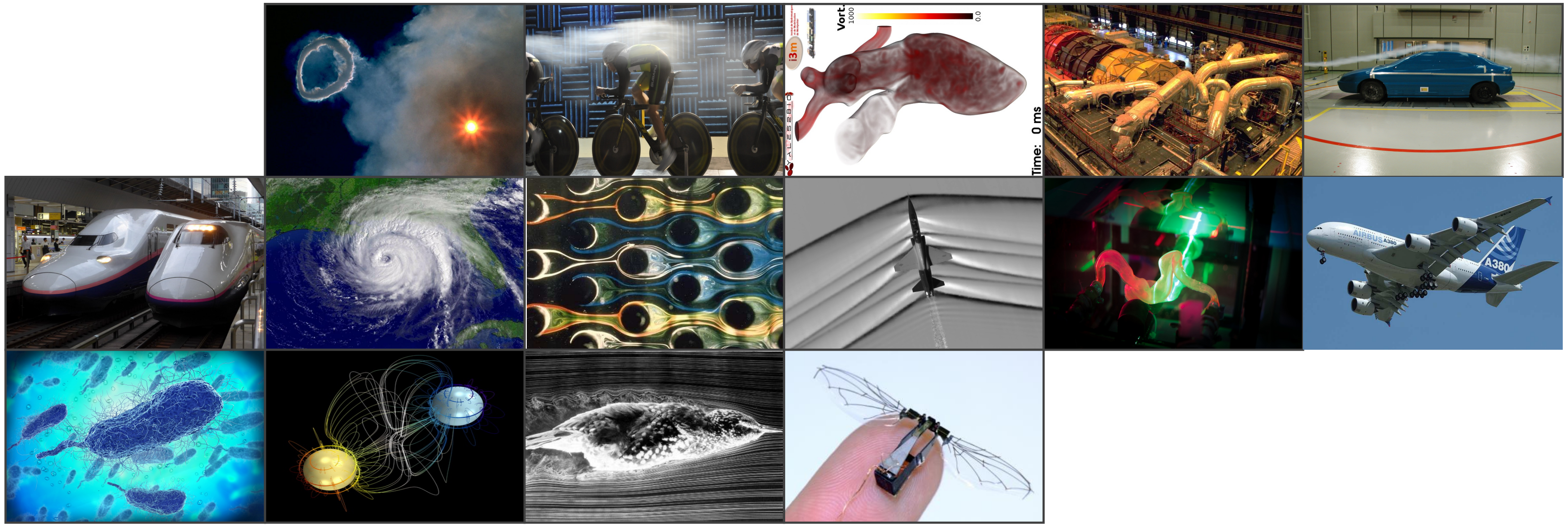
1 - Kinematic properties, fundamental laws, inviscid model



1 - Kinematic properties, fundamental laws, inviscid model

Description of fluid motion

- Microscopic and macroscopic scales
- Macroscopic quantities
- Material domain
- Fluid particle
- Streamlines and pathlines
- Reynolds transport theorem
- Incompressibility condition

Conservation of mass

- Integral and local formulations
- Reformulation of the Reynolds theorem

Forces applied to a domain

- Surface and volume forces
- Stress tensor

Conservation of momentum

- Integral and local formulations

Hydrostatics

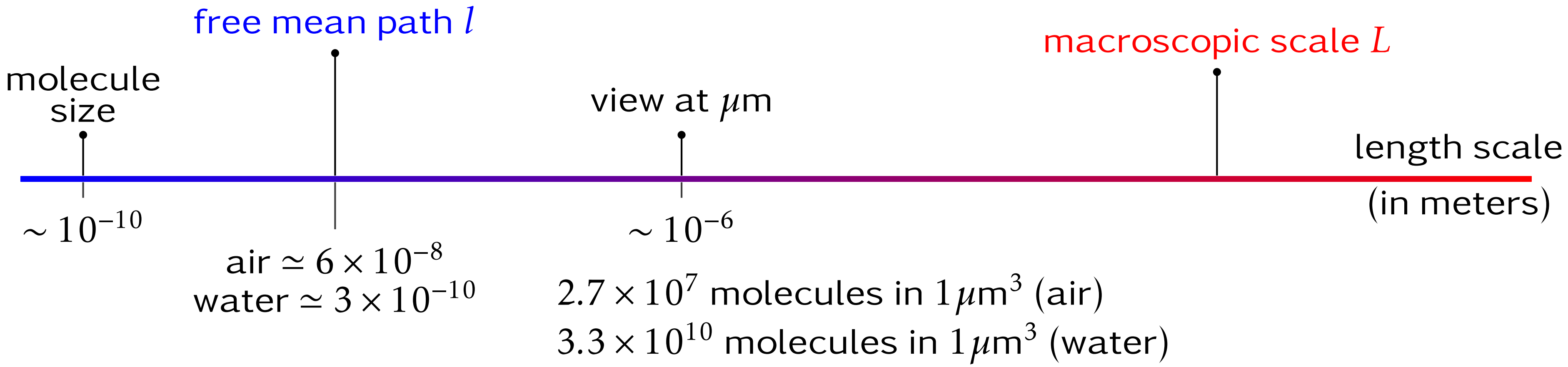
- Hydrostatic equation
- Buoyancy force

Inviscid model

- Euler's equations of motion
- Bernoulli's equation

Key results

Microscopic and macroscopic scales



A gas or liquid flow can thus be regarded as a continuous medium, all the relevant length scales are very large with respect to the microscopic scale l.

The continuous medium is usually characterized by its Knudsen number Kn, Kn $= l/L \ll 1$

Avogadro's number $N_A = 6.022\times10^{23}$ molecules in 1 mole
Normal conditions $T_0 = 0^\circ$ C and $P_0 = 101325$ Pa
For a diatomic molecule, $l = \sqrt{\gamma\pi/2}\,(\nu/c)$

Macroscopic quantities

Let us consider a small fluid volume $\mathcal{V}$ of length size d such as $l \ll d \ll L$. The macroscopic properties are determined by averaging over all the molecules contained in $\mathcal{V}$. Hence,

density $\quad \rho = \dfrac{m}{\mathcal{V}} \qquad (\mathrm{kg/m^3})$

macroscopic velocity $\quad \boldsymbol{U} = \dfrac{\sum_\alpha m_\alpha \boldsymbol{u}_\alpha}{m} \qquad (\mathrm{m.s^{-1}})$

where m_α and $\boldsymbol{u}_\alpha$ are the mass and velocity of species α
$m = \sum_\alpha m_\alpha$ is the total mass of the volume $\mathcal{V}$

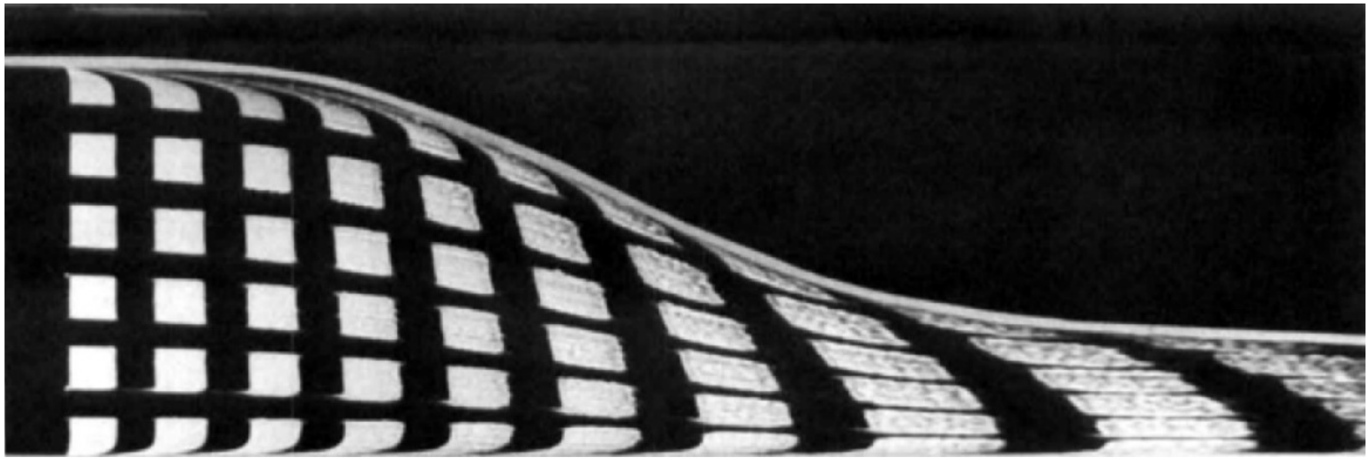
These variables are well defined since the number of molecules is huge : the macroscopic description used in fluid dynamics is based on this assumption.

Macroscopic quantities are functions of time t and position $\boldsymbol{x}$, and the macroscopic length scale L is associated with the space variations of these quantities.

Material domain

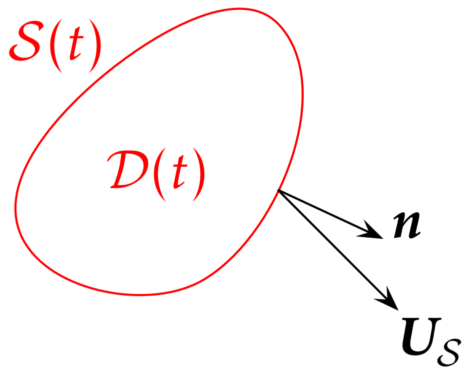
The flow is here described by its velocity $\boldsymbol{U}(\boldsymbol{x},t)$, its density $\rho(\boldsymbol{x},t)$ and an additional quantity $\varphi(\boldsymbol{x},t)$ of interest (temperature for instance)

A material point $\boldsymbol{x}(t)$ is defined as a point moving with the fluid, that is $d\boldsymbol{x}/dt \equiv \boldsymbol{U}$ A material domain $\mathcal{D}$ is a set of material points moving with the fluid (therefore $\mathcal{D}$ contains the same fluid particles during fluid motion)



Time-streak marker technique (hydrogen bubbles are used as tracers) applied to the steady flow in a contraction (Schraub *et al.*, 1965)

Material domain (cont.)



Given domain $\mathcal{D}$ bounded by the surface $\mathcal{S}(t)$, and moving at an arbitrary velocity $\boldsymbol{U}_{\mathcal{S}}$

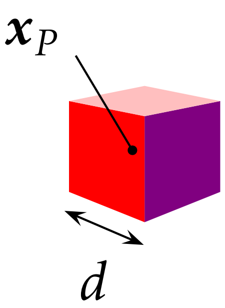
$\boldsymbol{U}_{\mathcal{S}}$ is the local velocity of a point on $\mathcal{S}(t)$ (this velocity is only defined on $\mathcal{S}$)

$\mathcal{D}$ is a material domain if $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$

$\boldsymbol{n}$ is the unit normal vector pointing outward from the surface S

Fluid particle

A fluid particle is the elementary material domain used for discretization done at the macroscopic scale for the observer



The size d of a fluid particle is by definition small $(d \to 0)$ wrt the macroscopic scale L

The fluid particle is usually associated with a material point $\boldsymbol{x}_P$, and the fluid properties are assumed to be constant in its volume $\mathcal{V}$, meaning that $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x}_P, t)$ and $\rho = \rho(\boldsymbol{x}_P, t)$ for instance.

A material domain $\mathcal{D}$ is by definition a set of fluid particles

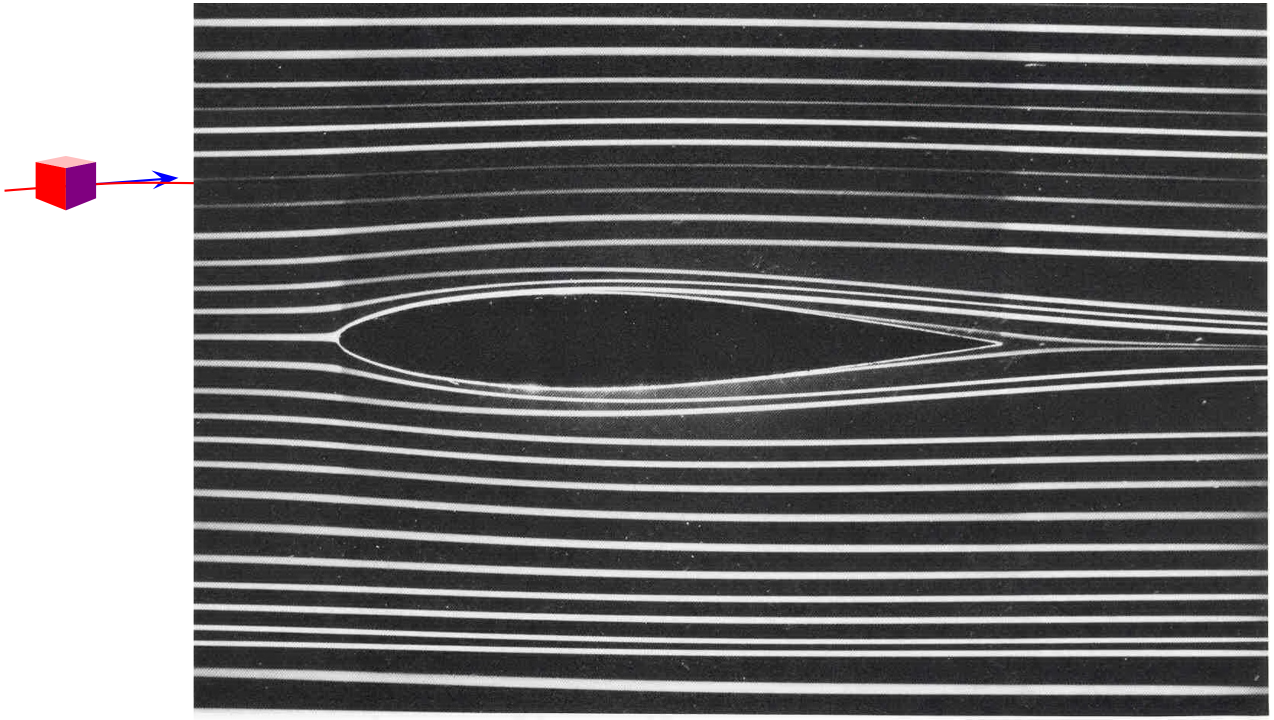
The mass M of a domain $\mathcal{D}$ is defined by the volume integral

$$M = \int_{\mathcal{D}} \rho \, d\mathcal{v}$$

and the mass of a fluid particle is simply $m = \rho(\boldsymbol{x}_P, t)\mathcal{V}$

Streamlines and pathlines

The streamlines of a fluid flow are the (imaginary) curves tangential to the instantaneous velocity at a given time t and at every point, determined by $\boldsymbol{U} \times d\boldsymbol{x} = 0$. That corresponds to an Eulerian description of the flow, where $\boldsymbol{x}$ and t are independant variables.



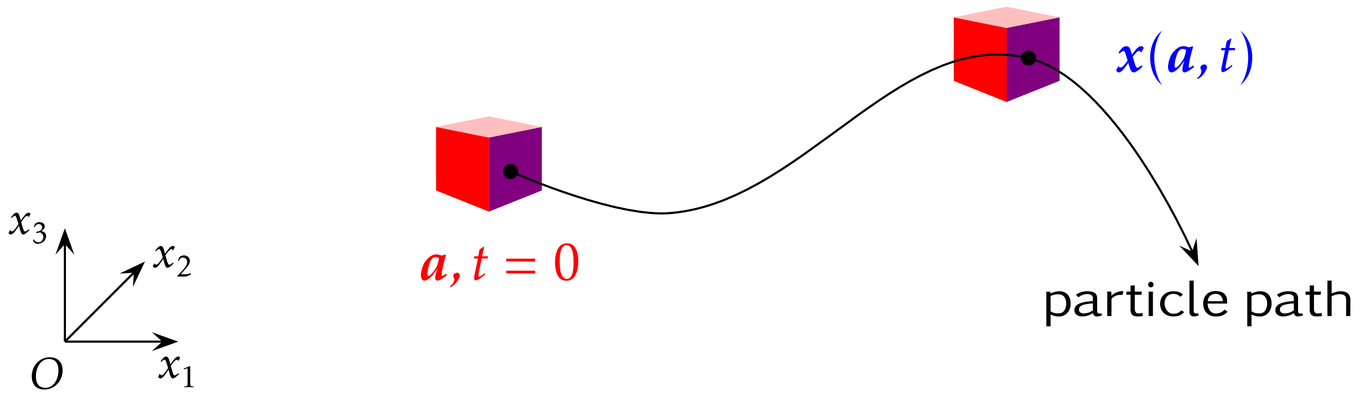
Flow past an airfoil NACA 64A015 (water tunnel, $Re_c = 7 \times 10^3$, zero incidence). The flow is laminar, and appears to be unseparated (small separation region near the trailing edge)

Werlé (1974)
in Van Dyke (1982, Fig. 23)

Streamlines and pathlines (cont.)

In the Lagrangian description, attention is focused on a particular fluid particle : $\boldsymbol{a}$ and t are independant variables, where $\boldsymbol{a}$ is its initial position, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$.

The trajectory (or particle path) is provided by $d\boldsymbol{x}/dt = \boldsymbol{U}(\boldsymbol{x}, t)$ with $\boldsymbol{x} = \boldsymbol{a}$ at $t = 0$ as initial condition.



There is a particularly interesting case to consider when the flow is steady, that is $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x})$. The fluid particles then follow streamlines, which coincide with pathlines.

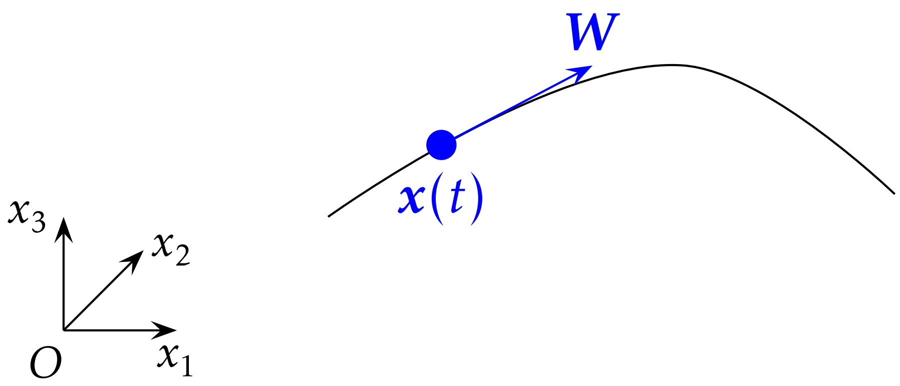
Material derivative

By writing the total differential of an arbitrary flow variable $\varphi = \varphi(\boldsymbol{x}, t)$, and making use of the summation convention

$$d\varphi = \frac{\partial \varphi}{\partial t} dt + \frac{\partial \varphi}{\partial x_i} dx_i = \frac{\partial \varphi}{\partial t} dt + d\boldsymbol{x} \cdot \nabla \varphi$$

For a point $\boldsymbol{x}(t)$ of a given motion, thus satisfying

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{W} \quad \Longrightarrow \quad \frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \boldsymbol{W} \cdot \nabla \varphi$$



Let us introduce the material derivative D/Dt for a material point $\boldsymbol{x} = \boldsymbol{x}(t)$, with thus $d\boldsymbol{x}/dt = \boldsymbol{U}$. By definition,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}$$

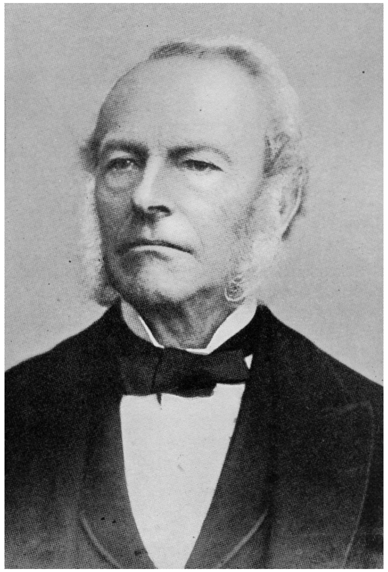
Material derivative (cont.)

The time variation of a flow variable $\varphi = \varphi(\boldsymbol{x}(t), t)$ in following a material point (*i.e.* along the flow) is

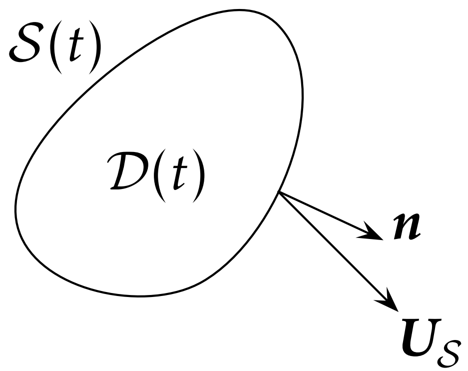
$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi = \underbrace{\frac{\partial \varphi}{\partial t}}_{\substack{\text{local rate of} \\ \text{change of } \varphi \text{ at } \boldsymbol{x}}} + \underbrace{U_i \frac{\partial \varphi}{\partial x_i}}_{\substack{\text{convective rate of change} \\ \text{of } \varphi \text{ induced by } \boldsymbol{U}}}$$

The notation D/Dt for the convective or material derivative was introduced by Stokes (1845)

Sir George Gabriel Stokes (1819-1903)



Reynolds transport theorem



Given domain $\mathcal{D}(t)$ bounded by the surface $\mathcal{S}(t)$ moving at an arbitrary velocity $\boldsymbol{U}_{\mathcal{S}}$

$\boldsymbol{U}_{\mathcal{S}}$ is the local velocity of the control surface $\mathcal{S}(t)$

$\boldsymbol{n}$ is the outward-pointing unit normal vector

Let us consider the integral quantity of a local quantity per unit volume $\varphi(\boldsymbol{x},t)$ over the domain $\mathcal{D}(t)$

$$\Phi(t) = \int_{\mathcal{D}} \varphi(\boldsymbol{x},t)\, dv$$

The Reynolds theorem states that

$$\frac{d\Phi}{dt} = \underbrace{\int_{\mathcal{D}(t)} \frac{\partial \varphi}{\partial t}\, dv}_{\text{time variation of } \varphi} + \underbrace{\int_{\mathcal{S}(t)} \varphi \boldsymbol{U}_{\mathcal{S}} \cdot \boldsymbol{n}\, ds}_{\text{motion of } \mathcal{S}} \qquad \begin{cases} \boldsymbol{U}_{\mathcal{S}} = \boldsymbol{0} & \text{fixed domain} \\ \boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U} & \text{material domain} \end{cases} \qquad (1)$$

Reynolds theorem (cont.)

Leibniz's rule

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(\xi,t)d\xi = \int_{a(t)}^{b(t)} \frac{\partial f(\xi,t)}{\partial t} d\xi + \frac{db}{dt} f(b,t) - \frac{da}{dt} f(a,t)$$

Gottfried Wilhelm von Leibniz
(1646 - 1716)

The Reynolds (transport) theorem is the 3-D generalization of the well-known Leibniz's rule for differentiating a one-dimensional integral with variable limits. A proof can be found in any textbook.

Incompressibility condition

Reynolds theorem applied to a material domain ($\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$) with $\varphi \equiv 1$

$$\int_{\mathcal{D}} \varphi \, dv = V(t) \qquad \text{(volume of the domain } \mathcal{D}\text{)}$$

$$\frac{dV}{dt} = 0 + \int_{\mathcal{S}} \boldsymbol{U} \cdot \boldsymbol{n} \, ds = \int_{\mathcal{D}} \nabla \cdot \boldsymbol{U} \, dv \qquad \text{(by using the divergence theorem)}$$

For the case of a fluid particle, where V is reduced to $\mathcal{V}$ in this limit case,

$$\frac{d\mathcal{V}}{dt} = \int_{\mathcal{D}} \nabla \cdot \boldsymbol{U} \, dv = (\nabla \cdot \boldsymbol{U})\mathcal{V} \qquad \Longrightarrow \qquad \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \nabla \cdot \boldsymbol{U}$$

$$\begin{cases} \nabla \cdot \boldsymbol{U} < 0 & \text{contraction} \\ \nabla \cdot \boldsymbol{U} > 0 & \text{expansion} \end{cases} \quad \text{of the fluid particle}$$

The relative rate of volume growth of a fluid particle $\nabla \cdot \boldsymbol{U}$ was derived by (Euler, 1755), and leads to the incompressibility condition $\boxed{\nabla \cdot \boldsymbol{U} = 0}$

$\rightsquigarrow$ the volume of the fluid particle remains constant during its motion

Fundamental principles

- conservation of mass
- conservation of momentum (Newton's second law)
- conservation of energy
  will be introduced in Chapter 7 (in the framework of compressible flows)

- **For a material domain $\mathcal{D}$**

$$M = \int_{\mathcal{D}} \rho \, dv \quad \text{and} \quad \frac{dM}{dt} = \frac{d}{dt}\int_{\mathcal{D}} \rho \, dv = 0$$

($\mathcal{D}$ always contains the same fluid particles)

Application of the Reynolds theorem, Eq. (1), with $\varphi = \rho$ and $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$

$$\frac{dM}{dt} = \frac{d}{dt}\int_{\mathcal{D}} \rho \, dv = \int_{\mathcal{D}} \frac{\partial \rho}{\partial t} dv + \int_{\mathcal{S}} \rho \boldsymbol{U} \cdot \boldsymbol{n} \, ds = \int_{\mathcal{D}} \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) \right\} dv$$

and by considering the limit for a fluid particle ($\mathcal{D} \rightarrow \mathcal{V}$), a local expression for the conservation of mass is thus obtained

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0} \qquad (2)$$

(2) also reads $\displaystyle \frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} = 0$

Alternative forms

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = \frac{\partial \rho}{\partial t} + \boldsymbol{U} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{U} = \boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{U} = 0} \qquad (3)$$

When the flow is incompressible, $\nabla \cdot \boldsymbol{U} = 0$:
the conservation of mass is then expressed as

$$\frac{D\rho}{Dt} = 0$$

In other words, density of a fluid particle remains constant, which can be simply interpreted by noting that

$$\rho = \frac{m}{\mathcal{V}} = \frac{\text{cst}}{\text{cst}} \qquad \begin{matrix}\text{(mass conservation)} \\ \text{(incompressibility)}\end{matrix}$$

Incompressibility condition (revisited)

In the general case, the mass conservation can be written

$$\nabla \cdot \boldsymbol{U} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

and it can be shown that

$$\frac{\Delta\rho}{\rho} \sim M_a^2 \quad \text{(see Chap. 7)} \qquad M_a \equiv \frac{U}{c} \quad \text{Mach number}$$

where c denotes the speed of sound. The incompressiblity condition is satisfied for low Mach number flows, that is $M_a \leq 0.3$ in practice.

- **Reformulation of the Reynolds theorem** to take into account the mass conservation equation

  The Reynolds theorem is now applied for the variable $\varphi = \rho\chi$ and an arbitrary domain $\mathcal{D}$. From Eq. (1)

$$\frac{d}{dt}\int_{\mathcal{D}} \rho\chi \, dv = \int_{\mathcal{D}} \frac{\partial(\rho\chi)}{\partial t} dv + \int_{\mathcal{S}} \rho\chi \boldsymbol{U}_{\mathcal{S}} \cdot \boldsymbol{n} \, ds$$

  After some algebra

$$\frac{\partial(\rho\chi)}{\partial t} = \rho\frac{\partial\chi}{\partial t} + \chi\frac{\partial\rho}{\partial t} = \rho\frac{\partial\chi}{\partial t} - \chi\nabla\cdot(\rho\boldsymbol{U}) = \underbrace{\rho\frac{\partial\chi}{\partial t} + \rho\boldsymbol{U}\cdot\nabla\chi}_{=\rho D\chi/Dt} - \nabla\cdot(\rho\chi\boldsymbol{U})$$

  Alternative form of the Reynolds theorem

$$\boxed{\frac{d}{dt}\int_{\mathcal{D}} \rho\chi \, dv = \int_{\mathcal{D}} \rho\frac{D\chi}{Dt} dv + \int_{\mathcal{S}} \rho\chi(\boldsymbol{U}_{\mathcal{S}} - \boldsymbol{U})\cdot\boldsymbol{n} \, ds} \qquad (4)$$

Reformulation of the Reynolds theorem (cont.)

For a material domain, $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$,

$$\frac{d}{dt}\int_{\mathcal{D}} \rho\chi \, dv = \int_{\mathcal{D}} \rho \frac{D\chi}{Dt} \, dv$$

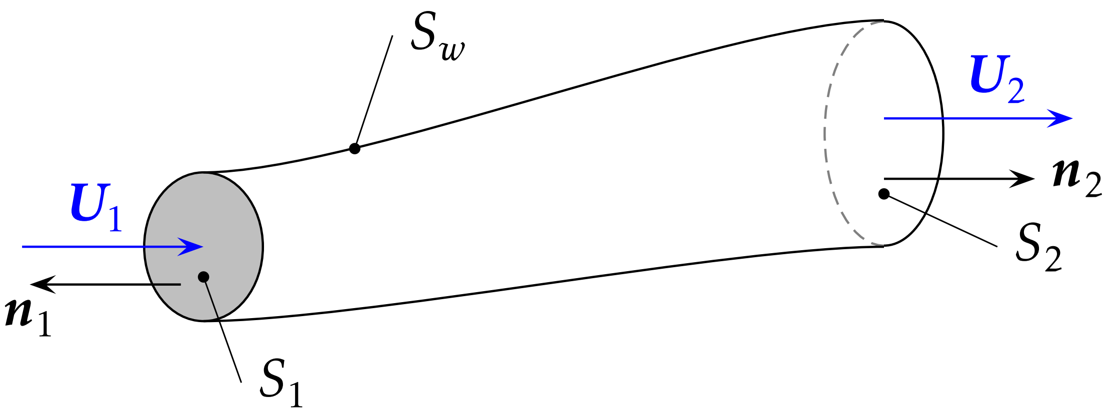
For a fixed domain, $\boldsymbol{U}_{\mathcal{S}} = 0$,

$$\frac{d}{dt}\int_{\mathcal{D}} \rho\chi \, dv = \int_{\mathcal{D}} \rho \frac{D\chi}{Dt} \, dv - \int_{\mathcal{S}} \rho\chi \boldsymbol{U} \cdot \boldsymbol{n} \, ds$$

In Eq. (4), $\boldsymbol{U} - \boldsymbol{U}_{\mathcal{S}}$ is the relative velocity of fluid crossing the control surface $\mathcal{S}$ for an observer attached to $\mathcal{D}$

A famous result can be already demonstrated

Conservation of mass for an unsteady incompressible flow inside a given duct



$$\nabla \cdot \boldsymbol{U} = 0 \Longrightarrow \int_{\mathcal{S}} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S} = 0$$

with $\mathcal{S} = S_1 \cup S_2 \cup S_w$

$$\int_{S_1} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S} + \int_{S_2} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S} + \underbrace{\int_{S_w} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S}}_{=0 \, (\text{no leaks, } \boldsymbol{U} \cdot \boldsymbol{n} = 0)} = 0$$

If the velocity is uniform at the inlet and outlet (1-D model), $Q_v = S_1 U_1 = S_2 U_2$

The volumetric flow rate Q_v ($m^3.s^{-1}$) is constant : the bulk velocity U_d decreases as the cross-sectional area S increases (Leonardo da Vinci, 1502)

$$U_d = \frac{1}{S} \int_S \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S}$$

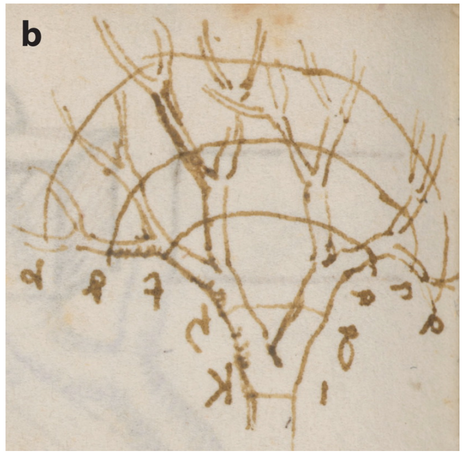
Contribution of Leonardo da Vinci (1452-1519) to fluid mechanics

"Water that flows through a pipe which is empty and fills first the whole of its flat part, will fill up all the other parts, straight and oblique, and moving with equal speed."

An interesting analogy related to the conservation of volume, which Leonardo applied between different natural world systems, is that of the flow through branches of a tree and through rivers or closed conduits.

"each year when the branches of the plants have exhausted their growth, they comprise together as much as the size of their trunk, and in each degree of their [branch] growth, you will find the size of said trunk as in .i.K. .g.h. .e.f. .CD. .a.b. All of them will be the same the tree not being damaged; otherwise the rule does not fail."

From Marusic & Broomhall, 2021, Leonardo da Vinci and Fluid Mechanics, *Ann. Rev. Fluid Mech.*, **53**



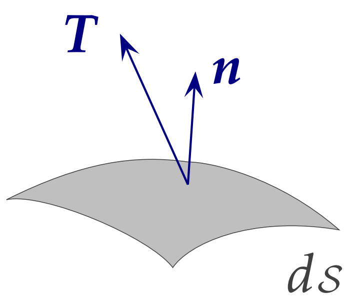
Introduction

The total force applied to a domain $\mathcal{D}$ can be split into a surface contribution $\boldsymbol{F}_s$ and a volume contribution $\boldsymbol{F}_v$, $\boldsymbol{F} = \boldsymbol{F}_s + \boldsymbol{F}_v$

$\boldsymbol{F}_v$ represents a possible body force acting in bulk, long-range electromagnetic and gravitational forces. For the gravitational attraction,

$$\boldsymbol{F}_v = \int_{\mathcal{D}} \rho \boldsymbol{g} \, d\mathcal{V}$$

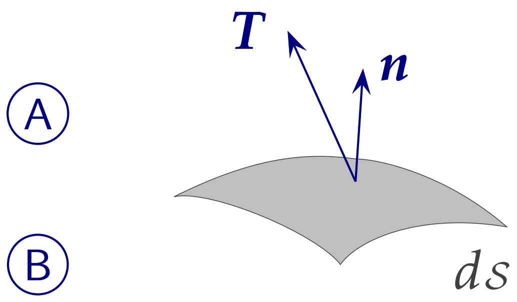
Surface forces $\boldsymbol{F}_s$ are associated with the microscopic interactions across the surface $\mathcal{S}$



$\boldsymbol{T}(\boldsymbol{x}, t, \boldsymbol{n})$ is the force per unit surface, *i.e.* the stress vector (in Pascal, Pa)

$$\boldsymbol{F}_s = \int_{\mathcal{S}} \boldsymbol{T} \, d\mathcal{S}$$

Surface force



$$dF_s = T(x,t,n)\,ds$$

is the force exerted by matter A to matter B across ds

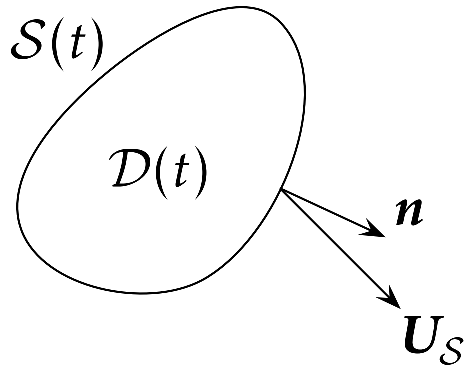
Action-reaction principle (third Newton's law)

$$T(x,t,-n) = -T(x,t,n)$$

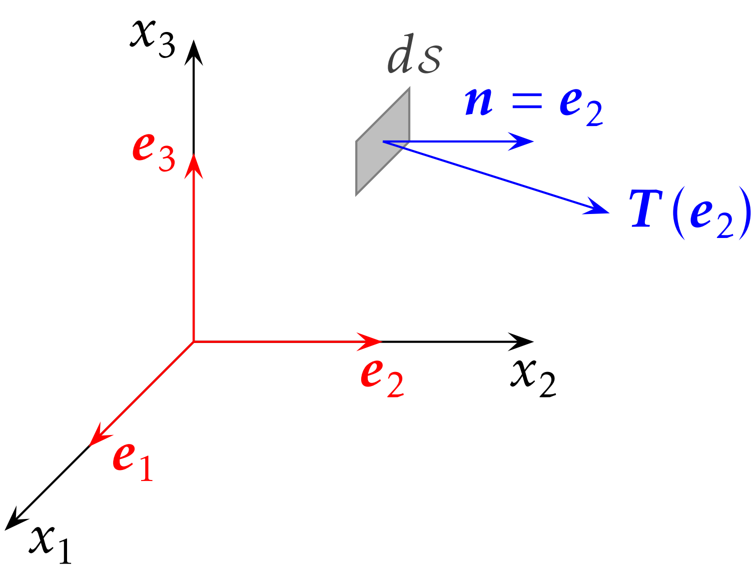
When only the pressure $p(x,t)$ contributes to the surface force, that is for a fluid at rest or for inviscid flow, $T = -p n$

$$F_s = -\int_{\mathcal{S}} p n \, ds \qquad \text{(with pressure } p \text{ in Pa)}$$

Pressure force is isotropic, *i.e.* direction-independent (no priviledged direction)



Stress tensor



$T(e_i) \equiv T(x, t, e_i)$

e_i unit vector along x_i

$e_2 = (0, 1, 0)$

$$\overline{\overline{\sigma}} \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad T(e_2) = \begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix} = \overline{\overline{\sigma}} \cdot e_2$$

Cauchy's stress theorem states that $T(x, t, n) = \overline{\overline{\sigma}} \cdot n$, *i.e.* a linear (and explicit) dependence of T wrt the normal vector n of ds. The stress tensor component σ_{ij} represents the i-th component of T along the direction j, that is $T_i(e_j)$

The stress tensor $\overline{\overline{\sigma}}$ is also necessarily symmetric, that is $\sigma_{ij} = \sigma_{ji}$

(Proofs are provided in Appendix)

Stress tensor (cont.)

Auguste (Louis) Cauchy
(1789-1857)

Expression of the surface force $\boldsymbol{F}_s$

$$\boldsymbol{F}_s = \int_{\mathcal{S}} \boldsymbol{T}\, d\mathcal{S} = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}\, d\mathcal{S} = \underbrace{\int_{\mathcal{D}} \nabla \cdot \overline{\overline{\boldsymbol{\sigma}}}\, d\mathcal{V}}_{\text{divergence theorem}}$$

When only the pressure contributes to $\boldsymbol{F}_s$ (inviscid flow),
$\boldsymbol{T} = -p\boldsymbol{n}$, $\boldsymbol{n}$ is an eigenvector of $\overline{\overline{\boldsymbol{\sigma}}}$ and thus $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$

$$\boldsymbol{F}_s = -\int_{\mathcal{S}} p\boldsymbol{n}\, d\mathcal{S} = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}\, d\mathcal{S}$$

$$\overline{\overline{\boldsymbol{\sigma}}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

$$\boldsymbol{T}_1 = -p\boldsymbol{e}_1 \qquad \boldsymbol{T}_2 = -p\boldsymbol{e}_2 \qquad \boldsymbol{T}_3 = -p\boldsymbol{e}_3$$

Equation of motion

Fundamental principle of dynamics applied to a fluid particle of mass $\rho\mathcal{V}$

$$\rho\mathcal{V}\frac{D\boldsymbol{U}}{Dt} = \underbrace{\int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n}\,d\mathcal{S} + \int_{\mathcal{D}} \rho\boldsymbol{g}\,d\mathcal{v}}_{\boldsymbol{F}_s + \boldsymbol{F}_v = \boldsymbol{F}}$$

Acceleration (material derivative of $\boldsymbol{U}$)

$$\left.\frac{D\boldsymbol{U}}{Dt}\right|_i = \frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + U_{\underline{j}}\frac{\partial U_i}{\partial x_{\underline{j}}}$$

$$\int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n}\,d\mathcal{S} = \int_{\mathcal{D}} \nabla\cdot\overline{\overline{\boldsymbol{\sigma}}}\,d\mathcal{v} = \mathcal{V}\,\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}}$$

$$\int_{\mathcal{D}} \rho\boldsymbol{g}\,d\mathcal{v} = \mathcal{V}\,\rho\boldsymbol{g}$$

Conservation of momentum

$$\boxed{\rho\frac{D\boldsymbol{U}}{Dt} = \nabla\cdot\overline{\overline{\boldsymbol{\sigma}}} + \rho\boldsymbol{g}} \qquad (5)$$

Integral equation of motion

Using the Reynolds transport theorem (4), the local formulation (5) can also be written as

$$\boxed{\frac{d}{dt}\int_{\mathcal{D}} \rho \boldsymbol{U} \, d\mathcal{V} = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n} \, d\mathcal{S} + \int_{\mathcal{D}} \rho \boldsymbol{g} \, d\mathcal{V} + \int_{\mathcal{S}} \rho \boldsymbol{U} (\boldsymbol{U}_{\mathcal{S}} - \boldsymbol{U}) \cdot \boldsymbol{n} \, d\mathcal{S}} \qquad (6)$$

(fixed domain $\boldsymbol{U}_{\mathcal{S}} = 0$; material domain $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$)

Focus on two particular cases

- Hydrostatics (and Archimedes′ principle)

  Equilibrium of a floating or submerged body : fluid is at rest ($\boldsymbol{U} = 0$), and pressure is the only force exerted by the fluid to the body

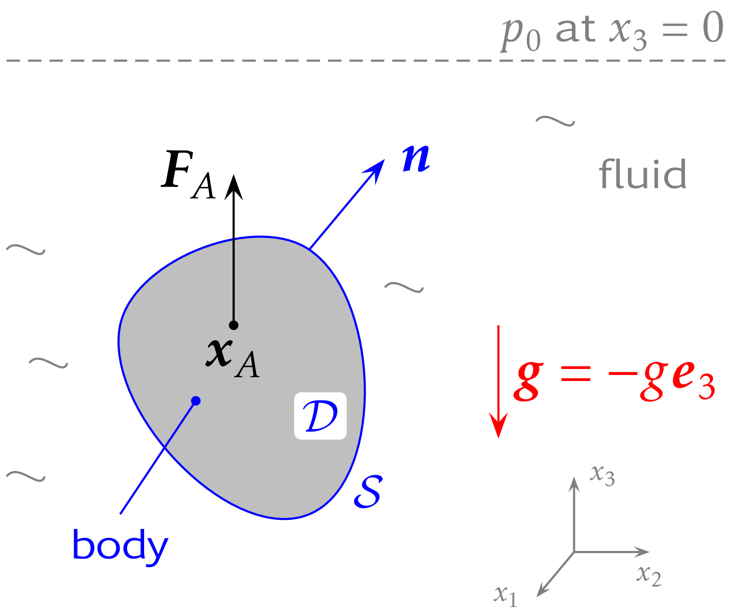
- Inviscid model

  All fluids are viscous, but viscous effects can be neglected in some cases (discussed later in Chapter 4). An inviscid fluid model can then be introduced : the viscosity is zero and there is no internal friction. The only force exerted on a fluid particle is again associated with pressure.

  For an inviscid flow model, $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$

Hydrostatic balance

Immersed body in a homogeneous fluid ($\rho = \text{cst}$) at rest ($\boldsymbol{U} = \boldsymbol{0}$), $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$



Conservation of momentum, $-\nabla p + \rho \boldsymbol{g} = 0$

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial x_2} = 0 \quad \frac{\partial p}{\partial x_3} = -\rho g \quad \Longrightarrow \quad p = p(x_3)$$

By integration, $p = p_0 - \rho g x_3$ ($x_3 < 0$ here)

For water, $\rho \simeq 10^3$ kg.m^{-3}, the pressure increases about one atmosphere for every 10 meters of water depth.

Elementary force $d\boldsymbol{F}_A$ exerted by the fluid on the body ($\boldsymbol{n}$ pointing into the fluid domain), $d\boldsymbol{F}_A = \boldsymbol{T}\, ds = \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}\, ds = -p\overline{\overline{\boldsymbol{I}}} \cdot \boldsymbol{n}\, ds$

$$\boldsymbol{F}_A = \int_{\mathcal{S}} -p\overline{\overline{\boldsymbol{I}}} \cdot \boldsymbol{n}\, ds = \int_{\mathcal{D}} -\nabla p\, dv = -\int_{\mathcal{D}} \rho \boldsymbol{g}\, dv = -\boldsymbol{g}\int_{\mathcal{D}} \rho\, dv = -\boldsymbol{g} M_f$$

Archimedes buoyancy force

Archimedes principle : the upward buoyant force $\boldsymbol{F}_A$ that is exerted on a body immersed in a fluid is equal to the weight of the fluid M_f that the body displaces.

What is the point of application of this force $\boldsymbol{F}_A$?
The torque $\boldsymbol{C}$ applied to the body reads

$$\boldsymbol{C} = \int_{\mathcal{S}} \boldsymbol{x} \times (-p\boldsymbol{n})\, ds = -\int_{\mathcal{D}} \boldsymbol{x} \times \nabla p\, dv = -\int_{\mathcal{D}} \rho \boldsymbol{x}\, dv \times \boldsymbol{g}$$

(refer to slide 11 for a demonstration)

By choosing the center of mass $\boldsymbol{x}_A$ of the displaced fluid defined by

$$M_f \boldsymbol{x}_A \equiv \int_{\mathcal{D}} \rho \boldsymbol{x}\, dv,$$

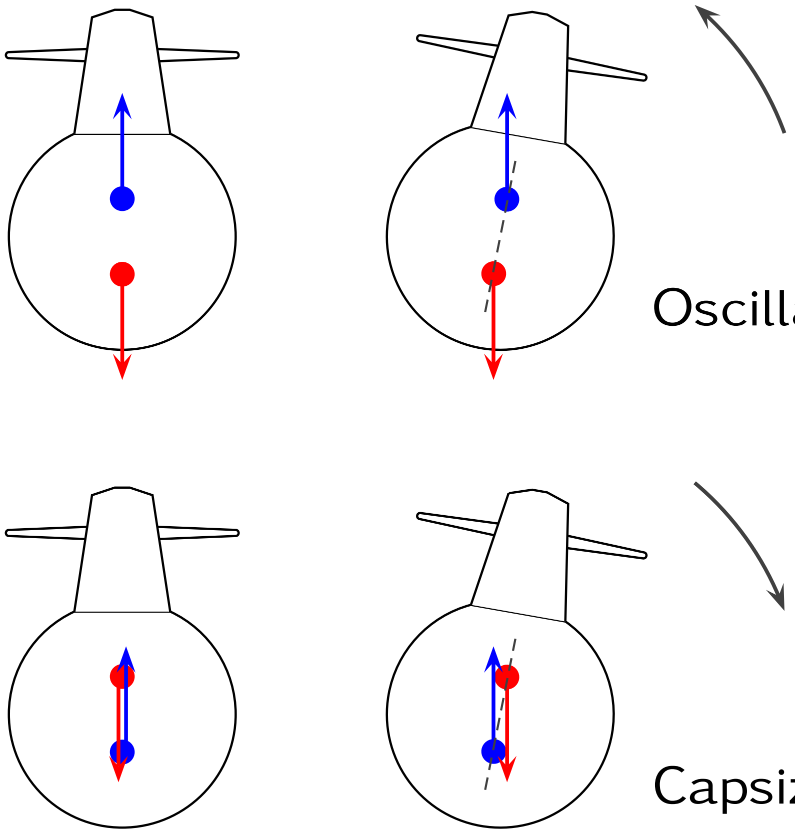
as the reference point to calculate the torque $\boldsymbol{C}$, that is $\boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{x}_A$, one has

$$\int_{\mathcal{D}} \rho(\boldsymbol{x} - \boldsymbol{x}_A)\, dv = 0 \qquad \text{and thus } \boldsymbol{C} = 0 \text{ (equilibrium)}$$

The buoyancy force $\boldsymbol{F}_A$ applies to the (fictif) center of mass $\boldsymbol{x}_A$ of the displaced fluid (the so-called center of buoyancy of the immersed body)

Archimedes boyancy force (cont.)

Stability of a (fully) immersed submarine



Oscillations (rolling)

Capsizing!

Center of mass $\boldsymbol{x}_A$ of the displaced fluid

$$\int_{\mathcal{D}_{\rm im}} \underline{\rho}(\boldsymbol{x}-\boldsymbol{x}_A)\,dv = 0$$

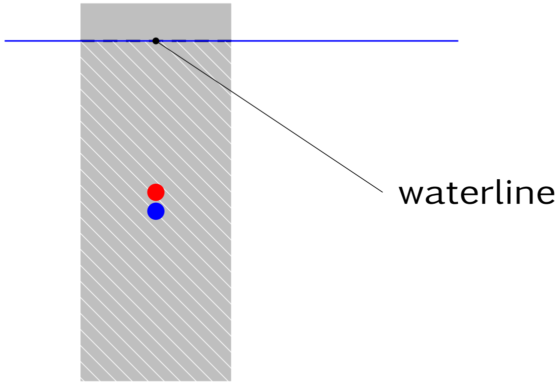
Center of gravity $\boldsymbol{x}_G$ of the body

$$\int_{\mathcal{D}} \underline{\rho_s}(\boldsymbol{x}-\boldsymbol{x}_G)\,dv = 0$$

The center of gravity $\boldsymbol{x}_G$ • ($\boldsymbol{F} = M_s\boldsymbol{g}$) of the body must be below the center of mass $\boldsymbol{x}_A$ • (Archimedes force $\boldsymbol{F}_A$)

Tip of the iceberg

salt water, $\rho_s \simeq 1025$ kg.m^{-3}
fresh water ice, $\rho_f \simeq 920$ kg.m^{-3}
$\mathcal{V}$ total volume and $\mathcal{V}_i$ immersed volume



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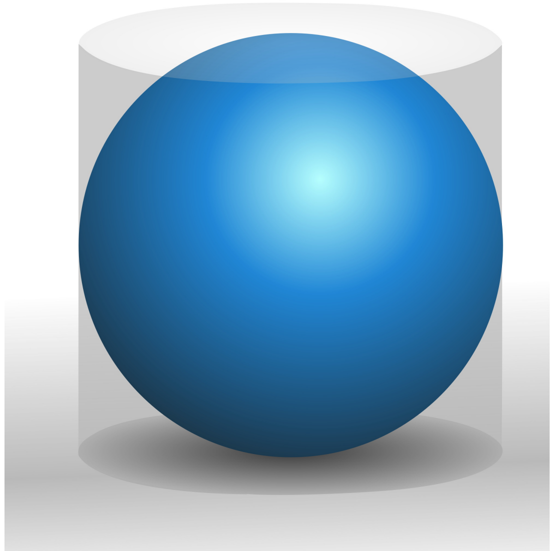
$$\rho_f g \mathcal{V} = \rho_s g \mathcal{V}_i \quad \Longrightarrow \quad \frac{\mathcal{V}_i}{\mathcal{V}} = \frac{\rho_f}{\rho_s} \simeq 0.9$$

Draw your own iceberg here !

See Pollack, 2019, *Phys. Today*, **72**(12), 70-71 (available on moodle) to continue the discussion regarding stability ...

- **Archimedes of Syracuse in Sicily (Greek, 287-212 BC)**
  Archimedes is considered one of the greatest scientists of all time, and the greatest mathematician of antiquity.





Every cylinder – whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere – is 3/2 of the sphere in volume

Archimedes' screw (worm screw)

Flamanville Nuclear Power Plant (EPR, 3rd generation nuclear of pressurized water reactor). Pumping station : the two Archimedean screws (15 m length, 1.5 m diameter, 6 tons ; $Q_v = 1500$ m^3/h) ensuring the upward flow of water collected in the pre-discharge structure, before its return to the sea.

(Électricité de France / Julien Goldstein)

Euler's equations of motion

An inviscid flow is a flow for which the fluid viscocity is equal to zero : there is no friction, no viscous effect. The only force to consider is due to pressure, and the stress tensor reads $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$

The flow is then governed by Euler's equations (1757), formulated here for an incompressible flow,

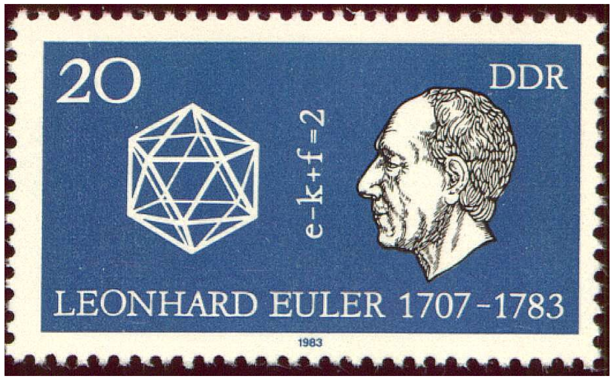
$$\begin{cases} \dfrac{D\boldsymbol{U}}{Dt} = -\dfrac{1}{\rho}\nabla p + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases} \tag{7}$$

In an inviscid model, the flow has a nonzero tangential velocity to only ensure impermeability, corresponding to a slip boundary condition at the wall, $\boldsymbol{U} \cdot \boldsymbol{n} = 0$

For an interface between two fluids,
$\boldsymbol{U}_A \cdot \boldsymbol{n} = \boldsymbol{U}_B \cdot \boldsymbol{n}$

Leonhard Euler (1707-1783)

Leonhard Euler is, after Isaac Newton (1643-1727), the founder of analytical mechanics (two-body problem for instance), and of fluid dynamics – thanks to the differential calculus also introduced by Gottfried Leibniz (1646-1716)



Bernoulli's equation

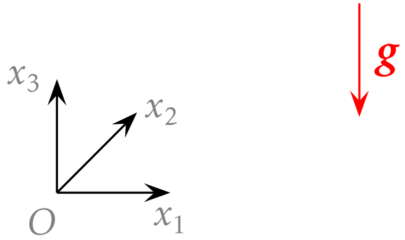
At this step, we can already establish one of the most famous result in fluid dynamics, simple to apply and useful in practice. To do so, the conservation of the kinetic energy is first derived from Euler's equation (7)

$$\boldsymbol{U}\cdot\frac{D\boldsymbol{U}}{Dt} = \boldsymbol{U}\cdot\left(-\frac{1}{\rho}\nabla p + \boldsymbol{g}\right)$$

For a conservative force such as gravity, it is always possible to involve a potential function to express this force†

$\boldsymbol{g} = -\nabla\Psi(\boldsymbol{x}) \qquad g_i = -g\delta_{3i} \qquad \Psi = -\boldsymbol{g}\cdot\boldsymbol{x} = g x_3$

$\boldsymbol{U}\cdot\boldsymbol{g} = -\boldsymbol{U}\cdot\nabla\Psi$



† existence of such a function Ψ requires that $\nabla\times\boldsymbol{g} = 0$

The conservation of the kinetic energy can then be recast as follows

$$\rho\frac{D}{Dt}\left(\frac{U^2}{2}\right) + \boldsymbol{U}\cdot\nabla p + \rho\boldsymbol{U}\cdot\nabla\Psi = 0$$

Bernoulli's equation (cont.)

Under the following assumptions

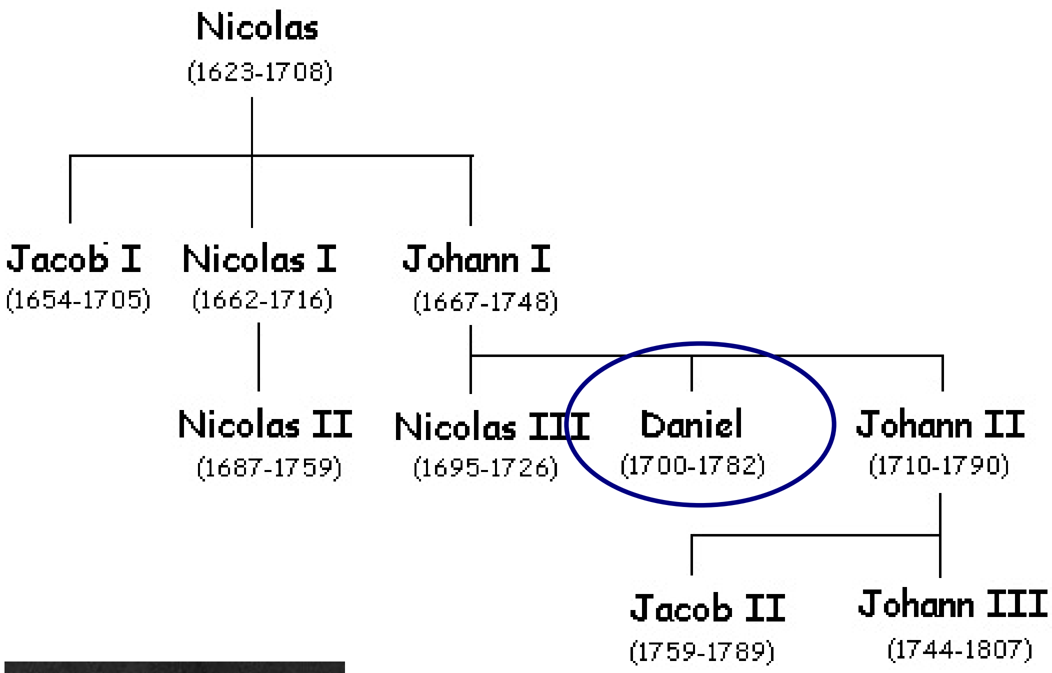
1. inviscid flow $(\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}})$
2. steady flow $(\partial_t = 0)$
3. incompressible flow $(\nabla \cdot \boldsymbol{U} = 0$ and $\rho = \text{cst})$
4. conservative forces $(\boldsymbol{g} = -\nabla\Psi$ here$)$

the previous equation is reduced to $\boldsymbol{U} \cdot \nabla\left(\rho U^2/2 + p + \rho\Psi\right) = 0$, and Bernoulli's Equation is then established

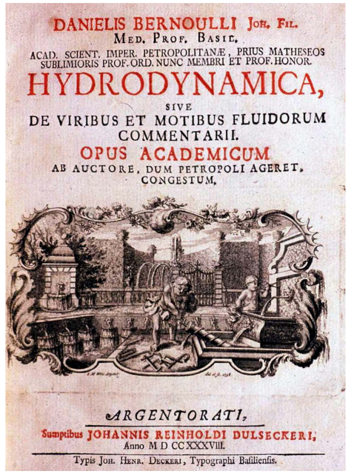
$$\boxed{\mathcal{H} \equiv p + \rho\Psi + \rho\frac{U^2}{2} = \text{cst} \quad \text{along a streamline}} \tag{8}$$

$\mathcal{H}$ takes different values for different streamlines. For a steady flow, streamlines are also pathlines and $\mathcal{H}$ is cst for a fluid particle. Eq. (8) can be interpreted as the conservation of mechanical energy (work done by forces is balanced by the change in kinetic energy, $\mathcal{H} \sim \text{J.m}^{-3}$ energy per unit volume)

The Bernoulli family



Daniel Bernoulli (1700-1782)

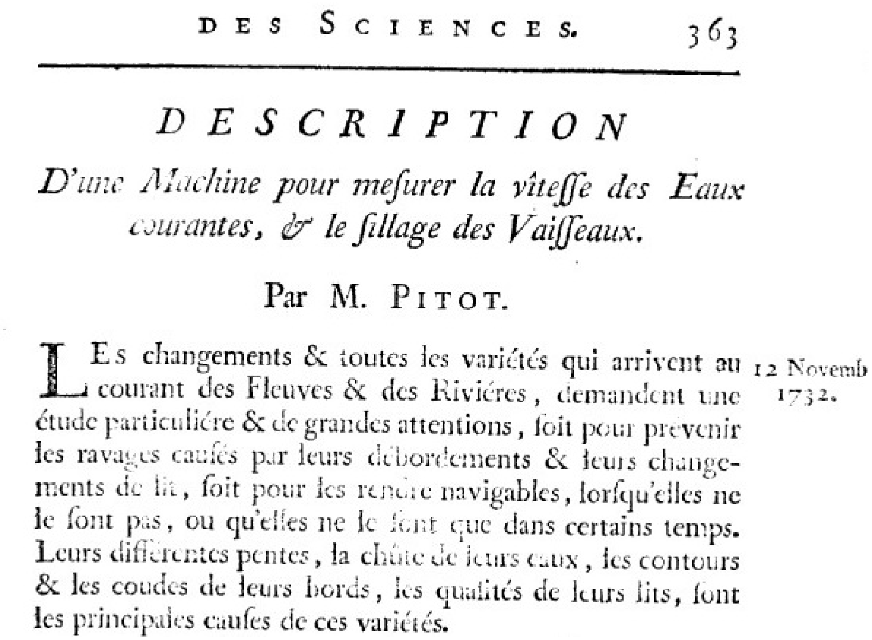


Hydrodynamica (1738)

(the term hydrodynamics was introduced by Bernoulli to merge the two domains of hydrostatics and hydraulics)

Pitot tube (measurement of total pressure)

Henri Pitot (1732) has developed a device for measuring the flow velocity of the Seine river in Paris (Pitot tube combined with a manometer)



DES SCIENCES. 363

DESCRIPTION

D'une Machine pour meſurer la vîteſſe des Eaux courantes, & le ſillage des Vaiſſeaux.

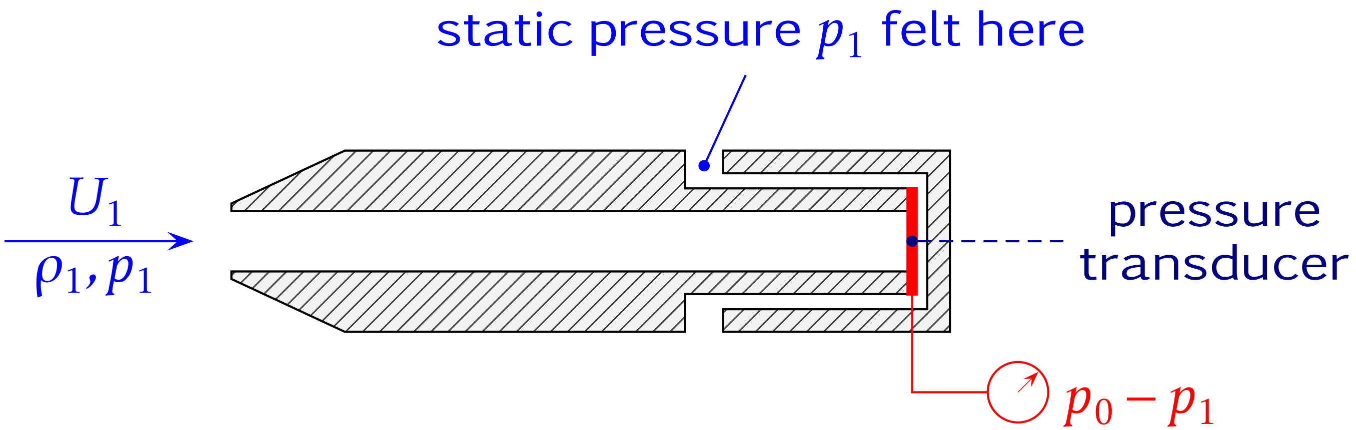
Par M. PITOT.

LES changements & toutes les variétés qui arrivent au courant des Fleuves & des Riviéres, demandent une étude particuliére & de grandes attentions, ſoit pour prévenir les ravages cauſés par leurs débordements & leurs changements de lit, ſoit pour les rendre navigables, lorſqu'elles ne le ſont pas, ou qu'elles ne le ſont que dans certains temps. Leurs différentes pentes, la chûte de leurs eaux, les contours & les coudes de leurs bords, les qualités de leurs lits, ſont les principales cauſes de ces variétés. 12 Novemb 1732.

Pitot tubes are used on aircraft to measure airspeed (along with static ports and angle-of-attack vane probes)

(Airbus A350 XWB)

Prandtl tube (Pitot - static tube : measurement of airspeed)



For an incompressible flow, the difference of pressure felt by the sensor is (application of Bernoulli's equation $p + \rho U^2/2 = \text{cst}$)

$$\begin{cases} \text{Left, total pressure } p_0 = p_1 + \dfrac{1}{2}\rho_1 U_1^2 \\ \text{Right, static pressure } p_1 \end{cases} \Longrightarrow \quad U_1 = \sqrt{\frac{2(p_0 - p_1)}{\rho_1}}$$

The term $q_1 = \frac{1}{2}\rho_1 U_1^2$ is called the dynamic pressure. The total pressure p_0 is the sum of the static pressure and of the dynamic pressure : valid for incompressible flow only ! A compressible version will be given later in the course (see slide 237)

Rake of Pitot-tubes

Formula 1 Pre-Season Testing 2020, Circuit de Barcelona, Williams FW43 driven by George Russell

y-250 vortex

(unknown source ...)

A rake of Pitot-tubes is used to measure the total pressure distribution (over a plane or along a line)

- Concept of fluid particle, material domain and material derivative
  (that is for a material point which satisfies $d\boldsymbol{x}/dt = \boldsymbol{U}$)

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{U} \cdot \nabla = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}$$

- Flow incompressibility $\nabla \cdot \boldsymbol{U} = 0$ (verified for $M_a \leq 0.3$ flows)

- Conservation of mass (continuity equation)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0$$

- Conservation of momentum

$$\rho \frac{D\boldsymbol{U}}{Dt} = \nabla \cdot \overline{\overline{\boldsymbol{\sigma}}} + \rho \boldsymbol{g} \qquad \boldsymbol{T} = \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n} \qquad \text{inviscid flow} : \boldsymbol{T} = -p\boldsymbol{n} \quad \overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$$

- Integral formulations for an arbitrary control volume through the Reynolds theorem Eqs (1), (4) & (6)

  Reynolds′ theorem

$$\frac{d}{dt}\int_{\mathcal{D}} \varphi \, dv = \int_{\mathcal{D}} \frac{\partial \varphi}{\partial t} \, dv + \int_{\mathcal{S}} \varphi \boldsymbol{U}_{\mathcal{S}} \cdot \boldsymbol{n} \, ds$$

$$\boxed{\frac{d}{dt}\int_{\mathcal{D}} \rho\chi \, dv = \int_{\mathcal{D}} \rho \frac{D\chi}{Dt} \, dv + \int_{\mathcal{S}} \rho\chi (\boldsymbol{U}_{\mathcal{S}} - \boldsymbol{U}) \cdot \boldsymbol{n} \, ds}$$

- Inviscid model : hydrostatic balance, Euler′s equations (7) and Bernoulli′s equation (8)

Outline of the course

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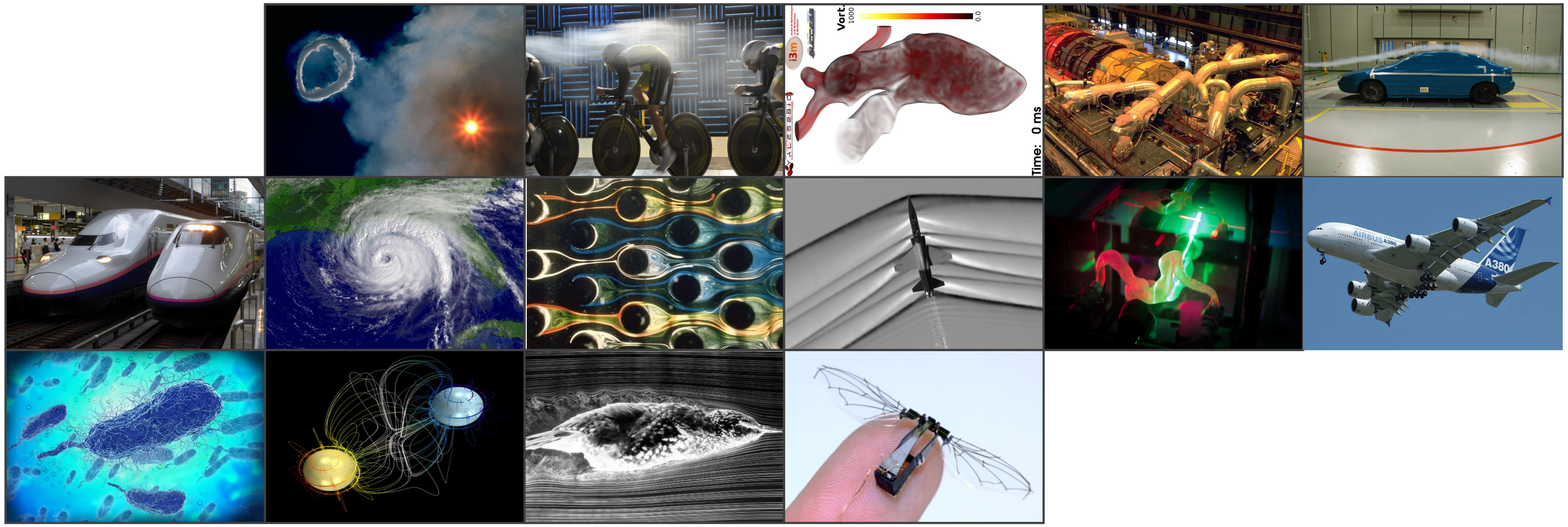
2 - Newtonian viscous fluid flow



2 - Newtonian viscous fluid flow

Viscous flow

- Newtonian law
- Deformation of a fluid particle
- Plane Couette flow
- Lift and drag forces

Equations of fluid dynamics

- General case
- Navier-Stokes equations
- Boundary conditions

Kinetic energy and viscous dissipation

- Integral form
- Viscous dissipation
- Dynamic viscosity
- Plane channel flow

Energetics of continuous-flow system

- Mechanical energy budget
- Energy head form

Key results

Newtonian law

Equation for the conservation of momentum, see Chapter 1, Eq. (5)

$$\rho \frac{D\boldsymbol{U}}{Dt} = \nabla \cdot \overline{\overline{\boldsymbol{\sigma}}} + \rho \boldsymbol{g}$$

where $\overline{\overline{\boldsymbol{\sigma}}}$ is the stress tensor (internal forces inside the fluid)

Fluid at rest (hydrostatics) or inviscid fluid flow,
$\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}}$ (only contribution of the pressure, isotropic stress tensor)

In general, $\boxed{\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}}$ where $\overline{\overline{\boldsymbol{\tau}}}$ is the viscous stress tensor

$$\rho \frac{D\boldsymbol{U}}{Dt} = -\nabla p + \nabla \cdot \overline{\overline{\boldsymbol{\tau}}} + \rho \boldsymbol{g} \qquad \begin{cases} -\nabla p & \text{pressure force} \\ \nabla \cdot \overline{\overline{\boldsymbol{\tau}}} & \text{viscous force exerted on the fluid particle} \end{cases}$$

i-th component of the vector $\nabla \cdot \overline{\overline{\boldsymbol{\sigma}}}$

$$\nabla \cdot \overline{\overline{\boldsymbol{\sigma}}}\Big|_i = \frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial (p\delta_{ij})}{\partial x_j} + \frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \text{(implicit summation over } j\text{)}$$

Newtonian law

Classical constitutive relation for common fluids such as air (for incompressible flow) or water,

$$\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}}$$

μ dynamic (shear) viscosity, intrinsic fluid property

$\overline{\overline{\boldsymbol{D}}}$ deformation tensor (or velocity strain tensor) $\quad D_{ij} \equiv \dfrac{1}{2}\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j}{\partial x_i}\right)$

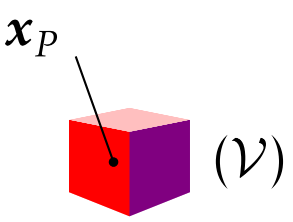
This relation reflects the local and instantaneous link between the viscous stress $\overline{\overline{\boldsymbol{\tau}}}$ and the deformation tensor, *i.e.* the velocity gradients $\overline{\overline{\boldsymbol{D}}}$

$\overline{\overline{\boldsymbol{D}}}$ is directly associated with the deformation of fluid particles

The fluid will be assumed to be a Newtonian fluid for the remainder of this course

Deformation of a fluid particle

Taylor series for the velocity in the vicinity of a fluid particle at $\boldsymbol{x}_P$
(for a given time t)



$$\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x}_P) + \nabla\boldsymbol{U}(\boldsymbol{x}_P)\cdot(\boldsymbol{x}-\boldsymbol{x}_P) + \cdots$$

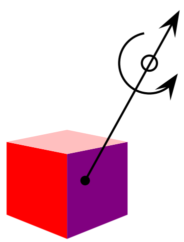
$$U_i(\boldsymbol{x}) = U_i(\boldsymbol{x}_P) + \left.\frac{\partial U_i}{\partial x_j}\right|_{x_P}(x_j - x_{Pj}) + \cdots$$

$$\frac{\partial U_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)}_{D_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

$\left\{\begin{array}{l} \overline{\overline{\boldsymbol{D}}} \text{ symmetric part of } \nabla\boldsymbol{U} \\ \overline{\overline{\boldsymbol{\Omega}}} \text{ antisymmetric part of } \nabla\boldsymbol{U} \end{array}\right.$

Ω_{ij} is associated
with the (solid-body)
rotation of
the fluid particle

$$\overline{\overline{\boldsymbol{\Omega}}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$



$$\boldsymbol{\Omega} = \frac{1}{2}\nabla\times\boldsymbol{U}$$

$(\boldsymbol{\Omega} \neq \overline{\overline{\boldsymbol{\Omega}}})$

Deformation of a fluid particle (cont.)

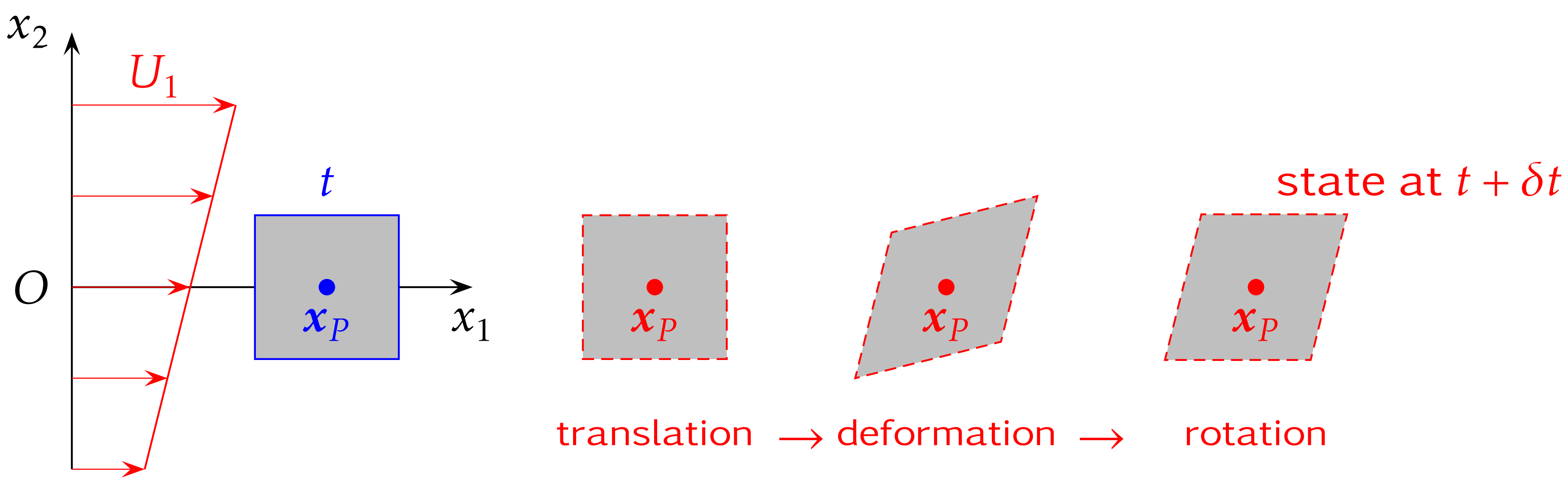
$$\boldsymbol{U}(\boldsymbol{x}) \simeq \underbrace{\boldsymbol{U}(\boldsymbol{x}_P)}_{\text{translation}} + \underbrace{\overline{\overline{\boldsymbol{D}}} \cdot (\boldsymbol{x} - \boldsymbol{x}_P)}_{\text{deformation}} + \underbrace{\overline{\overline{\boldsymbol{\Omega}}} \cdot (\boldsymbol{x} - \boldsymbol{x}_P)}_{\text{rotation at } \boldsymbol{\Omega}} + \cdots$$

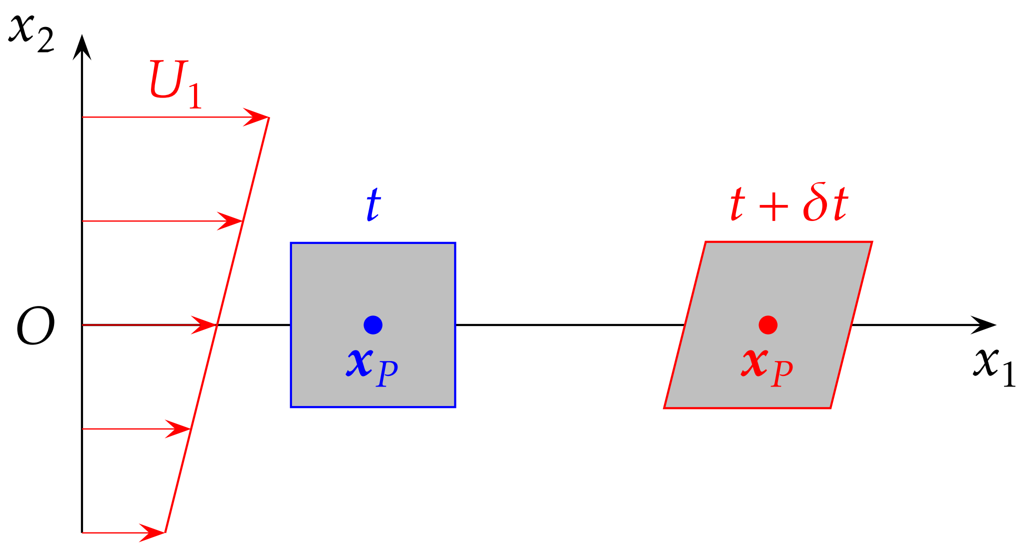
Neither translation nor rotation deform the fluid particle :
deformations are only induced by the well named deformation tensor $\overline{\overline{\boldsymbol{D}}}$

Illustration with the deformation of a fluid particle in a $2-D$ shear flow $U_1 = Sx_2$ and $U_2 = U_3 = 0$; In compressible flow $\nabla \cdot \boldsymbol{U} = 0$ and

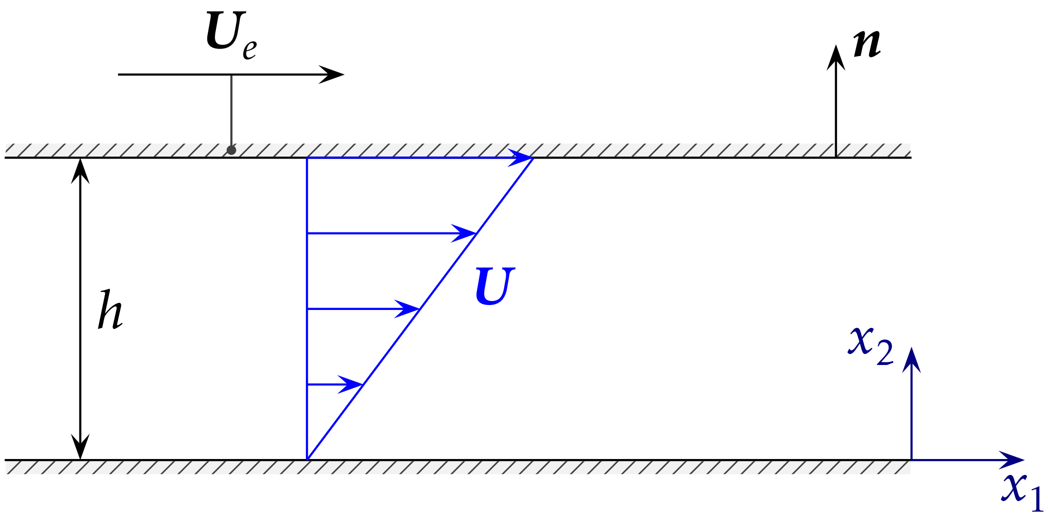
$$\frac{\partial U_i}{\partial x_j} = S \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\frac{S}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{D_{ij}} + \underbrace{\frac{S}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\Omega_{ij}} \qquad \boldsymbol{\Omega} = \frac{S}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

- **Deformation of a fluid particle (cont.)**





Plane Couette flow



U_e constant velocity imposed to the upper plane

$\boldsymbol{U} = (U_1(x_2), 0, 0)$
$U_1 = U_e x_2/h$ (shown later)

$$\nabla \boldsymbol{U}\Big|_{ij} = \frac{\partial U_i}{\partial x_j} = \begin{pmatrix} 0 & U_e/h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D_{ij} = \begin{pmatrix} 0 & U_e/(2h) & 0 \\ U_e/(2h) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Stress tensor $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$

$$\overline{\overline{\sigma}} = \begin{pmatrix} -p & \mu U_e/h & 0 \\ \mu U_e/h & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

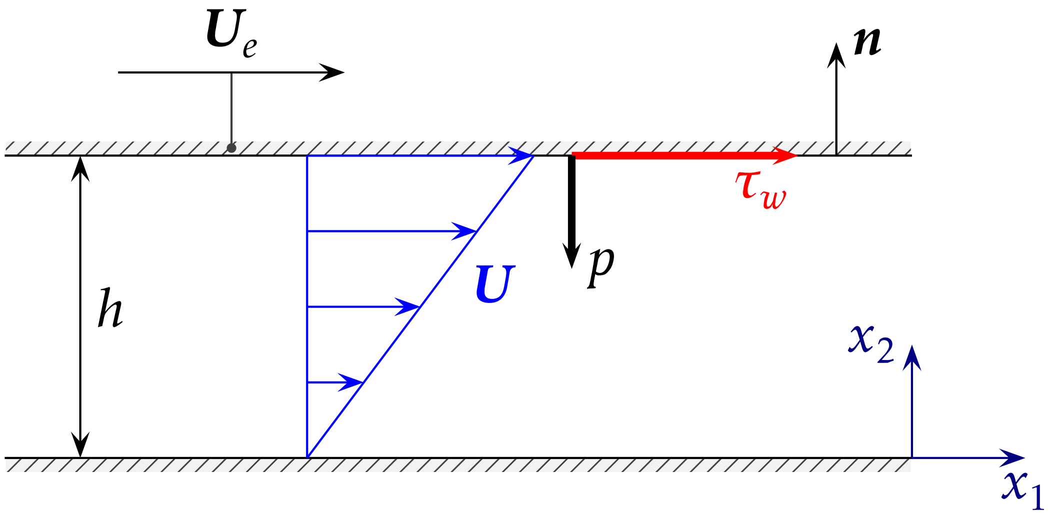
The shear stress is found to be constant here,

$\tau_{12} = \mu \dfrac{U_e}{h} = \text{cst}$

Plane Couette flow (cont.)

Force $dF = T\,ds$ exerted by the upper wall on the fluid?
By considering the unit outward normal vector $n = (0, 1, 0)$

$$T = \overline{\overline{\sigma}} \cdot n\Big|_{x_2=h} = \begin{pmatrix} -p & \mu U_e/h & 0 \\ \mu U_e/h & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu U_e/h \\ -p \\ 0 \end{pmatrix}$$



$$\tau_w = \tau_{12}|_{\text{wall}} = \frac{\mu U_e}{h}$$

wall shear stress exerted by the moving plane on the fluid

Plane Couette flow (cont.)

Rotation of the fluid particle

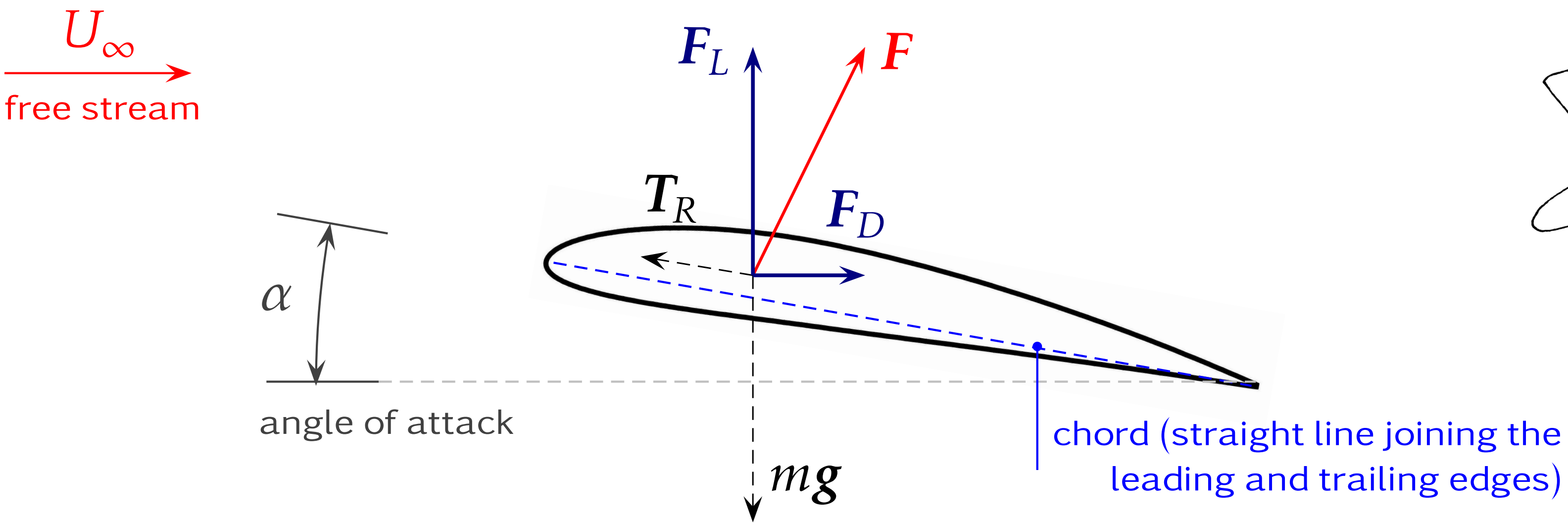
$$\Omega_{ij} = \begin{pmatrix} 0 & U_e/(2h) & 0 \\ -U_e/(2h) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{\Omega} = \begin{pmatrix} 0 \\ 0 \\ -U_e/(2h) \end{pmatrix}$$

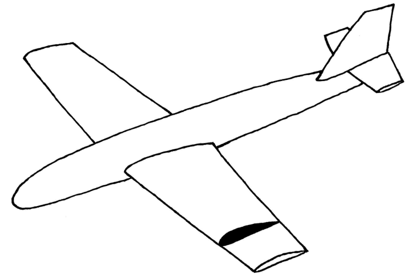
(clockwise direction)

Maurice Couette (1858-1943)
(*Rhéologie*, **8**, 2005)

Lift and drag forces

Camber airfoil as an illustration

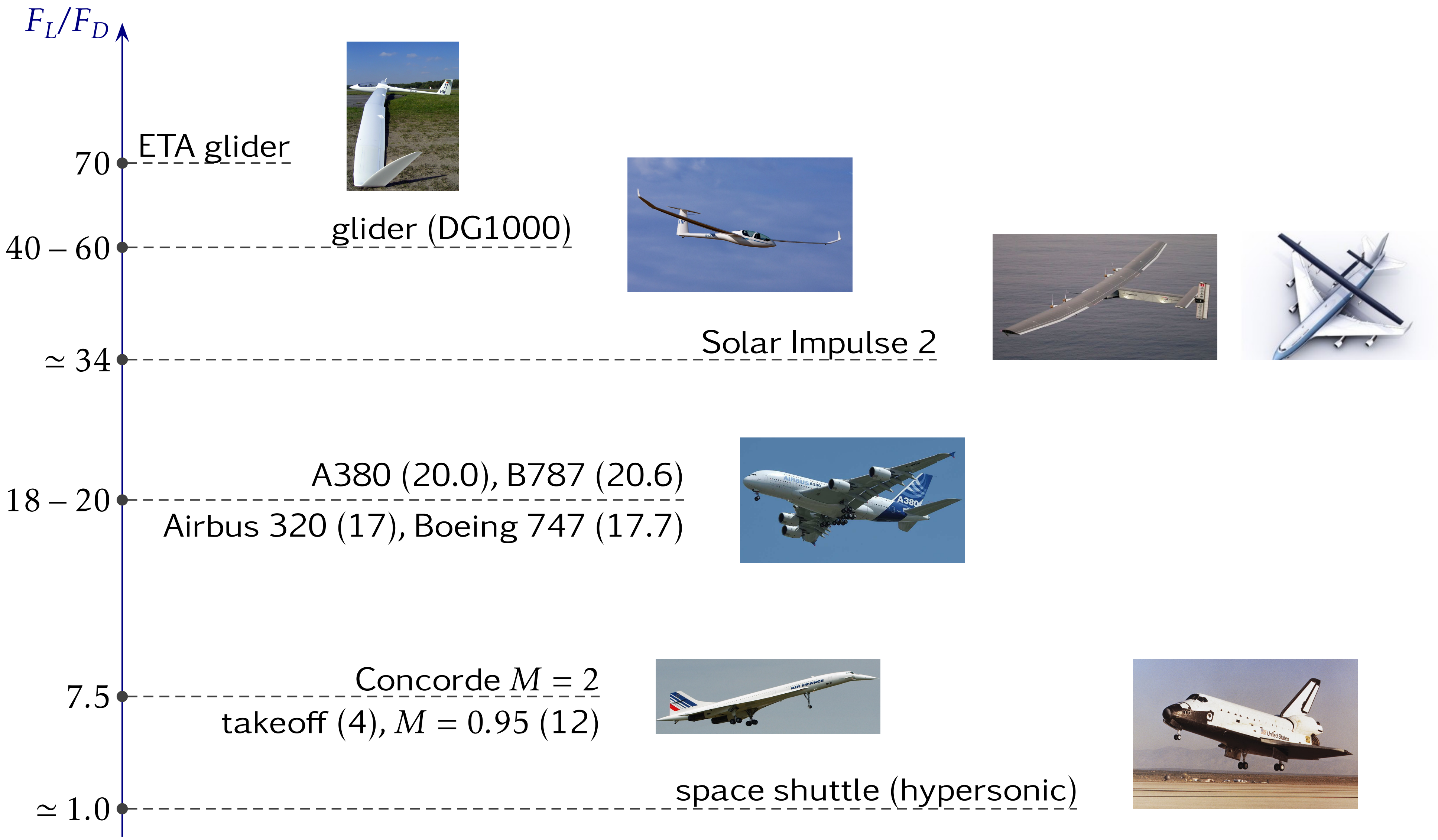




The resulting aerodynamic force $\boldsymbol{F}$ exerted by the flow to the airfoil is usually decomposed into the lift force $\boldsymbol{F}_L$, defined as the component perpendicular to the oncoming flow $\boldsymbol{U}_\infty$, and the drag force $\boldsymbol{F}_D$ acting opposite the motion of the airfoil.

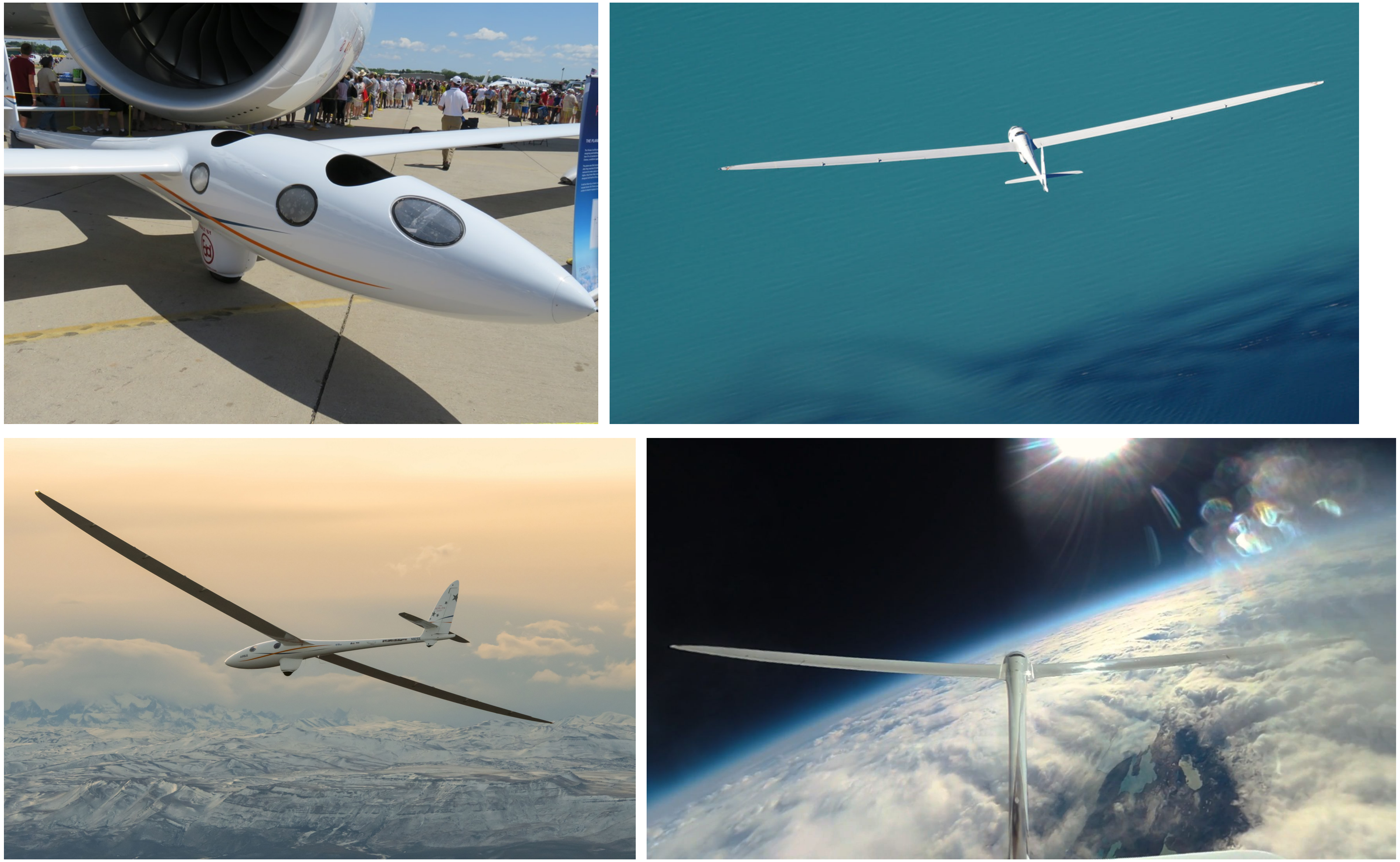
$\boldsymbol{T}_R$ is the thrust induced by the propulsion system, and $m\boldsymbol{g}$ the weight

- **Lift-to-drag ratio F_L/F_D**, flight performance indicator (« finesse » in french)



Solar Impulse 2

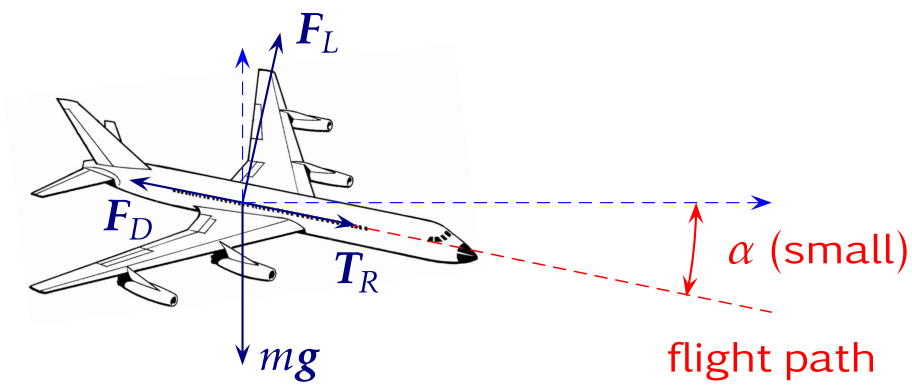
- **Stratospheric glider :** Perlan project
  glider of 26-meter wingspan, 10-meter length, $U_f \leq 194$ m.s^{-1}
  76 124 ft (23.2 km) on Sept. 2, 2018, ultimate goal of 90 000 ft (27000 m)



Button, K., *Aerospace America*, nov. 2019, 14-17 (Perlan 2 versus Airbus A350 XWB for scale in background)

An interpretation of the lift-to-drag ratio F_L/F_D

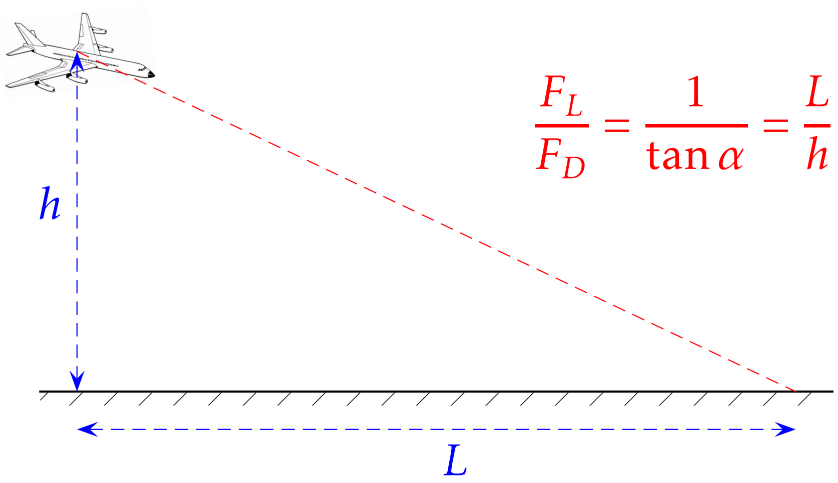
Aircraft flying at constant speed, $0 = \boldsymbol{T}_R + \boldsymbol{F}_L + \boldsymbol{F}_D + m\boldsymbol{g}$



$$\begin{cases} -F_D + mg\sin\alpha + T_R = 0 \\ F_L - mg\cos\alpha = 0 \end{cases}$$

With the engines off ($T_R = 0$),

$$\frac{F_L}{F_D} = \frac{1}{\tan\alpha}$$



Modern aircraft at cruise altitude
$h \simeq 13150$ m, $F_L/F_D \simeq 20$

glide angle $\alpha = 2.86°$
distance flown $L \simeq 263$ km

General case

$$\left\{\begin{array}{l}\dfrac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{U})=0\\ \rho\dfrac{D\boldsymbol{U}}{Dt}=-\nabla p+\nabla\cdot\overline{\overline{\boldsymbol{\tau}}}+\rho\boldsymbol{g}\\ \overline{\overline{\boldsymbol{\tau}}}=2\mu\overline{\overline{\boldsymbol{D}}}\end{array}\right.$$

conservation of mass

conservation of momentum

constitutive relation for Newtonian fluids

System of differential equations for $\rho(\boldsymbol{x},t)$ and $\boldsymbol{U}(\boldsymbol{x},t)$, not closed for the pressure $p(\boldsymbol{x},t)$

Assumption of incompressible flow $\nabla\cdot\boldsymbol{U}=0$ here,
but conservation of energy + Equation Of State (EOS) in the general case

We will consider an incompressible ($\nabla\cdot\boldsymbol{U}=0$) and (sometimes implicitly) homogeneous fluid flow ($\rho=\text{cst}$) up to Chapter 7

Incompressible and homogeneous fluid flow

Incompressibility condition : $\nabla \cdot \boldsymbol{U} = 0$
Homogeneous flow : $\rho = \text{cst}$ and $\mu = \text{cst}$
$\Longrightarrow$ conservation of mass is thus automatically verified

The viscous stress tensor is given by $\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}}$, and its divergence (that is the viscous force in the momentum equation) may be recast as $\nabla \cdot \overline{\overline{\boldsymbol{\tau}}} = \mu\nabla^2\boldsymbol{U}$

We thus obtained well-known Navier-Stokes′ equations
(where $\nu \equiv \mu/\rho$ is the kinematic viscosity)

$$\boxed{\begin{cases} \dfrac{D\boldsymbol{U}}{Dt} = -\dfrac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{U} + \boldsymbol{g} \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases}} \qquad (9)$$

Navier & Stokes

MÉMOIRE

SUR LES LOIS DU MOUVEMENT DES FLUIDES;

PAR M. NAVIER.

Lu à l'Académie royale des Sciences, le 18 mars 1822.

On voit donc en premier lieu que les équations indéfinies du mouvement du fluide deviendront respectivement

$$P - \frac{dp}{dx} = \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right) - \varepsilon\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right),$$

$$Q - \frac{dp}{dy} = \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right) - \varepsilon\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right),$$

$$R - \frac{dp}{dz} = \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right) - \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\right).$$

Navier, C.L.M.H., 1823, Mémoire sur les lois du mouvement des fluides (lu à l'Académie le 18 mars 1822), *Mémoires de l'Académie Royal des Sciences de l'Institut de France*, 389-440

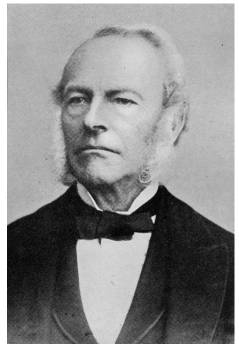
XXII. *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids. By* G. G. STOKES, M.A., *Fellow of Pembroke College.*

* The same equations have also been obtained by Navier in the case of an incompressible fluid, (*Mém. de l'Institut*, T. VI.) but his principles differ from mine still more than do Poisson's.

$$\rho\left(\frac{Du}{Dt} - X\right) + \frac{dp}{dx} - \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) - \frac{\mu}{3}\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) = 0, \ \&c. \ldots\ldots(12)$$

Stokes, G. G., 1845, On the theory of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids, *Transactions of the Cambridge Philosophical Society*, **8**, 287-305

Navier (1785-1836)



Stokes (1819-1903)

Boundary conditions

The flow is determined by solving Navier-Stokes′ equations (9), associated with appropriate boundary conditions

Solid wall :

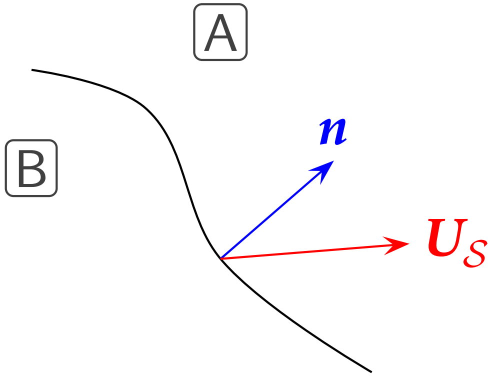
no-slip condition $\boldsymbol{U}_{\text{fluid}} = \boldsymbol{U}_{\text{wall}}$ (*e.g.* plane Couette flow)

Interface between two fluids A et B (without mass transfer and by neglecting surface tension) :

no-slip condition, $\boldsymbol{U}_A = \boldsymbol{U}_B$

continuity of efforts $\overline{\overline{\boldsymbol{\sigma}}}_A \cdot \boldsymbol{n} = \overline{\overline{\boldsymbol{\sigma}}}_B \cdot \boldsymbol{n}$

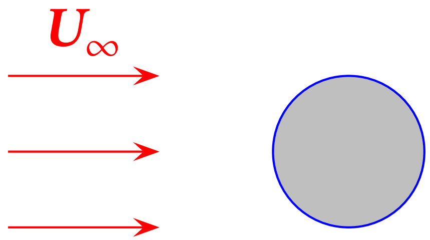
impermeability of the interface, $\boldsymbol{U}_{\mathcal{S}} \cdot \boldsymbol{n} = \boldsymbol{U} \cdot \boldsymbol{n}$



- **Formulation of the boundary value problem : flow around a body**

Navier-Stokes' equations

$$\begin{cases} \dfrac{D\boldsymbol{U}}{Dt} = -\dfrac{1}{\rho}\nabla(p+\rho\Psi) + \nu\nabla^2\boldsymbol{U} \\ \nabla\cdot\boldsymbol{U} = 0 \end{cases}$$



Boundary conditions : $\boldsymbol{U} = \boldsymbol{0}$ on the surface of the body,
and $\boldsymbol{U} \to \boldsymbol{U}_\infty$ far from the body.

The fluid itself is characterized by ν and ρ

The geometry of the body and the free stream velocity $\boldsymbol{U}_\infty$ are involved in the boundary conditions. The gravitation effect appears as a simple correction of the pressure field through $p+\rho\Psi$ (a so-called effective pressure)

After a transient, an established or developed flow is obtained : the initial conditions are forgot. This flow is however not necessary steady (independent of time)

Integral form of the kinetic energy equation

$$\boldsymbol{U}\cdot\left\{\rho\frac{D\boldsymbol{U}}{Dt}=\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}}+\rho\boldsymbol{g}\right\}\quad\Longrightarrow\quad\rho\frac{D}{Dt}\left(\frac{U^2}{2}\right)=\boldsymbol{U}\cdot\left(\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}}+\rho\boldsymbol{g}\right)$$

By applying the Reynolds theorem (4) with $\chi = U^2/2$

$$\frac{d}{dt}\int_{\mathcal{D}}\rho\frac{U^2}{2}\,dv=\underbrace{\int_{\mathcal{D}}\boldsymbol{U}\cdot\left(\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}}+\rho\boldsymbol{g}\right)dv}_{=\text{(a)}}+\int_{\mathcal{S}}\rho\frac{U^2}{2}(\boldsymbol{U}_{\mathcal{S}}-\boldsymbol{U})\cdot\boldsymbol{n}\,ds$$

and after some algebra,

$$\boldsymbol{U}\cdot(\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}})=\nabla\cdot(\boldsymbol{U}\cdot\overline{\overline{\boldsymbol{\sigma}}})-\overline{\overline{\boldsymbol{\sigma}}}:\nabla\boldsymbol{U}=\nabla\cdot(\boldsymbol{U}\cdot\overline{\overline{\boldsymbol{\sigma}}})-\overline{\overline{\boldsymbol{\sigma}}}:\overline{\overline{\boldsymbol{D}}}$$

$$\text{(a)}=\begin{cases}P_{\text{ext}}\equiv\displaystyle\int_{\mathcal{S}}\boldsymbol{U}\cdot(\overline{\overline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n})\,ds+\int_{\mathcal{D}}\rho\boldsymbol{U}\cdot\boldsymbol{g}\,dv & \text{external forces } \boldsymbol{F}_s+\boldsymbol{F}_v\\[2ex] P_{\text{int}}\equiv-\displaystyle\int_{\mathcal{D}}\overline{\overline{\boldsymbol{\sigma}}}:\overline{\overline{\boldsymbol{D}}}\,dv & \text{internal forces}\end{cases}$$

Short break ...

... to examine the expression $\overline{\overline{\sigma}} : \overline{\overline{D}}$

Scalar product between two tensors (inner product),
the result is thus a scalar (term-by-term multiplication and summation)

$$\overline{\overline{\sigma}} : \overline{\overline{D}} = \sigma_{ij} D_{ij} = \sigma_{11} D_{11} + \sigma_{12} D_{12} + \cdots + \sigma_{32} D_{32} + \sigma_{33} D_{33}$$

... to demonstrate the vectorial identity

$$\boldsymbol{U} \cdot (\nabla \cdot \overline{\overline{\sigma}}) = \nabla \cdot (\boldsymbol{U} \cdot \overline{\overline{\sigma}}) - \overline{\overline{\sigma}} : \overline{\overline{D}}$$

$$\begin{aligned} U_i \frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}(U_i \sigma_{ij}) - \sigma_{ij} \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j}(U_i \sigma_{ij}) - \frac{1}{2}\sigma_{ij} \frac{\partial U_i}{\partial x_j} - \frac{1}{2}\sigma_{ij} \frac{\partial U_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}(U_i \sigma_{ij}) - \sigma_{ij} D_{ij} \qquad (i \leftrightarrow j, \sigma_{ij} = \sigma_{ji}) \end{aligned}$$

Integral form of the kinetic energy equation (cont.)

Let us consider a materiel domain $\mathcal{D}$ (and thus $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}$)

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \frac{U^2}{2} d\mathcal{V} = P_{\text{ext}} + P_{\text{int}} \tag{10}$$

The external power P_{ext} is split into the increase of kinetic energy inside the domain $\mathcal{D}$ and the energy P_{int} required to deform the fluid particles inside $\mathcal{D}$.

The internal work P_{int} can be recast as follows

$$-P_{\text{int}} = \int_{\mathcal{D}} \overline{\overline{\boldsymbol{\sigma}}} : \overline{\overline{\mathbf{D}}} \, d\mathcal{V} = \underbrace{\int_{\mathcal{D}} \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\mathbf{D}}} \, d\mathcal{V}}_{\text{viscous dissipation} > 0} \tag{11}$$

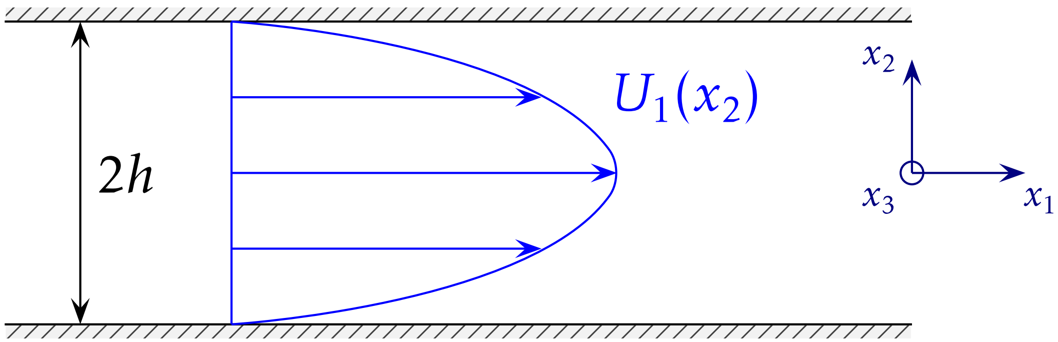
The red term represents the work of viscous forces acting on fluid particles. This viscous dissipation is induced by the degradation of mechanical energy into heat energy by friction

Viscous dissipation

For incompressible flows,

$$\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} = 2\mu\overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}} = \frac{\mu}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)^2 > 0 \qquad (\mathrm{W.m^{-3}})$$

- **Laminar flow in a plane channel (flow between two parallel walls aka Poiseuille flow)**



The velocity field is given by $\boldsymbol{U} = (U_1(x_2), 0, 0)$ with

$$U_1 = \frac{U_0}{h^2}(h^2 - x_2^2)$$

(see later slide 119)

Viscous friction force exerted by the fluid on the channel walls ?

The stress tensor is $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}$ with $\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}}$

$$\overline{\overline{\boldsymbol{D}}} = \begin{pmatrix} 0 & -\frac{U_0 x_2}{h^2} & 0 \\ -\frac{U_0 x_2}{h^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \overline{\overline{\boldsymbol{\tau}}} = 2\mu \begin{pmatrix} 0 & -\frac{U_0 x_2}{h^2} & 0 \\ -\frac{U_0 x_2}{h^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \overline{\overline{\boldsymbol{\sigma}}} = \begin{pmatrix} -p & -2\mu\frac{U_0 x_2}{h^2} & 0 \\ -2\mu\frac{U_0 x_2}{h^2} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

Laminar flow in a plane channel (cont.)

On the upper wall, $x_2 = h$, $\boldsymbol{n} = (0,1,0)$, stress vector $\boldsymbol{T} = \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}$

$$\boldsymbol{T} = \begin{pmatrix} -2\mu \dfrac{U_0}{h} \\ -p \\ 0 \end{pmatrix} \qquad d\boldsymbol{F} = \boldsymbol{T}\, ds \qquad \text{(force exerted by the upper wall to the flow)}$$

The viscous force exerted by the flow to the upper wall per unit surface area is thus given by $\boldsymbol{F} = 2(\mu U_0/h)\boldsymbol{e}_1$. By including the contribution of the lower wall, the total force is $\boldsymbol{F} = 4\mu(U_0/h)\boldsymbol{e}_1$

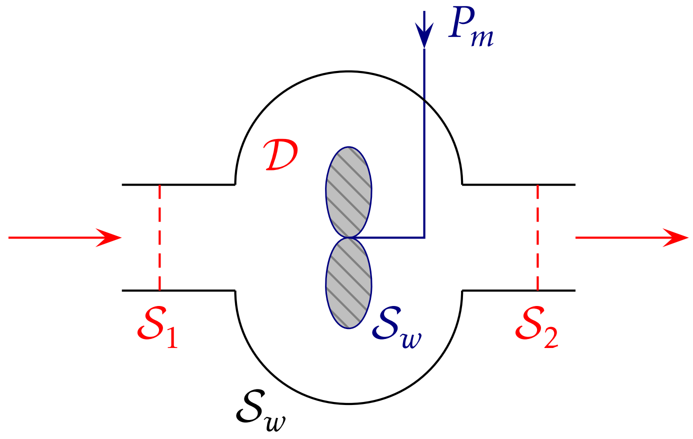
Viscous dissipation?

$$\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} = 2\mu \overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}} = 2\mu(D_{12}^2 + D_{21}^2) = 4\mu\left(\frac{U_0 x_2}{h^2}\right)^2$$

$$\int_{-h}^{+h} \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}\, dx_2 = \frac{8\mu U_0^2}{3h} \quad \text{W.m}^{-2} \quad \textit{i.e.} \text{ per unit surface area in the plane } (x_1, x_3)$$

$\rightsquigarrow$ viscous dissipation proportional to U_0^2

Mechanical energy budget



$\mathcal{D}$ control domain, with $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_w$

$\mathcal{S}_w$ impermeable (but possibly moving) surfaces, $\mathcal{S}_1$ and $\mathcal{S}_2$ are fixed surfaces

P_m mechanical power provided to the fluid flow through the shaft work (the shaft crossing the control surface)

Assumptions : incompressible flow, viscous stresses negligible at $\mathcal{S}_1$ and $\mathcal{S}_2$ (will be justified later), permanent regime

Conservation of mass (incompressible flow)

$$\int_{\mathcal{D}} \nabla \cdot \boldsymbol{U} \, dv = \int_{\mathcal{S}} \boldsymbol{U} \cdot \boldsymbol{n} \, ds = \int_{\mathcal{S}_1 \cup \mathcal{S}_2} \boldsymbol{U} \cdot \boldsymbol{n} \, ds = Q_{v2} - Q_{v1} = 0$$

$$Q_{v1} = Q_{v2} = Q_v \quad (\mathrm{m}^3.\mathrm{s}^{-1})$$

Mechanical energy budget (cont.)

Conservation of the kinetic energy

By noting that for a conservative force, here the gravity force, $\boldsymbol{g} = -\nabla\Psi(\boldsymbol{x})$ with $\Psi = -\boldsymbol{g}\cdot\boldsymbol{x} = g x_3$, the kinetic energy balance (see slide 82) can be rearranged as follows,

$$\boldsymbol{U}\cdot\boldsymbol{g} = -\boldsymbol{U}\cdot\nabla\Psi = -\frac{D\Psi}{Dt} \quad\Longrightarrow\quad \rho\frac{D}{Dt}\left(\frac{U^2}{2}+\Psi\right) = \boldsymbol{U}\cdot\nabla\cdot\overline{\overline{\boldsymbol{\sigma}}} \qquad (12)$$

We then introduce the power of the external surface force,

$$P_{\text{ext}}^{\mathcal{S}} \equiv \int_{\mathcal{S}} \boldsymbol{U}\cdot(\overline{\overline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n})\, ds$$

which leads to the corresponding integral form of the kinetic energy conservation by considering a fixed domain for the inlet $\mathcal{S}_1$ and outlet $\mathcal{S}_2$ surfaces ($\boldsymbol{U}_{\mathcal{S}} = 0$)

$$\frac{d}{dt}\int_{\mathcal{D}}\rho\left(\frac{U^2}{2}+\Psi\right)dv = P_{\text{ext}}^{\mathcal{S}} + P_{\text{int}} - \int_{\mathcal{S}_1\cup\mathcal{S}_2}\rho\left(\frac{U^2}{2}+\Psi\right)\boldsymbol{U}\cdot\boldsymbol{n}\, ds \qquad (13)$$

Mechanical energy budget (cont.)

By methodically examining each term of Eq. (13),

$$
\begin{cases}
P_{\text{ext}}^{\mathcal{S}} = \displaystyle\int_{\mathcal{S}} \boldsymbol{U} \cdot (\overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n})\, d\mathcal{s} = \underbrace{P_m}_{\text{through } \mathcal{S}_w} - \displaystyle\int_{\mathcal{S}_1 \cup \mathcal{S}_2} p\boldsymbol{U} \cdot \boldsymbol{n}\, d\mathcal{s} \\[2em]
P_{\text{int}} = -\underbrace{\displaystyle\int_{\mathcal{D}} \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}\, d\mathcal{v}}_{\substack{\text{viscous dissipation} \\ > 0}} = -\mathcal{D}_v
\end{cases}
$$

(no details regarding the machine work P_m is desired at this step)

Mechanical energy budget (cont.)

$$\frac{d}{dt}\int_{\mathcal{D}} \rho\left(\frac{U^2}{2}+\Psi\right)d\mathcal{V} = P_m - \mathcal{D}_\nu - \underbrace{\int_{\mathcal{S}_1\cup\mathcal{S}_2} \mathcal{H}\boldsymbol{U}\cdot\boldsymbol{n}\,ds}_{(a)}$$

where $\boxed{\mathcal{H} = p + \rho U^2/2 + \rho\Psi}$ is the local energy per unit volume, refer to Eq. (8). The term (a) is the energy flux between $\mathcal{S}_2$ (outlet) and $\mathcal{S}_1$ (inlet). By taking the time average of the previous equation,

$$\underbrace{\bar{P}_m}_{\text{mean power provided to the flow}} = \underbrace{\int_{\mathcal{S}_1\cup\mathcal{S}_2} \overline{\mathcal{H}\boldsymbol{U}\cdot\boldsymbol{n}}\,ds}_{\text{increase of energy flux between inlet/outlet}} + \underbrace{\bar{\mathcal{D}}_\nu}_{\text{viscous dissipation}}$$

Energy head form

A one-dimensional model of the system is commonly used in engineering by introducing the energy head H defined as,

$$\bar{Q}_v H \equiv \int_{\mathcal{S}} \overline{\mathcal{H} \boldsymbol{U} \cdot \boldsymbol{n}}\, ds$$

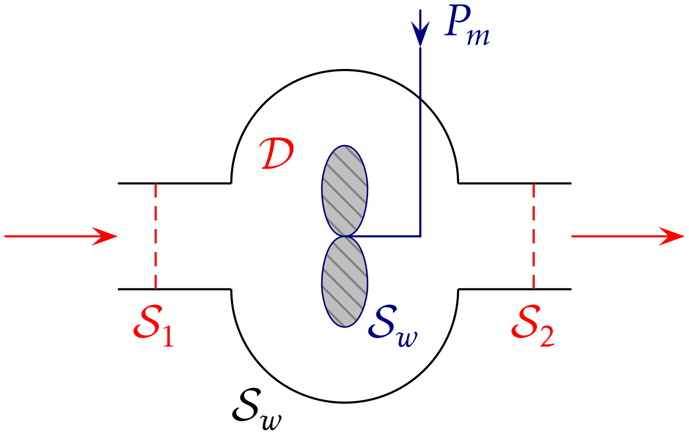
H is an average of the local quantity $\mathcal{H}$ in time (turbulent regime for instance), and over the cross section $\mathcal{S}$ weighted by the mean flow rate $\bar{Q}_v$. The conservation of mechanical energy may be recast in a compact form,

$$\boxed{\bar{P}_m + \bar{Q}_v(H_1 - H_2) = \bar{\mathcal{D}}_\nu} \qquad (\sim \mathrm{J.s}^{-1} \sim \mathrm{W}) \tag{14}$$

For a passive system (no machine, $\bar{P}_m = 0$), $\bar{Q}_v(H_1 - H_2) = \bar{\mathcal{D}}_\nu > 0$
a head loss is observed during fluid motion, due to viscous dissipation inside the fluid domain $\mathcal{D}$ (see later slide 119 for an explicit expression)

Energy head form (cont.)

By introducing the efficiency η of the machine, the power transmitted by the shaft is given



For a pump or a fan blade by

$Q_v(H_2 - H_1) = (P_m - \mathcal{D}_v) = \eta P_m > 0$

$Q_v \Delta H_P = \eta P_m > 0$

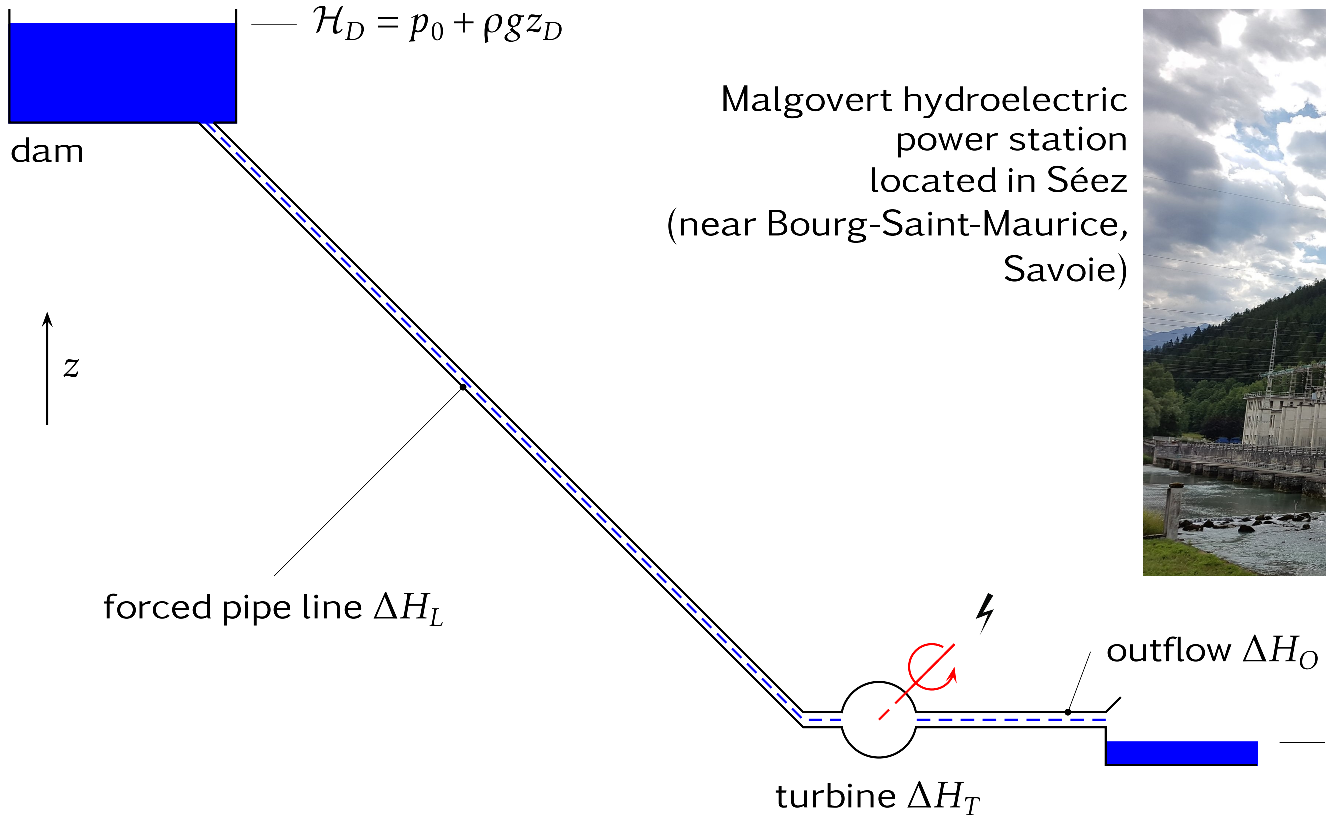
For a turbine by

$P_T = -P_m = Q_v(H_1 - H_2) - \mathcal{D}_v = \eta Q_v(H_1 - H_2) > 0$

$P_T = \eta Q_v \Delta H_T > 0$

(by abuse of language, the bars over are often omitted)

Case of a mountain dam



Malgovert hydroelectric power station located in Séez (near Bourg-Saint-Maurice, Savoie)

$$H_D - H_R = \rho g(z_D - z_R) = \Delta H_L + \Delta H_T + \Delta H_O$$

$\Delta H_L, \Delta H_O$ head losses

$$P_T = \eta_T Q_v \Delta H_T = \eta_T Q_v \left[\rho g(z_D - z_R) - \Delta H_L - \Delta H_O\right]$$

Case of a mountain dam (cont.)

Remark - To determine the energy head H_D at the surface of the dam lake $z = z_D$, we first notice that $\mathcal{H}_D = p_0 + \rho g z_D + \rho U_D^2/2 \simeq p_0 + \rho g z_D = \text{cst}$. The volumetric flow rate $Q_v = U_D \times \mathcal{S}$ is indeed obtained from a large surface area $\mathcal{S}$ and a very small value of the velocity U_D. Consequently,

$$H_D = \frac{\int_{\mathcal{S}} \mathcal{H}_D \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S}}{Q_v} \simeq \mathcal{H}_D \frac{\int_{\mathcal{S}} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S}}{Q_v} \simeq \mathcal{H}_D$$

- Stress tensor for a viscous fluid $\overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}$
- Newtonian fluid, viscous stress tensor for an incompressible flow $\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}}$
- Kinetic energy budget
- Navier-Stokes′ equations (9)
- Boundary conditions
- Energetics of continuous-flow system and its energy head form (14)

Outline

General information 2
Chapter 1 : Kinematic properties, fundamental laws, inviscid model 16
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Chapter 3 : Dimensional analysis - Reynolds number 98
Chapter 4 : Regimes and flow structures as a function of Reynolds number 129
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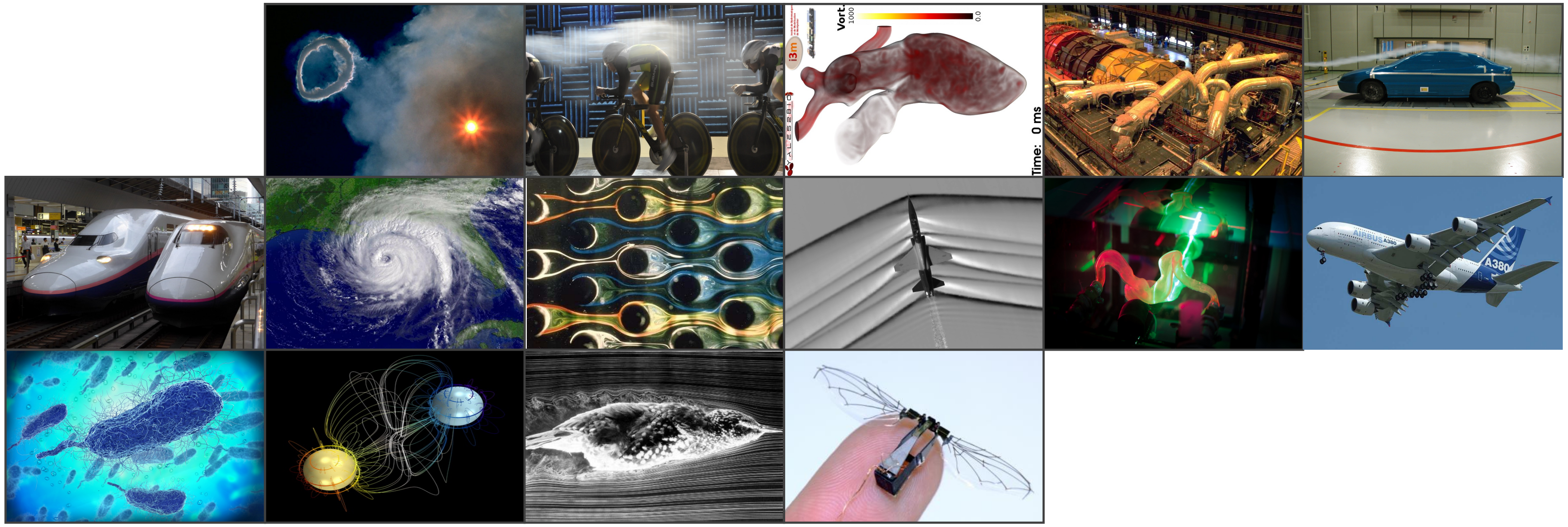
3 - Dimensional analysis - Reynolds number



3 - Dimensional analysis - Reynolds number

Dimensional analysis

- Motivation
- Units and dimensions
- Π-theorem
- Illustration

Reynolds number

- Definition
- Interpretation

Flow around a body

- Dimensional analysis
- Lift and drag coefficients
- Drag coefficient for a smooth sphere
- Reynolds number and mock-up
- Nondimensional Navier-Stokes eqs.

Applications of the Π-theorem

- Drag on a ship hull
- Rowing shell

Duct flow

- Poiseuille flow
- Skin-friction coefficient
- 1-D model

To know more about it

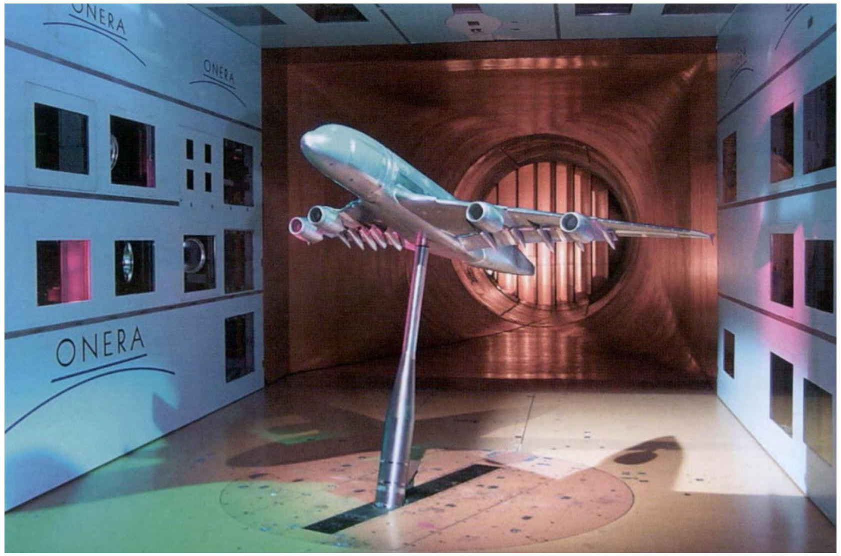
Key results

Motivation

Can the performance of a model aircraft in a wind tunnel be used to predict its aerodynamic performances at full scale (similitude requirements)?

Can parameters governing the general behavior of a particular flow (wake behind a cylinder for instance) be identified in a systematic way, avoiding to make a lot of experiments?

The answer is yes, based on dimensional analysis which provides scaling laws (linked to similitude & self-similar solutions)



Mock-up of the Airbus A380 in the F1 wind tunnel at ONERA Fauga-Mauzac

(courtesy of Jean Délery)

Units and dimensions

Primary dimensions	[ ]	unit
mass	$[M]$	kg
length	$[L]$	m
time	$[T]$	s
temperature	$[\Theta]$	K

(choice of units is quite arbitrary)

velocity $[V] = LT^{-1}$
force $[F] = MLT^{-2}$
pressure $[P] = ML^{-1}T^{-2}(= [F]/L^2)$
kinematic viscosity $[\nu] = L^2T^{-1}$
density $[\rho] = ML^{-3}$
dynamic viscosity $[\mu] = ML^{-1}T^{-1}(= [\rho][\nu])$
energy $[E] = ML^2T^{-2}$
power $[P] = ML^2T^{-3}$

Vaschy - Buckingham theorem (Π-theorem)

Vaschy (*Annales Télégraphiques,* 1892) & Buckingham (*Phys. Rev. Let.,* 1914)

The Π-theorem is a general statement for predicting the number of dimensionless quantities in a given problem

- n dimensional physical quantities a_i directly associated with the considered problem, including the particular quantity of interest a_1 (the drag for instance) : an implicit physical relation $\mathcal{F}(a_1, a_2, ..., a_n) = 0$ is assumed
- r is the rank of the matrix formed with the exponents of the variable dimensions a_i

$$\left[a_i\right] = L^{\alpha_i}\, T^{\beta_i}\, M^{\gamma_i}\, \Theta^{\delta_i}$$

dimensionless parameters

$$\left[a_1\right]^{\lambda_1} \cdots \left[a_n\right]^{\lambda_n} = L^0\, T^0\, M^0\, \Theta^0$$

Values of $\lambda_1, ..., \lambda_n$?

$$\begin{array}{c|ccc} & [a_1] & \cdots & [a_n] \\ \hline L & & & \\ T & & & \\ M & & & \\ \Theta & & & \end{array} \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \\ \beta_1 & & \beta_n \\ \gamma_1 & & \gamma_n \\ \delta_1 & \cdots & \delta_n \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \cdots \\ \lambda_n \end{pmatrix} = 0$$

Vaschy - Buckingham theorem (cont.)

- The problem can be reduced to a dimensionless law of $n - r$ dimensionless parameters Π_i, that is

$$\Pi_1 = \mathcal{F}_0(\Pi_2, \ldots, \Pi_{n-r})$$

where Π_1 is the dimensionless number associated with the quantity of interest a_1. If $n - r = 1$, then $\Pi_1 = \text{cst}$.

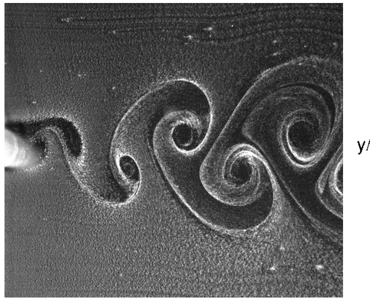
An operational formulation of the Π-theorem

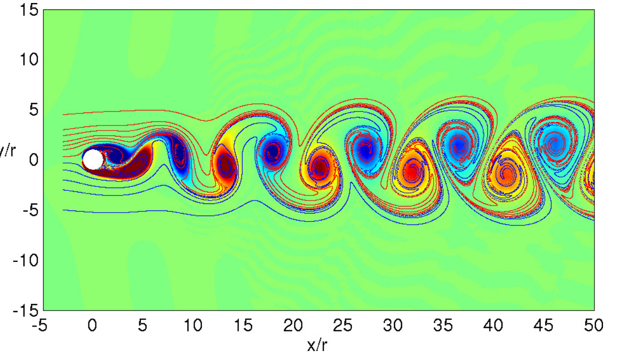
For a problem involving n variables, with r independent dimensions occurring in those variables, a dimensionless physical law of $n - r$ dimensionless parameters Π_i can be found

$$\mathcal{F}(\Pi_1, \Pi_2, ..., \Pi_{n-r}) = 0 \qquad \text{or equivalently} \qquad \Pi_1 = \mathcal{F}_0(\Pi_2, \ldots, \Pi_{n-r})$$

Application of the Π-theorem

Frequency f of the vortex shedding behind a cylinder





$\mathcal{F}(f, U_\infty, \rho, \mu, D) = 0 \qquad \Longrightarrow n = 5$

	U_∞	ρ	μ	D	f
L	1	−3	−1	1	0
T	−1	0	−1	0	−1
M	0	1	1	0	0
Θ	0	0	0	0	0

$\Longrightarrow r = 3$ (c4 = c3-c2-c1 ; c4+c5 = c1)

$n - r = 2$ dimensionless parameters Π_i to identify

$$\Pi_1 = \frac{fD}{U_\infty} = \text{St} \quad \text{Strouhal number}$$

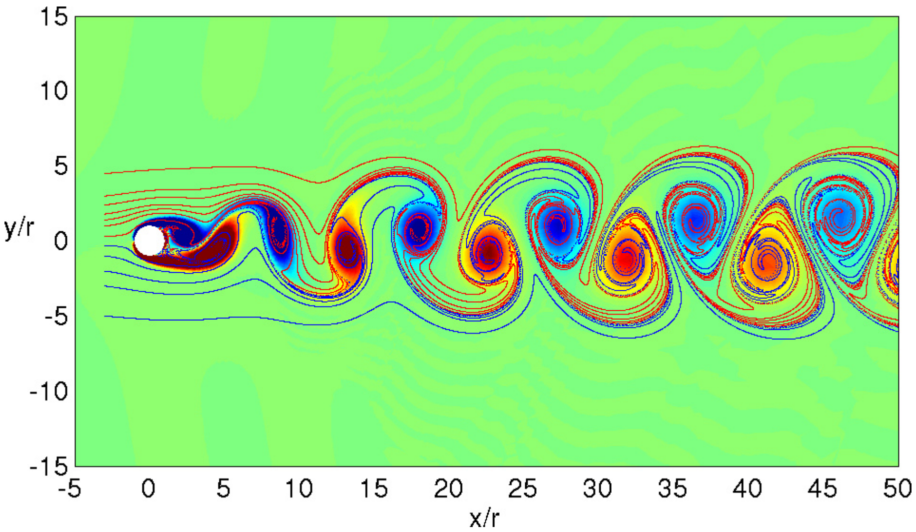
$$\Pi_2 = \frac{\rho U_\infty D}{\mu} = \text{Re} \quad \text{Reynolds number}$$

$$\Longrightarrow \quad \text{St} = \mathcal{F}_0(\text{Re})$$

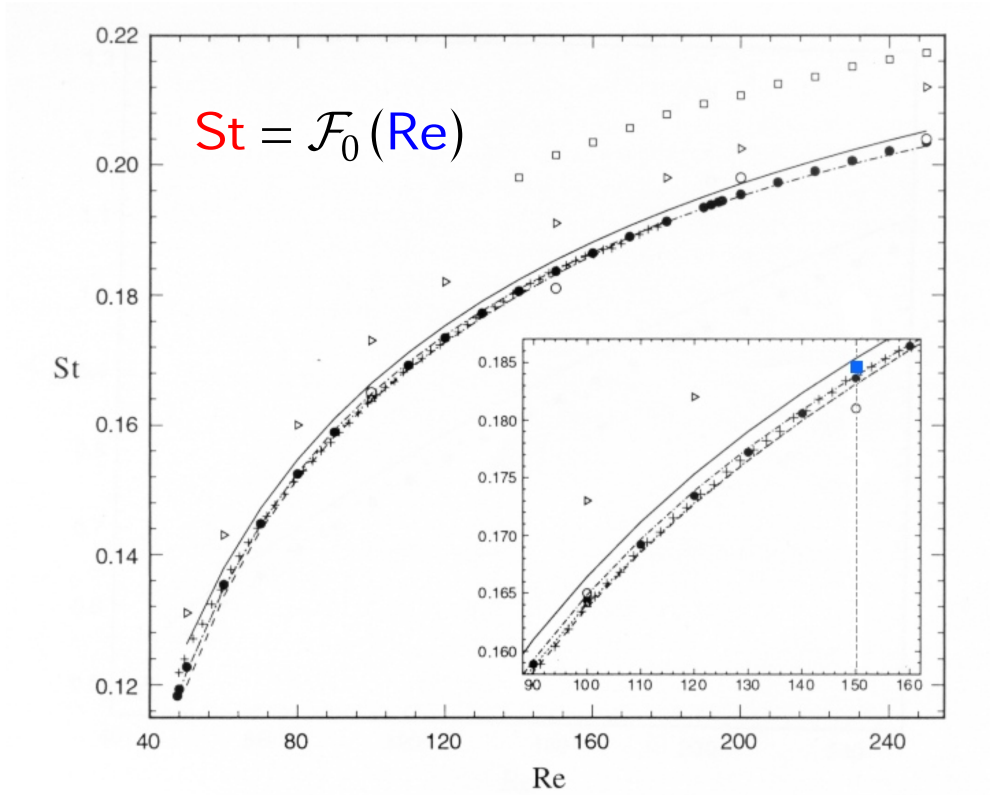
The same Strouhal number (dimensionless frequency) will be observed in two different experiments/simulations having the same Reynolds number Re

Low-Reynolds number flow around a cylinder

Re $= 150$, $M_\infty \simeq 0.33$, $D = 2 \times 10^{-5}$ m



Vorticity $\omega_3 = \partial U_2/\partial x_1 - \partial U_1/\partial x_2$ superimposed with fluid particle trajectories



Marsden *et al.*, *J. Comput. Acoust.*, 13(4), 2005

Definition and interpretation

U, L characteristic velocity and length scales of the flow
ν kinematic viscosity of the fluid

$\mathrm{Re} = \dfrac{UL}{\nu}$ The most important dimensionless parameter in fluid mechanics

From Navier-Stokes′ equation (9)

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} \;=\; -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{U} + \boldsymbol{g}$$

$$\sim \frac{U^2}{L} \qquad\qquad \sim \nu \frac{U}{L^2}$$

$$\frac{\text{convective term}}{\text{viscous term}} \sim \frac{UL}{\nu} = \mathrm{Re}$$

- **Interpretation (cont.)**

The Reynolds number is a measure of the ratio of inertial forces to viscous forces for a given flow.

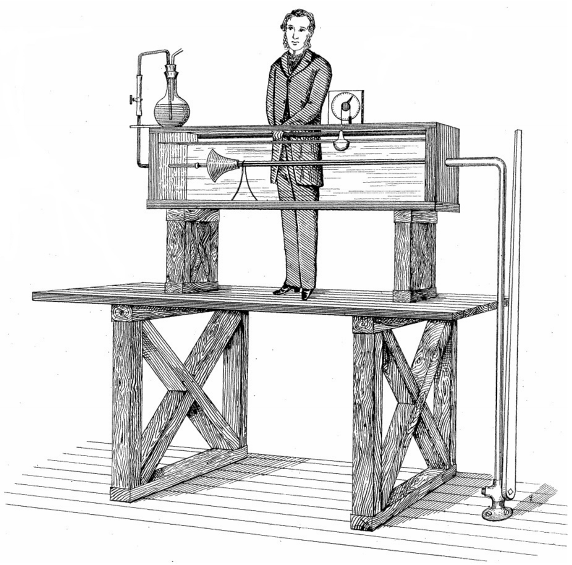
$\mathrm{Re} \ll 1$: convective term negligeable, Stokes or creeping flow
$\mathrm{Re} \gg 1$: necessary condition (only) to apply an inviscid model

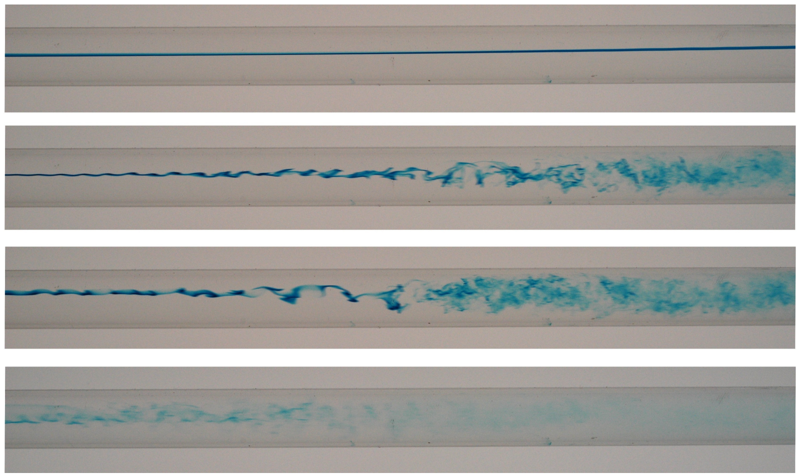
The Reynolds number can also be interpreted as the ratio of two characteristic times

$$\mathrm{Re} = \frac{L^2/\nu}{L/U} \sim \frac{\text{viscous time}}{\text{convective time}}$$

In other words, a small viscous time $L^2/\nu \ll L/U$ corresponds to efficient viscous effects : a small perturbation is damped during its convection, and the flow regime remains laminar

Experience of Reynolds : laminar versus turbulent regime

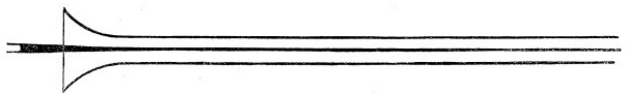




The general results were as follows :—

(1.) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, fig. 3.

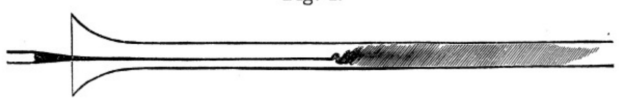
Fig. 3.



(2.) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

(3.) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in fig. 4.

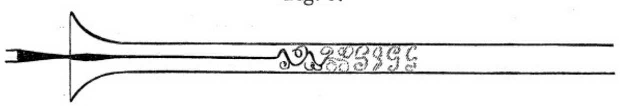
Fig. 4.



Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in fig. 5.

Fig. 5.



Reynolds, O., 1883, *Phil. Trans. Roy. Soc.*, **174**, 935-982
Osborne Reynolds (1842-1912)

Nondimensional Navier-Stokes equations

$$\tilde{U}_i = \frac{U_i}{U_\infty} \qquad \tilde{p} = \frac{p}{\rho U_\infty^2} \qquad \tilde{x}_i = \frac{x_i}{L} \qquad \boxed{\tilde{t} = \frac{t}{L/U_\infty}}$$

(t normalized by the convective time scale)

Navier-Stokes′ equations read

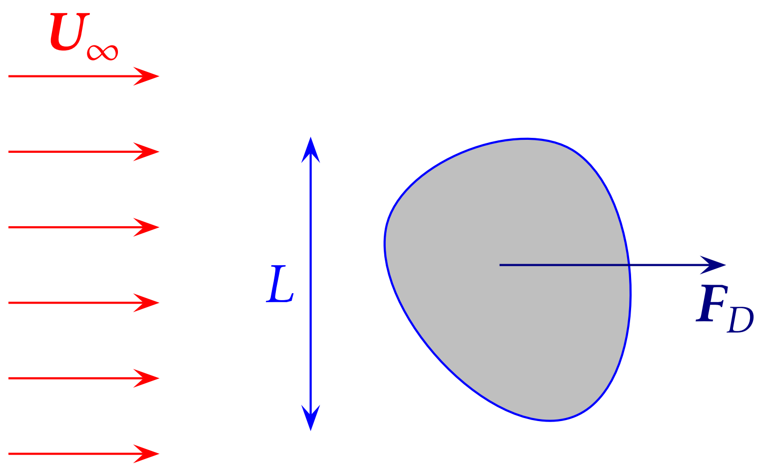
$$\begin{cases} \dfrac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} + \tilde{\boldsymbol{U}} \cdot \tilde{\nabla} \tilde{\boldsymbol{U}} = -\tilde{\nabla} \tilde{p} + \dfrac{1}{\mathrm{Re}} \tilde{\nabla}^2 \tilde{\boldsymbol{U}} \\ \tilde{\nabla} \cdot \tilde{\boldsymbol{U}} = 0 \end{cases} \qquad \mathrm{Re} = \frac{U_\infty L}{\nu}$$

Boundary conditions

$$\begin{cases} \tilde{\boldsymbol{U}} = 0 \ \text{ on the body surface} \\ \tilde{\boldsymbol{U}} \to (1,0,0) \ \text{ far from the body} \end{cases}$$

The fully developed flow (when initial conditions are forgotten) is only a function of the Reynolds number Re

Dimensional analysis for the drag force F_D exerted by the flow on the body



Flow parameters $F_D, \mu, \rho, U_\infty, L$

	F_D	μ	ρ	U_∞	L
L	1	−1	−3	1	1
T	−2	−1	0	−1	0
M	1	1	1	0	0
Θ	0	0	0	0	0

$(c_1 = c_2 + c_4 + c_5\,;\, c_2 = c_3 + c_4 - 2c_5)$

$n - r = 5 - 3 = 2 \quad \Longrightarrow \quad$ 2 parameters

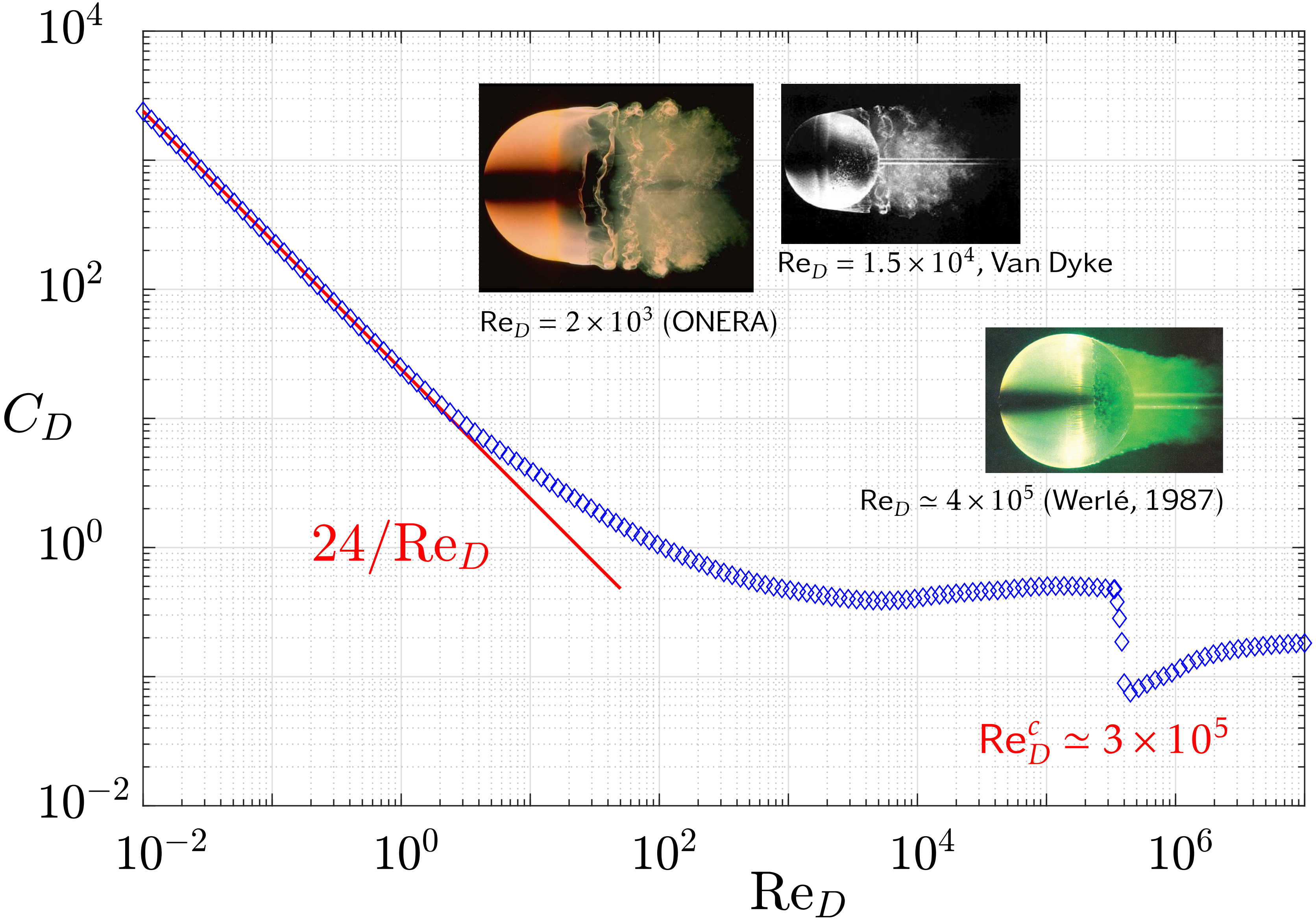
$$\Pi_1 = \frac{F_D}{q_\infty S} = C_D \qquad q_\infty \equiv \frac{1}{2}\rho U_\infty^2$$ (dynamic pressure see Pitot's tube) $\qquad S \propto L^2$

$$\Pi_2 = \frac{\rho U_\infty L}{\mu} = \text{Re}$$

$$\boxed{C_D = C_D(\text{Re})}$$ drag coefficient

Drag coefficient for a smooth sphere

(diameter D, $S = \pi D^2/4$, cont. in Chapter 4)



Reynolds number and mock-up

full scale $\mathrm{Re}' = \dfrac{\rho'_\infty U'_\infty c'}{\mu'_\infty}$ scaled mock-up $\mathrm{Re} = \dfrac{\rho_\infty U_\infty c}{\mu_\infty}$

In aeronautics, $U_\infty \simeq U'_\infty$ (Mach number similitude), but c is smaller than $c' \Longrightarrow$ need to increase Re to reach the nominal value Re′, and thus to correctly measure $C_D = C_D(\mathrm{Re}')$

How to increase the Reynolds number (for a perfect gas)?

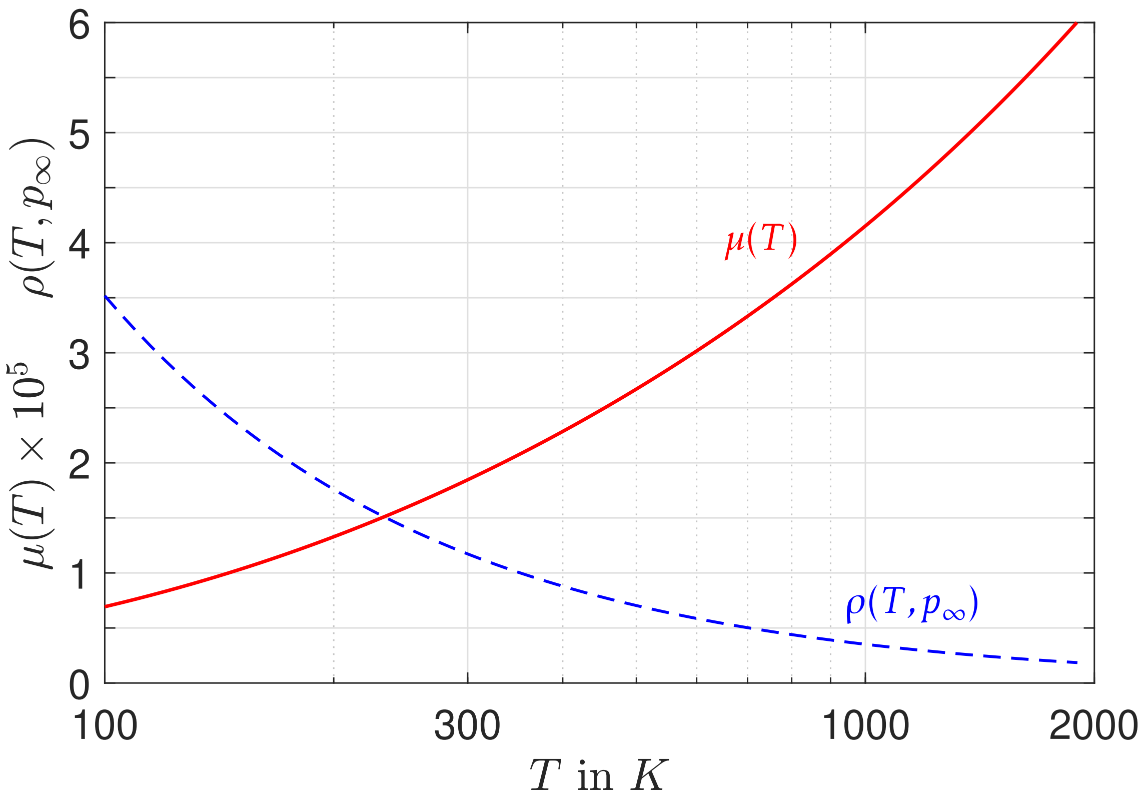
$$\mathrm{Re} = \frac{\rho_\infty U_\infty c}{\mu_\infty} \qquad \rho_\infty = \frac{p_\infty}{r T_\infty} \qquad q_\infty \equiv \frac{1}{2}\rho_\infty U_\infty^2 = \frac{\gamma}{2} p_\infty M_\infty^2$$

$p_\infty \nearrow$ pressurized wind tunnel (inducing mechanical constraints with the increase of the dynamic pressure q_∞), see the *Superpipe* in slide 122!

$T_\infty \searrow$ cryogenic wind tunnel ($\mu_\infty \searrow \ \rho_\infty \nearrow$)

A320 (180 passengers) in cruise condition
Reynolds number $\mathrm{Re}' = 3 \times 10^7$
At an altitude of 10000 m, $\mu'_\infty/\rho'_\infty = 3.53 \times 10^{-5}$ m^2.s^{-1}
$\mathrm{M}_\infty = 0.8$, $U'_\infty = 240$ m.s^{-1}, $c' = 4.4$ m

Dynamic viscosity μ for a gas



Sutherland's law (1893)

$$\mu(T) \simeq \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + T_s}{T + T_s}$$

Air

$T_0 = 273$ K $\quad T_s = 111$ K

$\mu_0 = 1.716 \times 10^{-5}$ kg/(m.s)

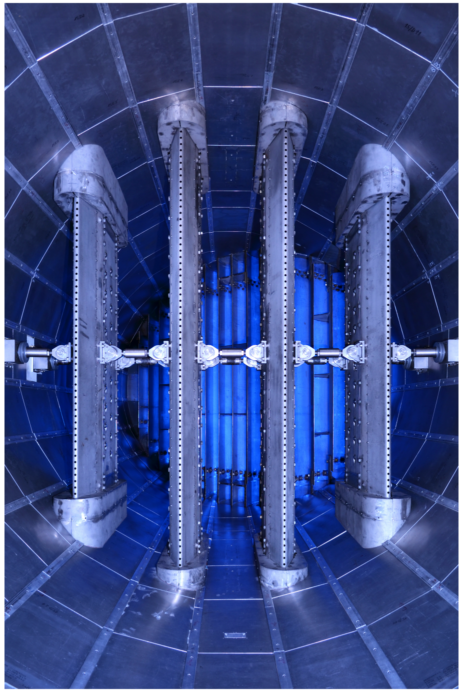
White (1988)

For common gases, $\mu = \mu(T)$ but the kinematic viscosity $\nu = \mu/\rho = \nu(p, T)$

Air at $T = 20^\circ$ C and $p_\infty = 1$ bar, $\nu = 1.5 \times 10^{-5}$ m^2.s^{-1}

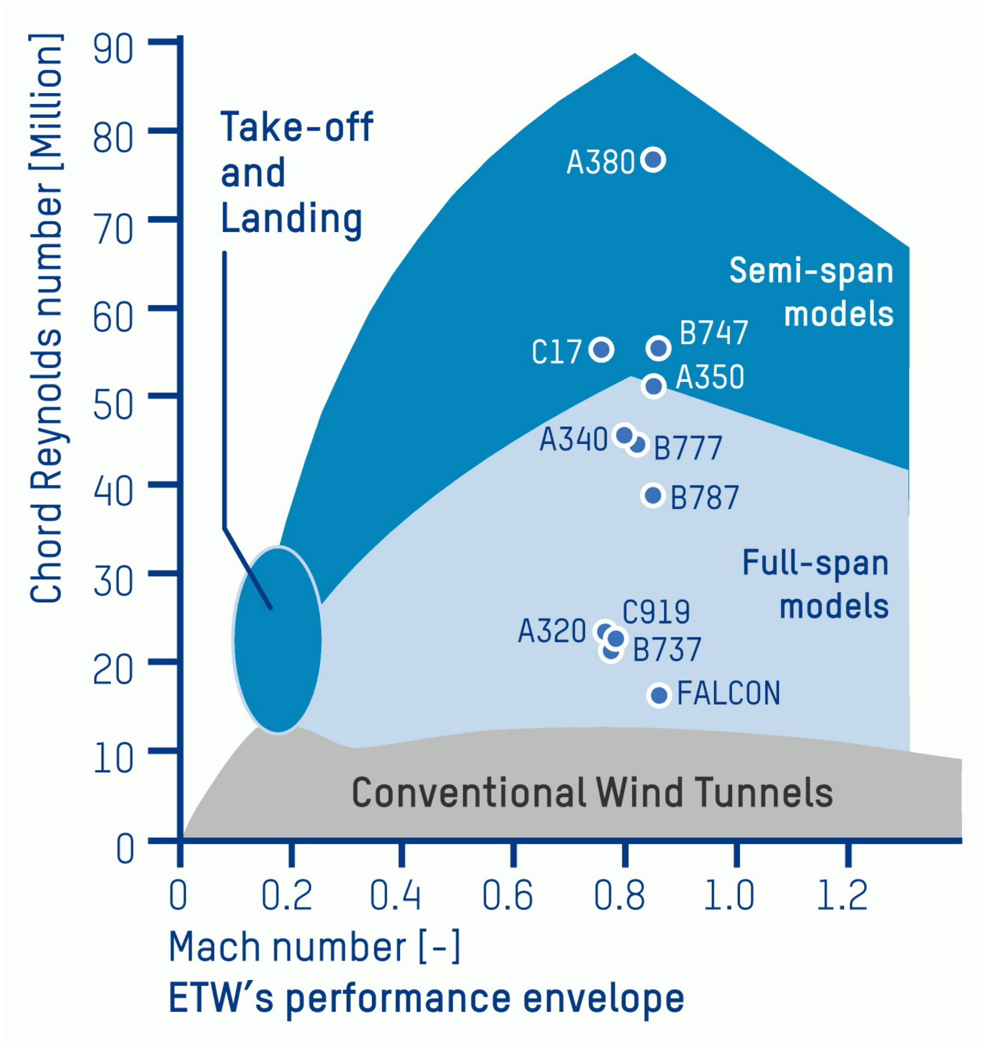
(As a reminder, $\nu = 10^{-6}$ m^2.s^{-1} for water)

- **European Transonic Windtunnel (ETW, Cologne, Germany)**



LN2 Injection Rakes

A340



Test Section : 2.4 m × 2.0 m ; length 9 m
Mach Number Range : 0.15 - 1.3
Pressure range : 1.25 - 4.5 bar
Temperature range : 110 - 313 K

http://www.etw.de/

Drag on a ship hull

USS John F. Kennedy aircraft carrier and accompanying destroyers

Ducks swimming across a lake

Same angle ($\simeq 39^\circ$) for the wake!

Flow parameters $F_D, \mu, \rho, U_\infty, L, g$

	F_D	μ	ρ	U_∞	L	g
L	1	−1	−3	1	1	1
T	−2	−1	0	−1	0	−2
M	1	1	1	0	0	0
Θ	0	0	0	0	0	0

$(c_1 = c_2 + c_4 + c_5\,;\, c_3 = c_2 - c_4 - 2c_5\,;\, c_6 = 2c_4 - c_5)$

$n - r = 6 - 3 = 3$ dimensionless parameters

$$\Pi_1 = \frac{F_D}{\frac{1}{2}\rho U^2 S}$$ (S = wetted-surface area)

$$\Pi_2 = \mathrm{Re} = \frac{\rho U L}{\mu}$$

$$\Pi_3 = \mathrm{Fr} = \frac{U}{\sqrt{gL}}$$ Froude number

Drag on a ship hull (cont.)

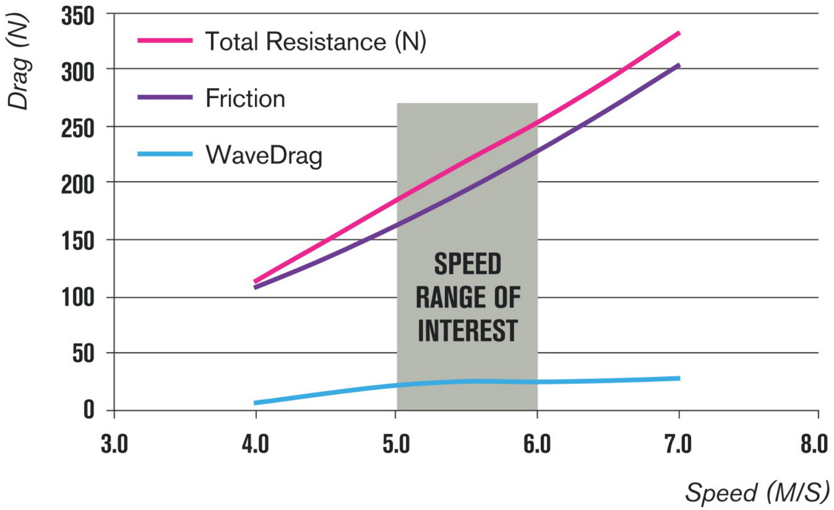
The drag is induced by surface gravity waves and by the (classical) viscous friction, $C_D = C_D(\mathrm{Re}, \mathrm{Fr})$. It is not possible - however - to preserve both Reynolds and Froude number similitude in experiments.

Froude's hypothesis $C_D \simeq C_F(\mathrm{Re}) + C_R(\mathrm{Fr})$
Friction drag coefficient C_F and residual (wave) drag coefficient C_R induced by two distinct physics

- For the model boat, the velocity is imposed by the Froude number and the model size : the total drag C_D is *measured*, and from the knowledge of $C_F(\mathrm{Re})$, the residual drag of the model C_R is then *calculated*.
- For the full scale ship, the residual drag C_R is known from previous experiments (same value of Fr), and the friction drag is again *calculated* as a function of the Reynolds number.

Rowing shell (McMahon, 1971)

How does the boat's speed change with the number of rowers ?



4 person shell, 88% of the drag due to wetted-surface friction

Killing, S., 2017, *Row360*, **009**

N identical oarsmen – same individual power P and weight, geometric similarity between boats, the drag is dominated by the friction drag $C_D \simeq C_F(\mathrm{Re})$ for a fitted body, and the value of the Reynolds number does not change with the number of rowers

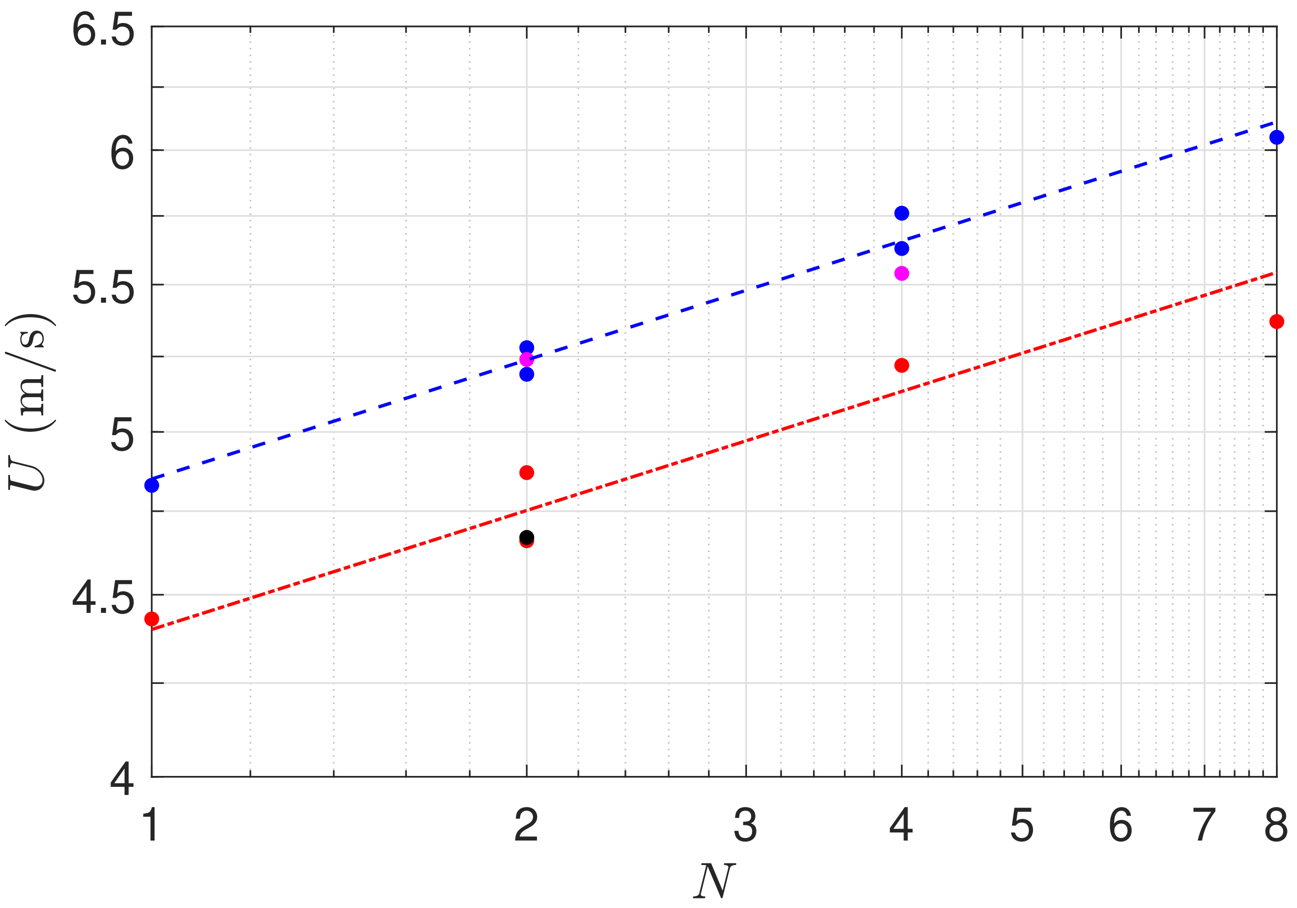
$$\text{power} \propto C_F \frac{1}{2}\rho U^2 S \times U \;\propto\; N \times P$$

The submerged volume is $\mathcal{V} \sim N$, leading to $S \sim N^{2/3}$ for the wetted surface

Finally, one gets $U \sim N^{1/9}$

Rowing shell (cont.)

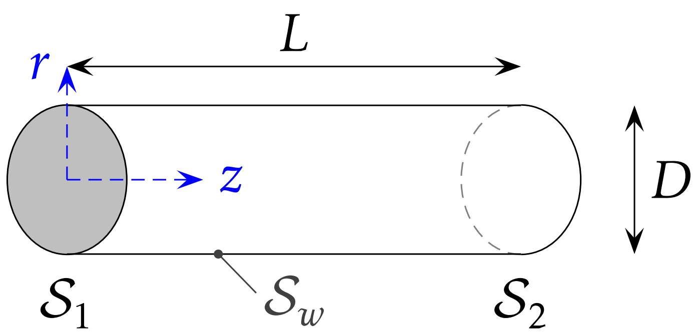
How does the boat's speed change with the number of rowers ?



Data from Lucerne 2010 A-finals (Sunday 11 July 2010)
http://sanderroosendaal.wordpress.com/

Poiseuille flow

Application to the laminar steady flow established in a circular pipe. Our objective is to derive a 1-D model from the analytical solution, see Eq. (14)



Conservation of mass, $Q_{v2} = Q_{v1} = Q_v$ (volumetric flow rate, $\sim$ m^3.s^{-1})

Navier-Stokes' equations (gravity force neglected, horizontal duct)

Cylindrical coordinates (r, θ, z)

$\boldsymbol{U} = (0, 0, U(r)) \quad \nabla \cdot \boldsymbol{U} = 0$

$$
\begin{cases}
0 = -\dfrac{\partial p}{\partial r} \\[2ex]
0 = -\dfrac{1}{r}\dfrac{\partial p}{\partial \theta} \\[2ex]
0 = -\dfrac{\partial p}{\partial z} + \mu \dfrac{1}{r}\dfrac{d}{dr}\left(r \dfrac{dU}{dr}\right)
\end{cases}
$$

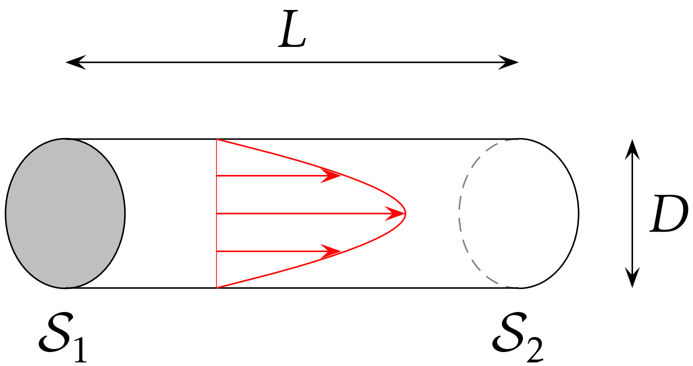
Poiseuille flow (cont.)

From Navier-Stokes' equations

$p = p(z)$ and $$\underbrace{\frac{dp}{dz}}_{\text{fct of } z} = \underbrace{\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right)}_{\text{fct of } r} = \text{cst} \equiv G$$

By integrating twice,

$$U(r) = -\frac{GR^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right] = U_0\left[1 - \left(\frac{r}{R}\right)^2\right]$$



$\rightsquigarrow$ parabolic velocity profile $U(r)$

and $p = Gz$, p decreases linearly along z ($G < 0$ if $U_0 \geq 0$)

The pressure drop is balanced by the wall viscous friction

Poiseuille flow (cont.)

Bulk velocity U_d

$$U_d \equiv \frac{1}{\mathcal{S}} \int_{\mathcal{S}} \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S} = \frac{1}{\pi R^2} \int_0^R U(r) 2\pi r dr = \frac{U_0}{2}$$

Wall shear stress τ_w

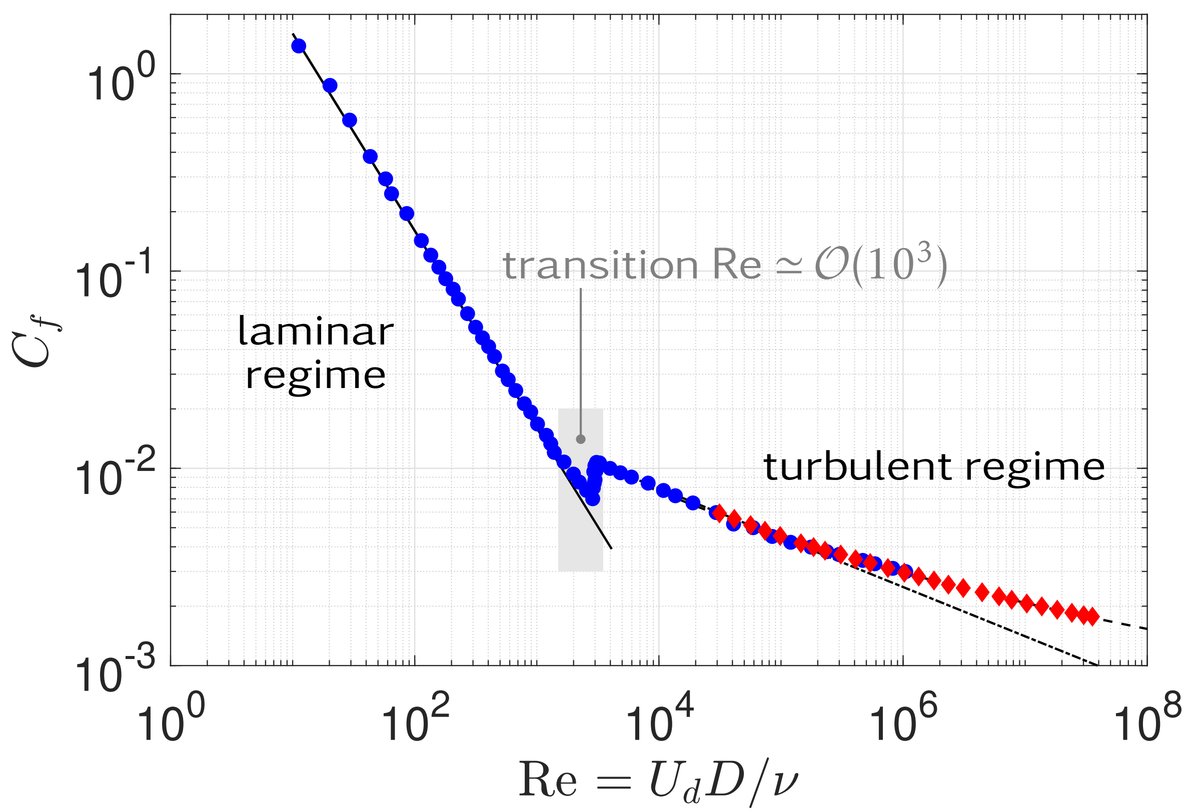
$$\tau_w = \tau_{zr}(r = R) = -2\mu \frac{U_0}{R}$$

Skin-fiction coefficient C_f ($\propto C_D$)

$$C_f \equiv \frac{|\tau_w|}{\rho U_d^2/2} = \frac{16}{\mathrm{Re}_D} \qquad \mathrm{Re}_D \equiv \frac{U_d D}{\nu}$$

The subscript D indicates that the Reynolds number Re_D is built by choosing D as length scale

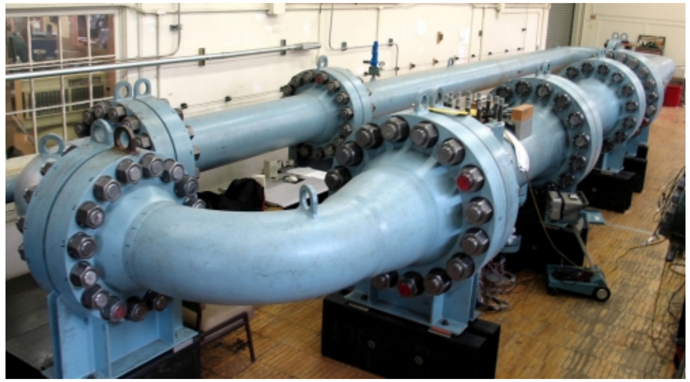
Skin-friction coefficient $C_f = \tau_w/(\rho U_d^2/2)$ for a circular pipe



- —— laminar regime $C_f = 16/\mathrm{Re}$
- -·-·- Blasius' relationship, $C_f \simeq 0.0791\,\mathrm{Re}^{-1/4}$
- - - - $1/C_f^{1/2} \simeq 3.860\log_{10}(\mathrm{Re}\,C_f^{1/2}) - 0.088$

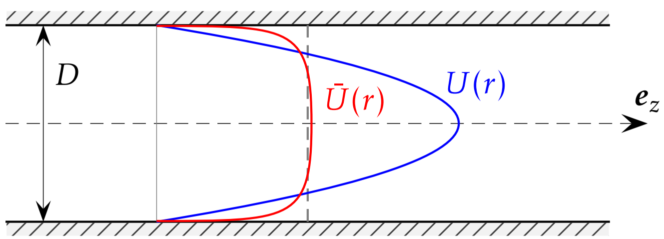
● Oregon facility

♦ Princeton *Superpipe*



McKeon *et al.* (2004) - *Superpipe*, the Reynolds number is increased through the pressure

Laminar versus turbulent regime



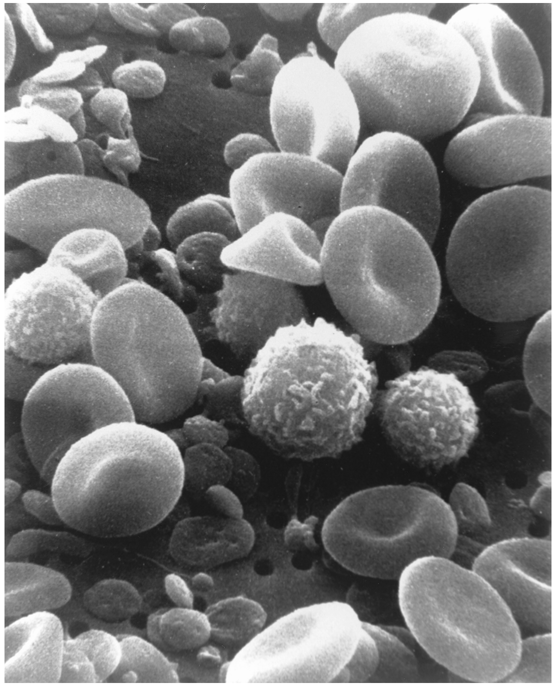
Poiseuille flow (cont.)

Jean-Léonard-Marie Poiseuille (1797-1869)

He was interested in the flow of human blood in narrow tubes :

– Recherches sur la force du coeur aortique (PhD 1828)

– Le mouvement des liquides dans les tubes de petits diamètres (book, 1840)



Scanning electron microscope image of blood cells

(National Cancer Institute, https://visualsonline.cancer.gov/... ...details.cfm?imageid=2129)

Development of a 1-D model for the Poiseuille flow

(energy budget of a continuous-flow system, refer to slide 91)

$$\int_{\mathcal{S}_1} \mathcal{H}\boldsymbol{U}\cdot\boldsymbol{n}\, d\mathcal{S} \equiv H_1 Q_v$$

with $\mathcal{H} \equiv p + \frac{1}{2}\rho U^2$ (local)

$$Q_v(H_1 - H_2) = \mathcal{D}_v$$

Mechanical energy is converted into heat by internal viscous friction

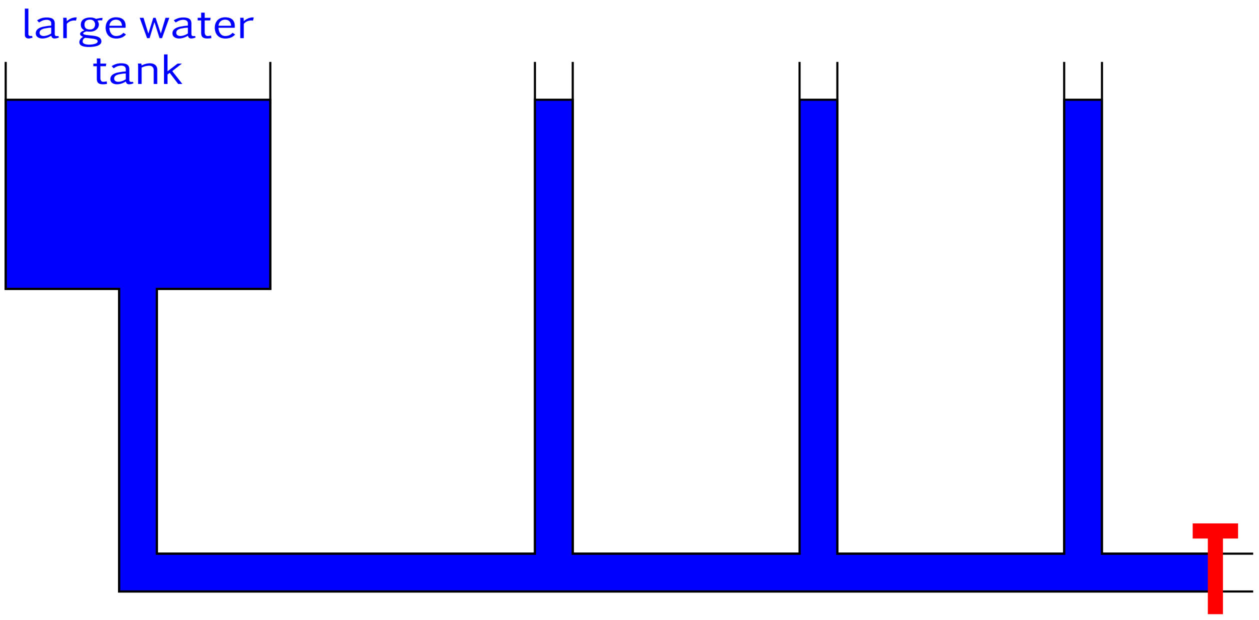
Development of a 1-D model for energy loss : only the pressure p contributes to the variations of H

$$\Delta H = [H]_2^1 = [p]_2^1 = \underbrace{-GL = \frac{4\mu U_0}{R^2}L}_{\text{Poiseuille flow}} = \underbrace{\frac{64}{\mathrm{Re}_D}}_{\psi(\mathrm{Re}_D)} \frac{L}{D}\frac{1}{2}\rho U_d^2$$

More generally,

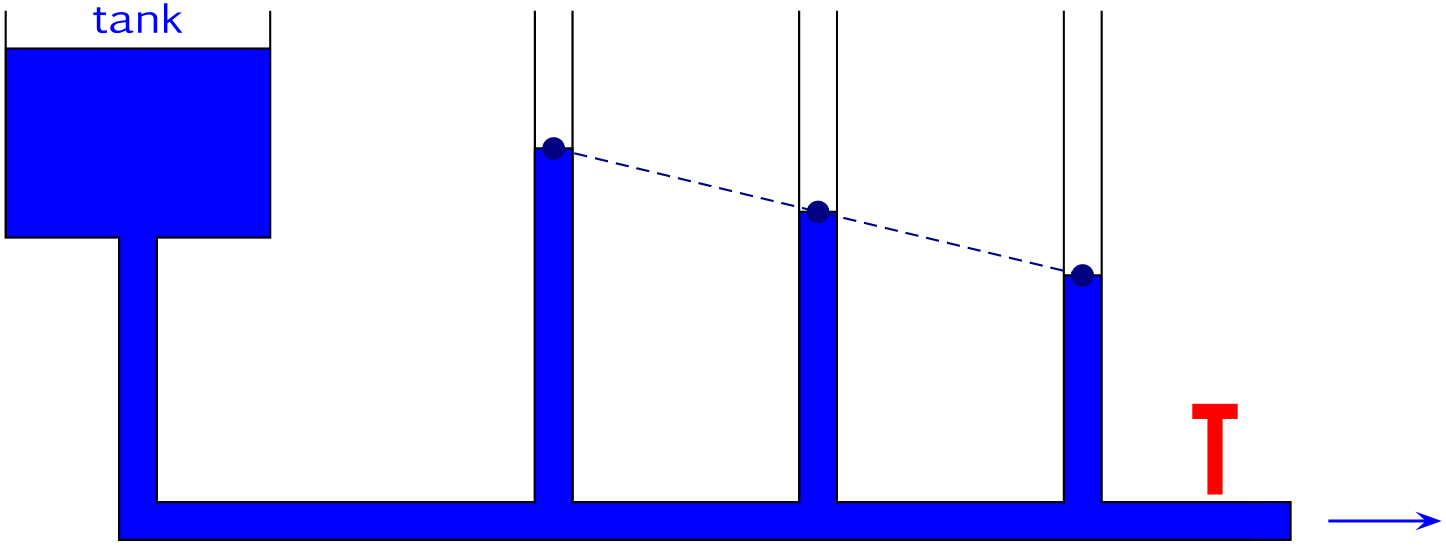
$$\Delta H = \psi(\mathrm{Re}_D)\frac{L}{D}\frac{1}{2}\rho U_d^2 \qquad \psi(\mathrm{Re}_D) = \frac{64}{\mathrm{Re}_D} \quad \text{for Poiseuille flow (laminar)}$$

Short quiz



Water height in each open tube when valve is open ?
(from firefighter training)

Short quiz (cont.)



An illustration of the linear head loss in a pipe!

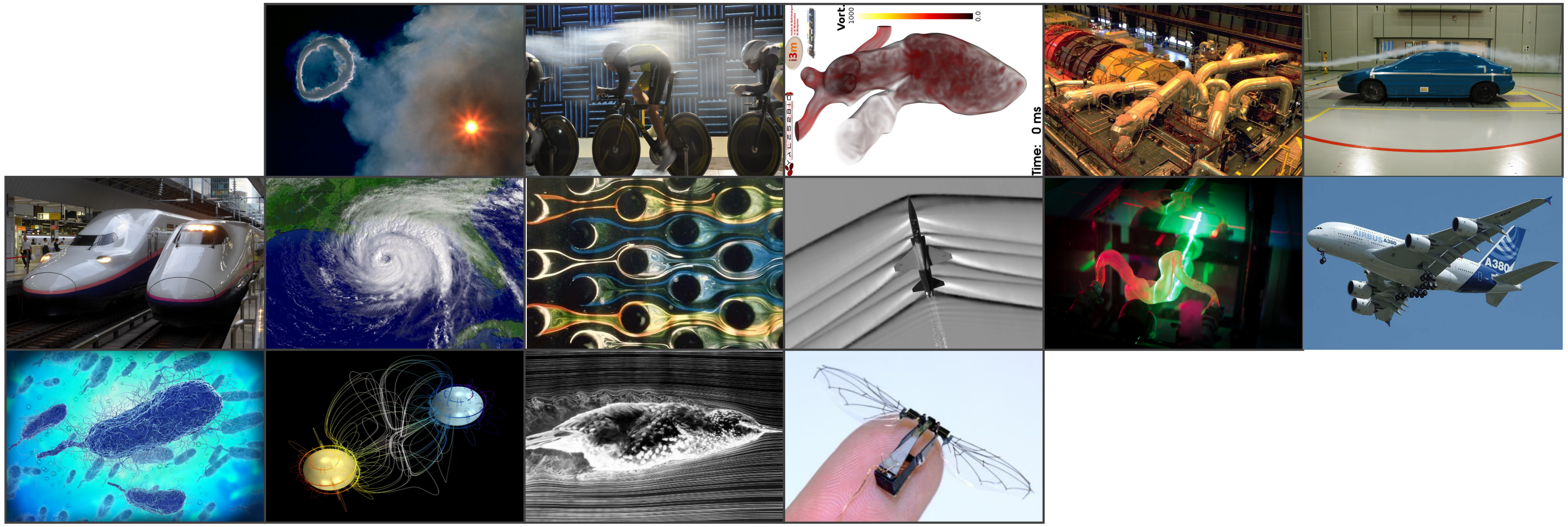
$$\Delta H = [\Delta p - \rho \boldsymbol{g} \cdot \boldsymbol{x}]_2^1 = \psi(\mathrm{Re}) \frac{L}{D} \frac{1}{2} \rho U_d^2$$

- Operational formulation of the Π-theorem
- Expression of some dimensionless parameters :
  Reynolds number, Strouhal number, Drag and Lift coefficients
- Clear interpretation of these dimensionless numbers
  in particular for the Reynolds number (laminar/turbulent regime)
- Poiseuille flow and associated 1-D model for the laminar regime

Outline

General information . . . 2
Chapter 1 : Kinematic properties, fundamental laws, inviscid model . . . 16
Chapter 2 : Newtonian viscous fluid flow . . . 62
Chapter 3 : Dimensional analysis - Reynolds number . . . 98
Chapter 4 : Regimes and flow structures as a function of Reynolds number . . . 129
Chapter 5 : Vorticity . . . 165
Chapter 6 : Turbulent flow . . . 186
Chapter 7 : Energy, thermodynamics and compressible flow . . . 212
Chapter 8 : Heat transfer . . . 246
Chapter 9 : Mixing of fluids . . . 280
Concluding remarks . . . 305
Appendices (online) . . . 307

4 - Flow regimes as a function of Reynolds number



4 - Flow regimes as a function of the Reynolds number

Low Reynolds number flow

- Reynolds number
- Stokes problem
- Stokes force
- Drag coefficient for a sphere

High Reynolds number flow

- Introduction
- Boundary layer
- Streamlined body
- Bluff body
- Delayed boundary layer separation
- Wake and free jet

Laminar boundary layer

- Boundary Layer approximations
- Prandtl equations
- Pressure gradient
- Blasius solution

Key results

Reynolds number

Navier-Stokes equation, see Eq. (9)

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{U}$$

$$\sim \frac{U^2}{L} \qquad \sim \nu \frac{U}{L^2} \qquad \frac{\text{convective term}}{\text{viscous term}} \sim \frac{UL}{\nu} = \mathrm{Re}_L$$

U, L characteristic velocity and length scales of the flow
ν kinematic viscosity of the fluid

The convective term can be neglected when $\mathrm{Re}_L \ll 1$

The Stokes problem ($\mathrm{Re}_L \ll 1$)

Alternative dimensionless form of Navier-Stokes equations ($\neq$ slide 109) by considering the diffusion time L^2/ν to build $\tilde{t}$

$$\tilde{U}_i = \frac{U_i}{U_\infty} \qquad \boxed{\tilde{t} = \frac{t}{L^2/\nu}} \qquad \tilde{p} = \frac{p}{\mu U_\infty / L} = \frac{p}{\rho U_\infty^2}\frac{\rho U_\infty L}{\mu} \qquad \mathrm{Re}_L = \frac{\rho L U_\infty}{\mu}$$

$$\frac{\partial \tilde{\boldsymbol{U}}}{\partial \tilde{t}} + \mathrm{Re}_L\, \tilde{\boldsymbol{U}} \cdot \tilde{\nabla} \tilde{\boldsymbol{U}} = -\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2 \tilde{\boldsymbol{U}}$$

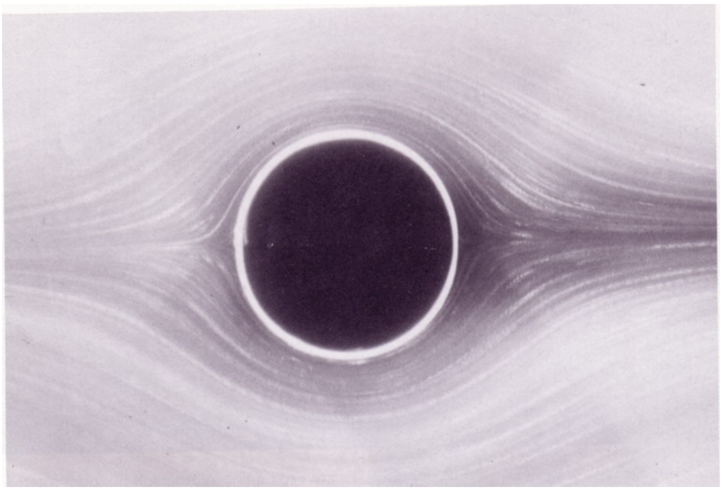
For low Reynolds number flows, the term $\mathrm{Re}_L\, \tilde{\boldsymbol{U}} \cdot \tilde{\nabla} \tilde{\boldsymbol{U}} \ll 1$ is assumed to be small (must be checked by the approximate solution at the end, very restrictive condition in practice) : creeping flow or Stokes flow assumption

Notations : $\tilde{\nabla}^2 \equiv \dfrac{\partial^2}{\partial \tilde{x}_1^2} + \dfrac{\partial^2}{\partial \tilde{x}_2^2} + \dfrac{\partial^2}{\partial \tilde{x}_3^2}$

Stokes problem (cont.)

Stokes equations
$$\begin{cases} \dfrac{\partial \boldsymbol{U}}{\partial t} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{U} \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases}$$

supplemented by initial and boundary conditions. For given boundary conditions, the fully developed flow can reach a steady state as $t \to \infty$.



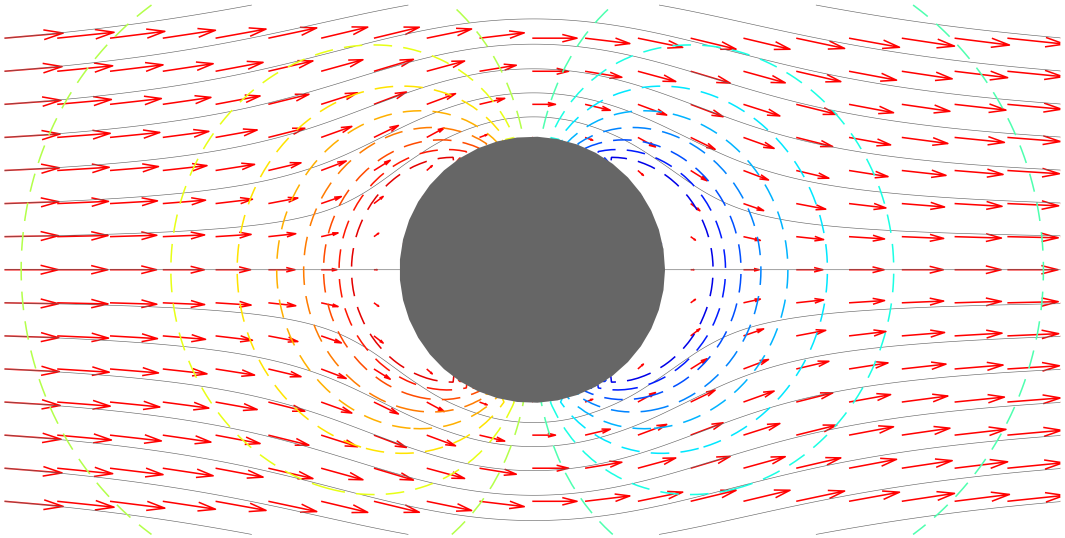
Uniform flow at U_∞ past a circular cylinder of diameter D, $\mathrm{Re}_D = 0.16$

Visualization of streamlines with a water flow and aluminium dust, by S. Taneda (fig. 6 in Van Dyke, 1982)

Steady Stokes flow (creeping flow)

$$\begin{cases} \mu \nabla^2 \boldsymbol{U} = \nabla p \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases} \quad + \text{(steady) boundary conditions in 3-D}$$

Mathematical and physical problem which is more simple to solve that the high Reynolds number case (but 2-D models must be developed with caution !)

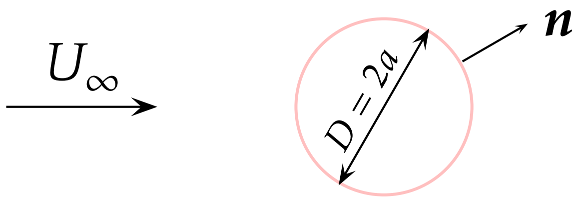


Flow past a sphere : 3-D analytical solution by Stokes (1851) :

velocity field, streamlines and iso-pressure lines are superimposed (from blue − to red +; pressure high in the front and low in the back)

Drag coefficient for a smooth sphere

$$\boldsymbol{F}_{\text{flow} \rightarrow \text{sphere}} = \int_{\mathcal{S}} -p\boldsymbol{n}\, d\mathcal{S} + \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\tau}}} \cdot \boldsymbol{n}\, d\mathcal{S}$$



drag force $F_D = \boldsymbol{F} \cdot \boldsymbol{e}_x$

= pressure drag (form drag) + skin friction drag

- For small Reynolds numbers, namely $\text{Re}_D = U_\infty D/\nu < 1$

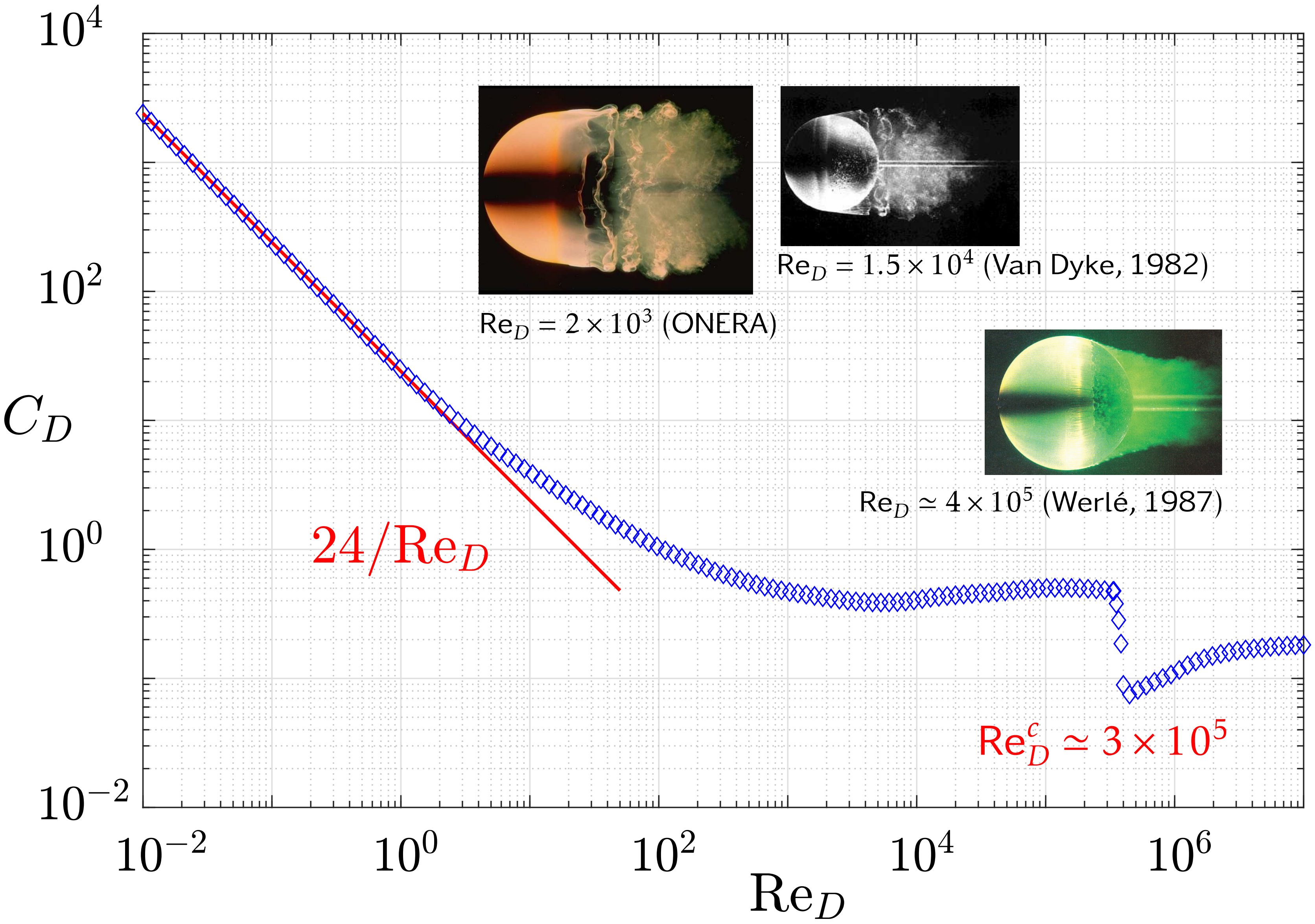
  Drag or Stokes force given by $F_D = 6\pi\mu a U_\infty$ with $D = 2a$

  and $6\pi = 2\pi + 4\pi$, 2π induced by the large front-to-back difference in pressure despite the flow symmetry

  Drag coefficient $C_D = C_D(\text{Re}_D)$

$$C_D \equiv \frac{F_D}{\rho U_\infty^2/2 \times S} = \frac{6\pi\mu a U_\infty}{\rho U_\infty^2/2 \times \pi a^2} = \frac{24}{\rho U_\infty D/\mu} = \frac{24}{\text{Re}_D}$$

- **Drag coefficient for a smooth sphere** (adapted from Clift, Grace, & Weber, 1978)



Application of the Stokes force

Relevant for small particles (dispersion of suspended particles) or high viscous flows (glacier flow)

- Seeding particles for optical measurements in fluid mechanics :

  drops of oliv oil
  $\rho = 970$ kg.m^{-3}, $\nu = 1.5 \times 10^{-5}$ m^2.s^{-1}, $d_p = 1.5\ \mu$m

  drag provided by Stokes force
  Re $= U_\infty d_p / \nu \leq 1 \Longrightarrow U_\infty < 9.7$ m.s^{-1}

- Famous application of the Stokes force : Millikan (1913) in a oil drop experiment to measure the elementary charge, that is the magnitude of the negative electric charge carried by a single electron.

  Further reading :
  Perry, M.F., 2007, Remembering the oil-drop experiment, *Phys. Today*, **56**(5)

Introduction

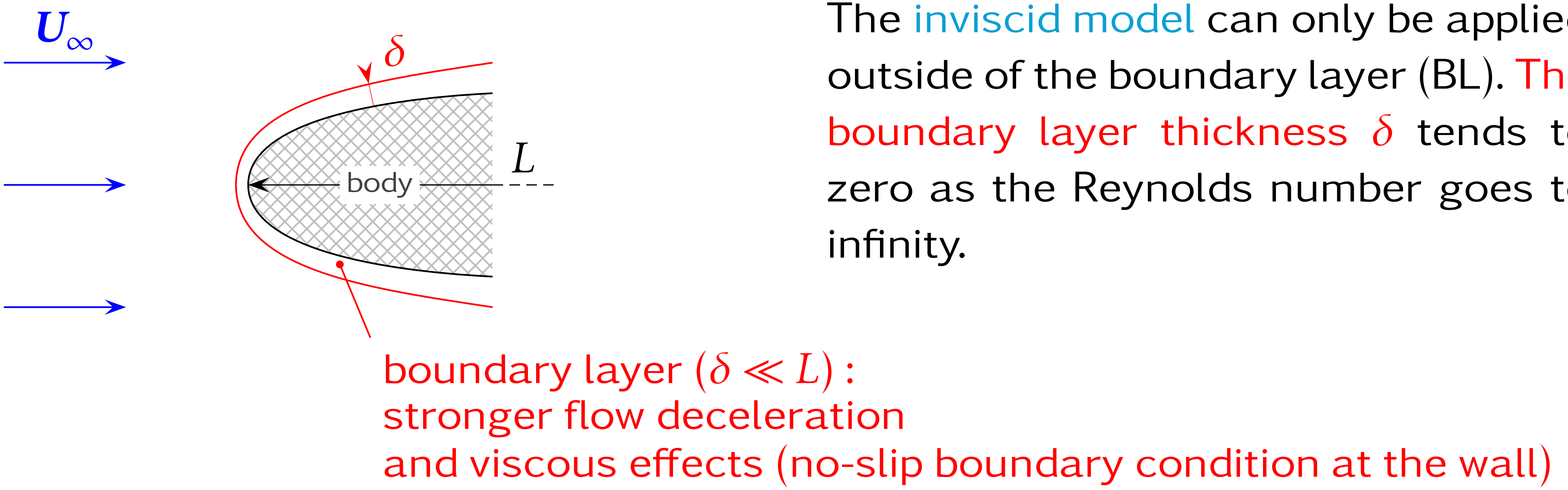
The viscous term is small with respect to the dominating nonlinear convective term, pressure variation scales as ρU^2 and the inviscid model may be applied

$$\begin{cases} \dfrac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\dfrac{1}{\rho} \nabla p \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases}$$

Euler equations

The assumption of a high Reynolds number $\mathrm{Re} \gg 1$ corresponds to an overall view of the flow, but some flow regions have strong velocity gradients associated with viscous effects, in particular near the wall due to the no-slip boundary condition : presence of a boundary layer (thin layer of fluid close to the surface in which viscous effects are dominant)

- **Concept of boundary layer** (introduced by Prandtl in 1904)



The inviscid model can only be applied outside of the boundary layer (BL). The boundary layer thickness δ tends to zero as the Reynolds number goes to infinity.

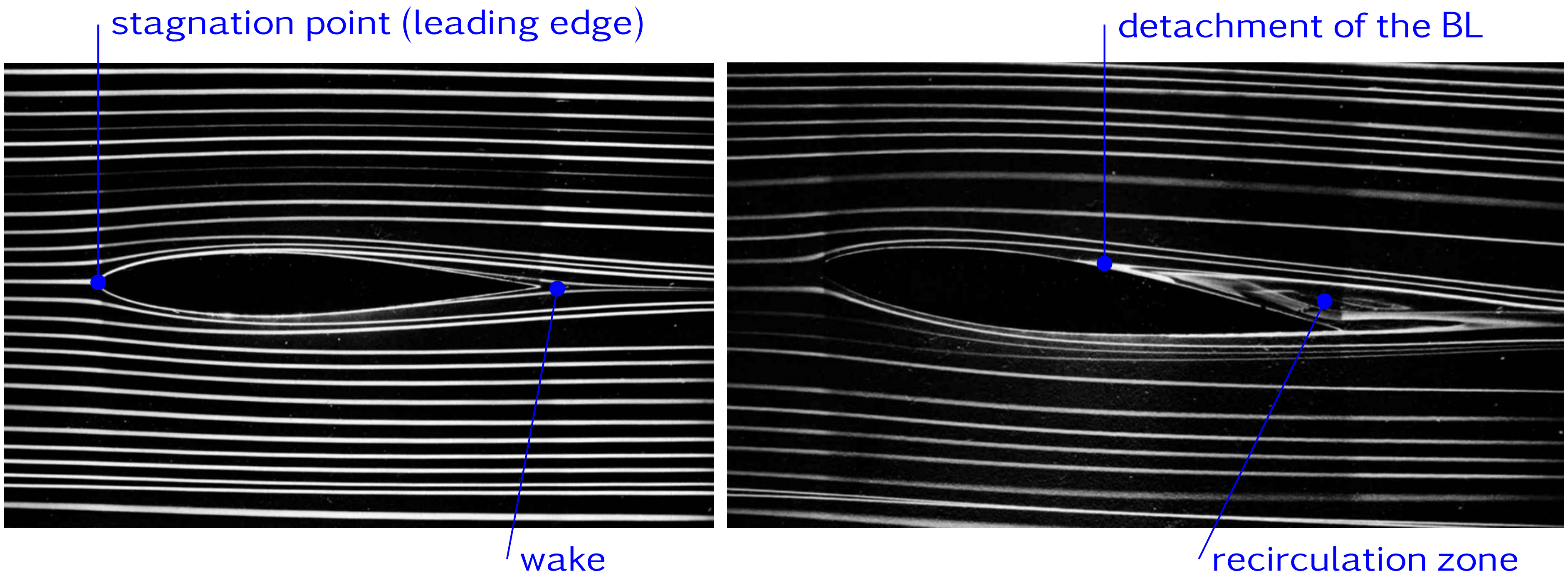
Two additional remarks :

When the Reynolds number decreases, the boundary layer thickens, and there is no particular structuration of the fluid flow for Re $\leq \mathcal{O}(1)$

When the Reynolds number is high enough, a transition from a laminar to a turbulent boundary layer occurs (leading to the drag crisis for instance)

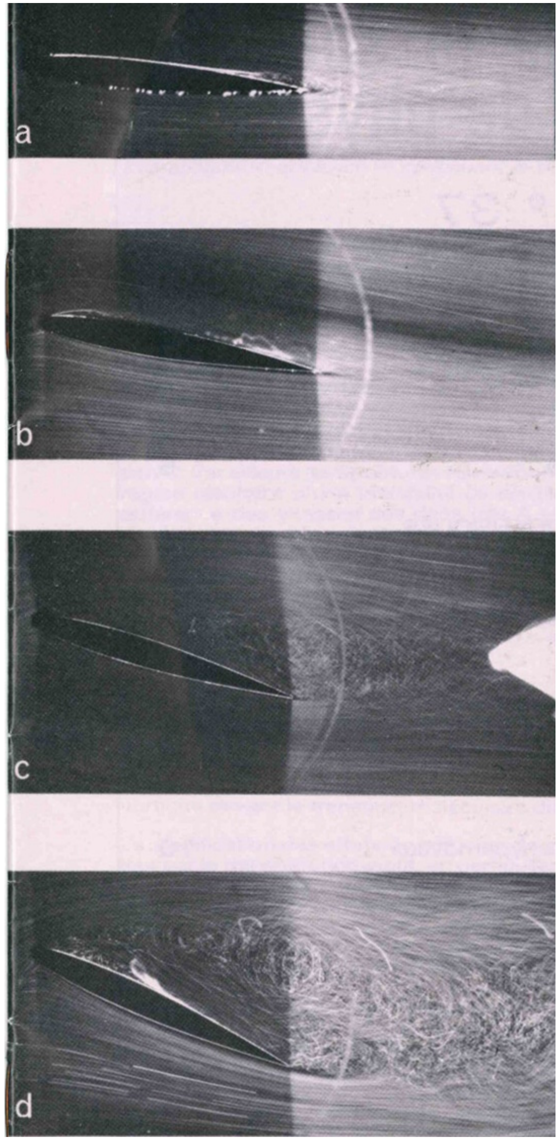
Flow around a streamlined body

For a streamlined or slender body, (a large part of) the boundary layer developing along the surface must remain attached to the body.



Visualization (dye streaks in water) of laminar separation from an airfoil 64A015, $Re_c = 7000$, with zero incidence, and with an angle of attack $\alpha = 5°$ (Werlé, 1974, ONERA)

Separation of the boundary layer

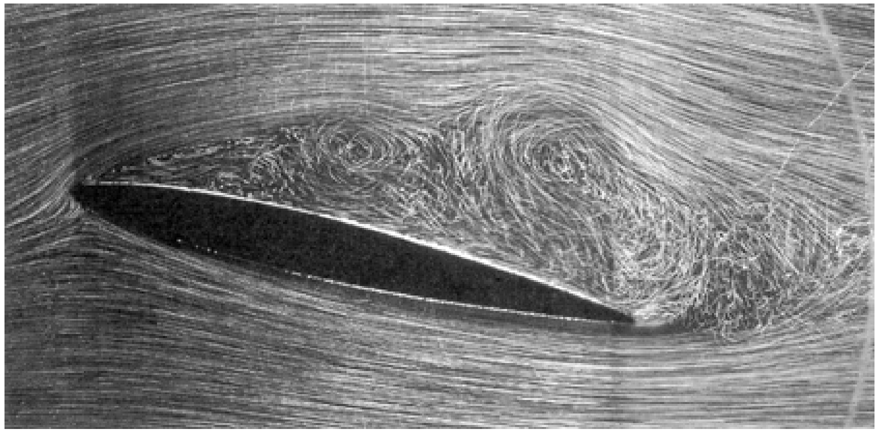


Airfoil (ONERA, D-profile)
chord $c = 100$ mm, $e/l = 10.5\%$
$10^4 \leq \mathrm{Re}_l \leq 5 \times 10^4$

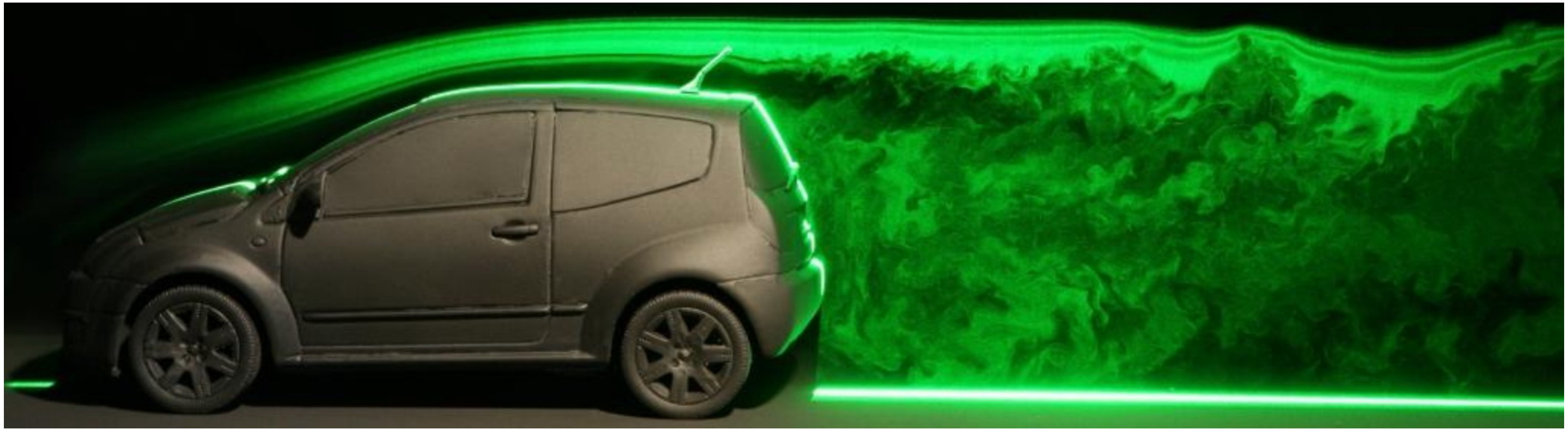
From Werlé, SFP **37** (1980)

streamlined body → bluff body

(a) $\alpha = 5°$
(b) $\alpha = 10°$
(c) $\alpha = 17.5°$
(d) $\alpha = 25°$



- **Characterisation of flow past real road vehicles with blunt afterbodies**



Citroën C2 at scale 1/24, square cross-section of 40 cm × 40 cm in an Eiffel wind tunnel, velocity of a few km/h ; overview of the detached flow by PIV (combined with a permanent lighting)

Courtesy of Mathieu Grandemange (PhD thesis, ENSTA-ParisTech, 2013)

Flow around a bluff body

Drag crisis – critical Reynolds number for which the flow pattern changes, leaving a narrower turbulent wake : the boundary layer on the front surface becomes turbulent

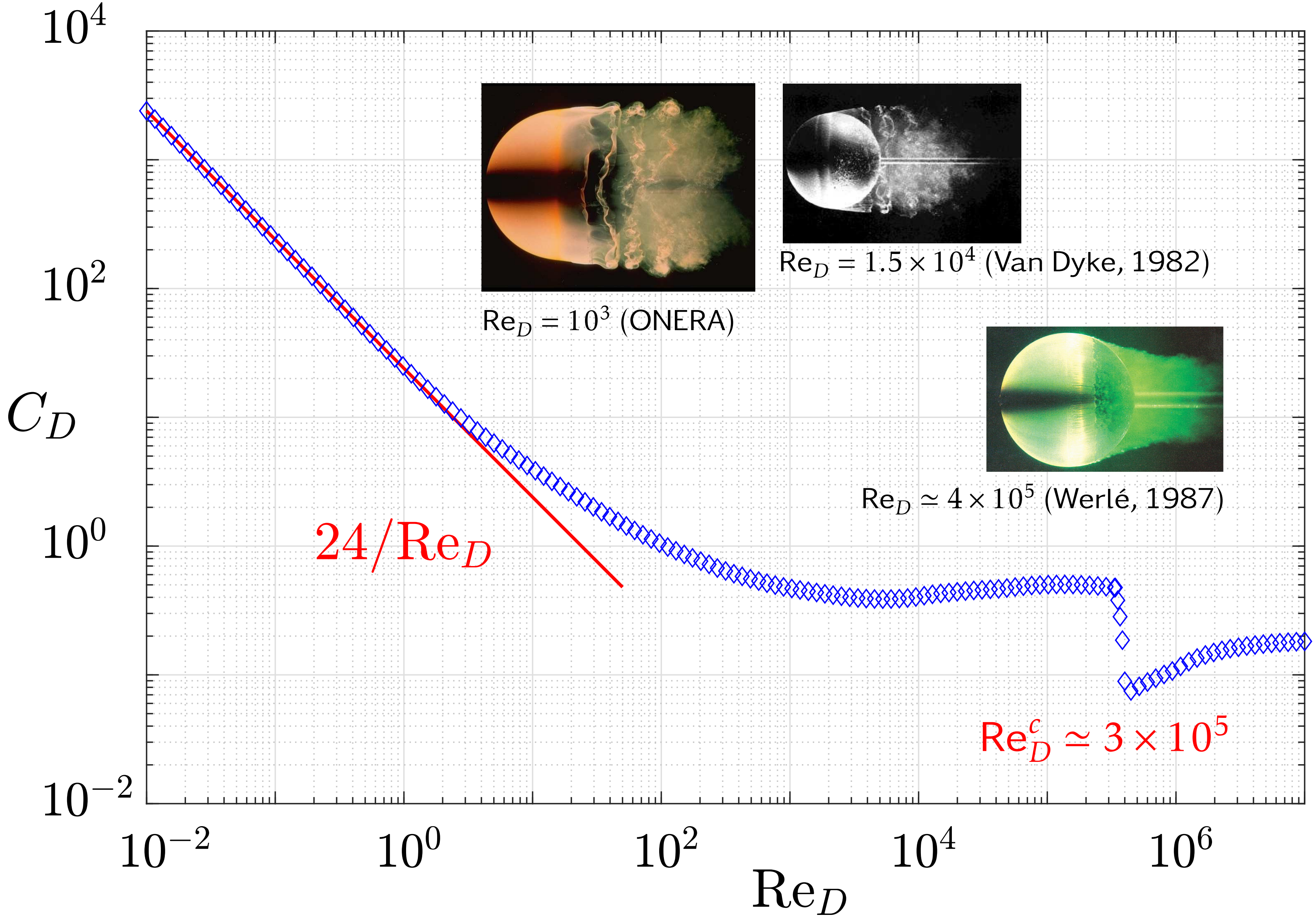
The reduction in form/pressure drag (induced by a narrower wake) is more important than the increase in friction drag (induced by the laminar-turbulent transition of the boundary layer)

$$\boldsymbol{F}_{\text{flow}\rightarrow\text{body}} = \int_{\mathcal{S}} -p\boldsymbol{n}\, d\mathcal{S} + \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\tau}}}\cdot\boldsymbol{n}\, d\mathcal{S} \qquad (\boldsymbol{n} \text{ outward normal of the body})$$

drag force $F_D = \boldsymbol{F}\cdot\boldsymbol{e}_x$

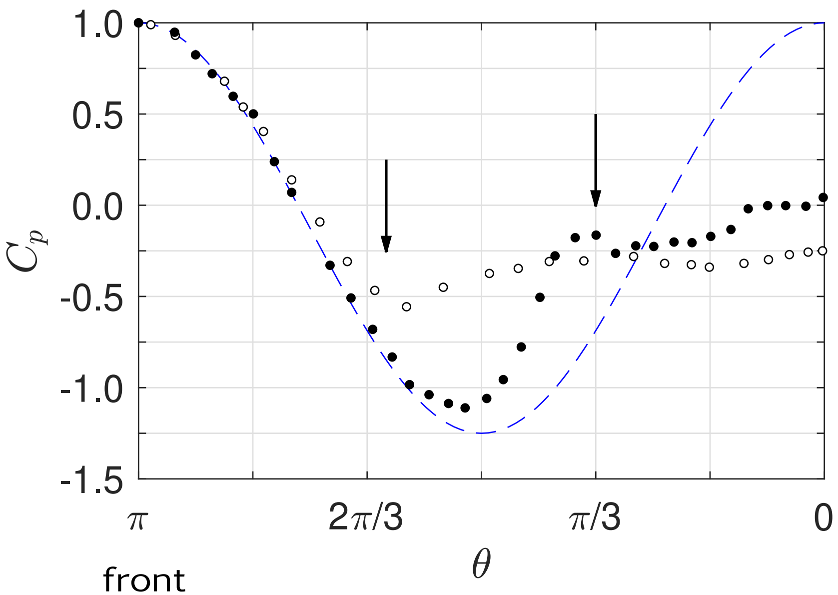
pressure drag (form drag) + skin friction drag

- **Drag coefficient for a smooth sphere** (adapted from Clift, Grace, & Weber, 1978)



Pressure distribution around a sphere

$$C_p = \frac{p - p_\infty}{q_\infty} \qquad q_\infty = \frac{1}{2}\rho U_\infty^2 \qquad C_p \text{ pressure coefficient}$$

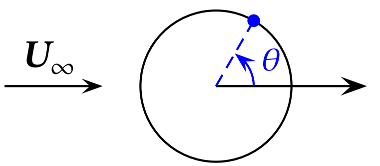


--- potential flow solution (inviscid solution), $C_p = 1 - (9/4)\sin^2\theta$

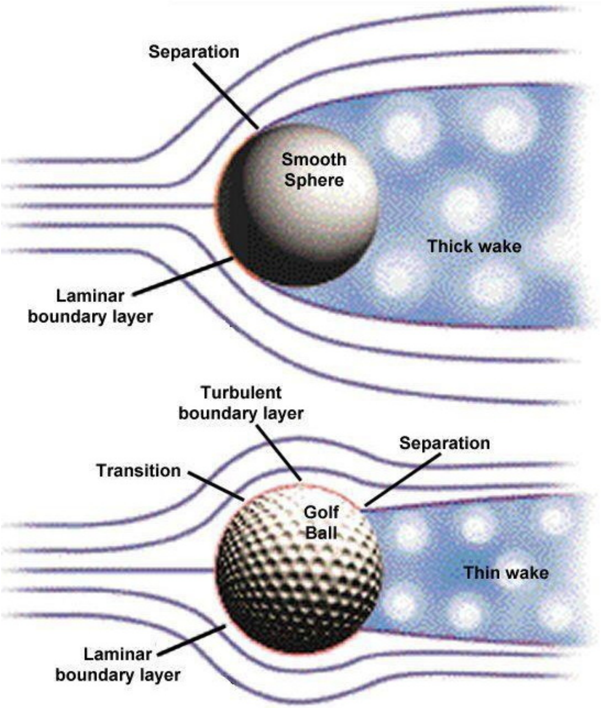
○ Expts by Achenbach (1972) for $\mathrm{Re}_D \simeq 1.65 \times 10^5$ (subcritical, before the drag crisis)

● Expts by Achenbach (1972) for $\mathrm{Re}_D > \mathrm{Re}_D^c$ (supercritical, after the drag crisis)

(See also viscous Stokes' solution for $\mathrm{Re} \ll 1$)



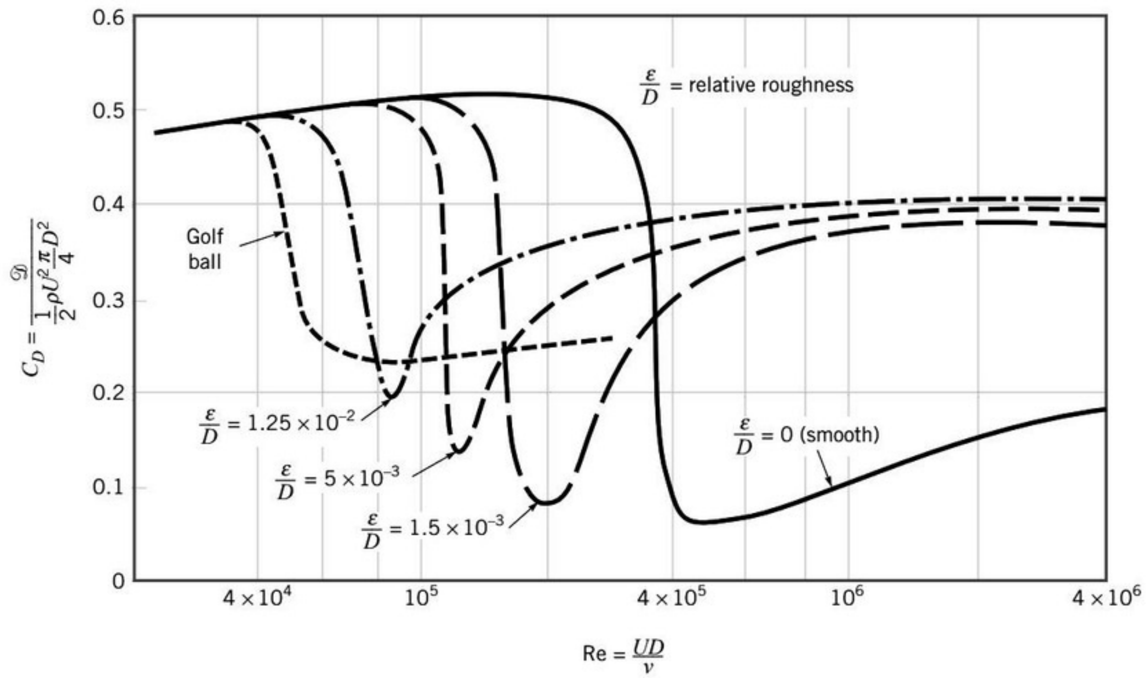
Why golf balls have dimples?



Moin & Kim, 1997, *Scientific American*
Goff, E., 2010, Power and spin in the beautiful game, *Phys. Today*, **67**(3)

Drag coefficient of spheres with varying surface roughness. The drag crisis or sudden drop in drag as Reynolds number increases occurs when the boundary layer transitions to turbulence upstream of separation

$D = 4.3$ cm, $U \simeq 67$ m.s^{-1}, Re $\simeq 1.9 \times 10^5$ (professional golfer)



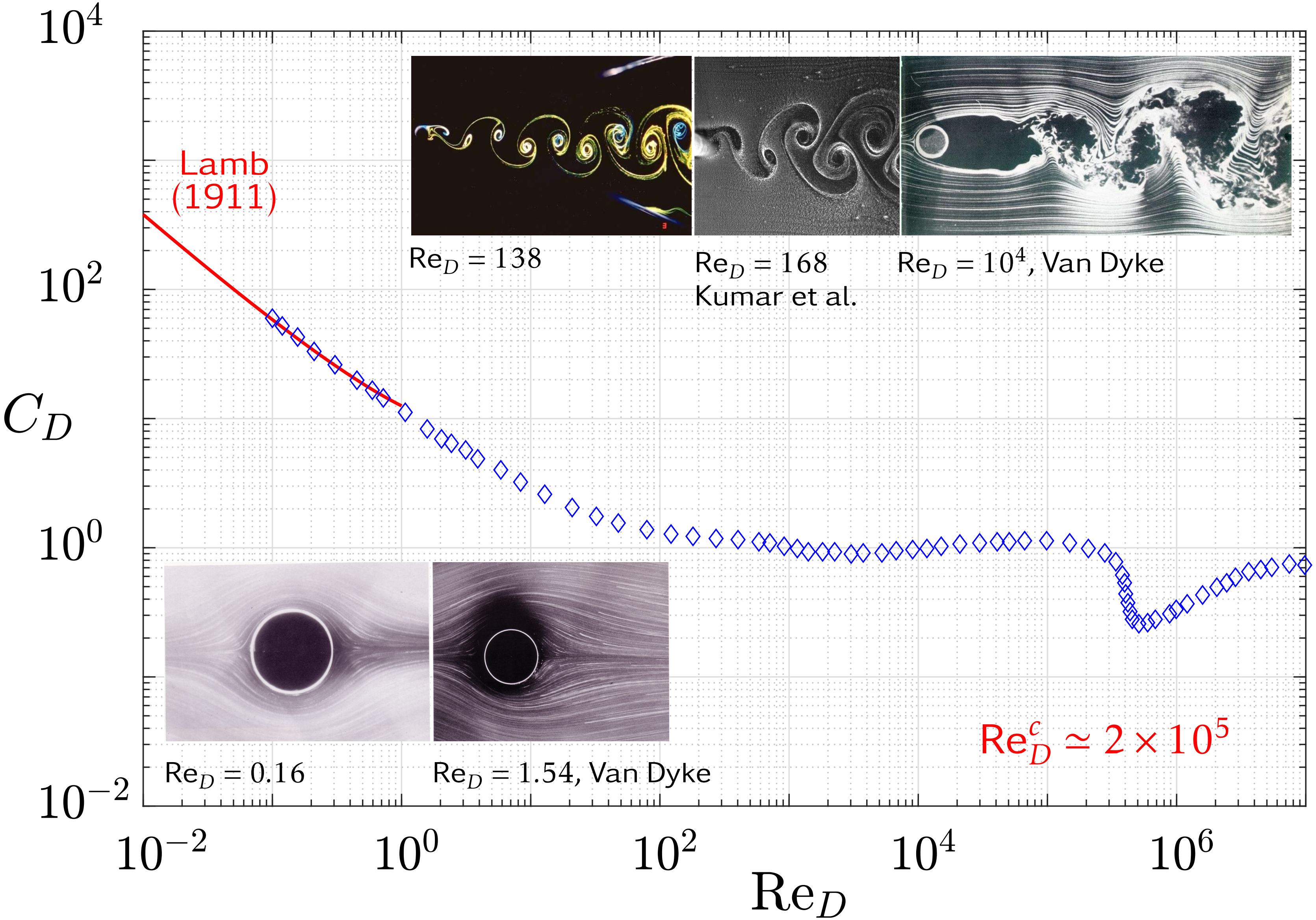
Munson *et al.*, 2014, *Fundamentals of fluid mechanics*

- **Vortex generators**
  for delaying boundary layer separation (generation of tip vortex which will promote transition)

Beechcraft Baron
(twin-engined piston aircraft)

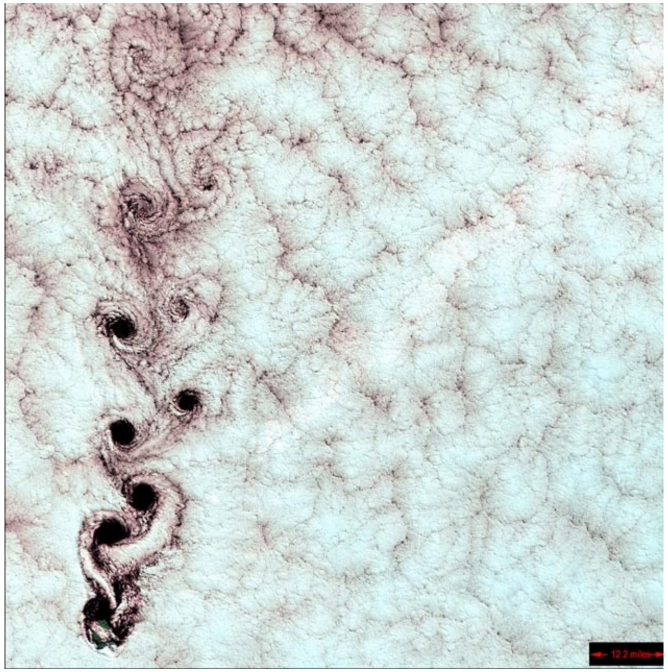
Boeing-777-3ZG-ER http://www.airliners.net/

- **Drag coefficient for a smooth cylinder**

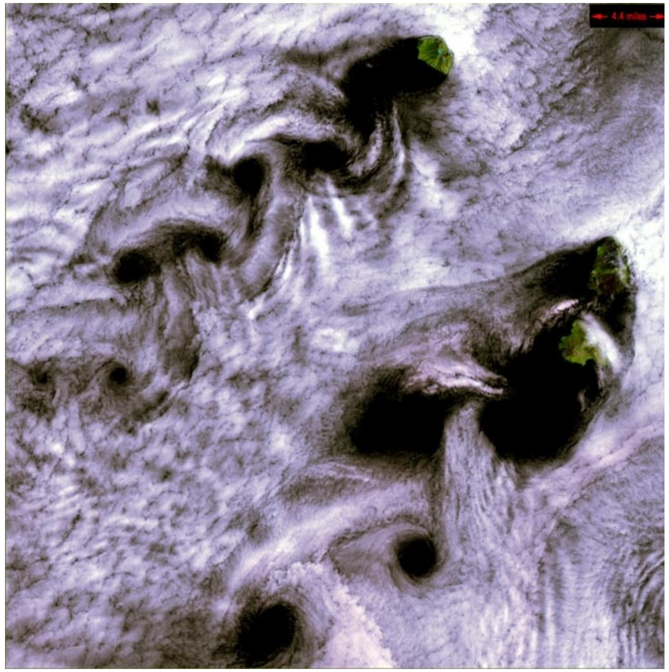


$\mathrm{Re}_D = 171$

Wake behind an obstacle : Kármán's vortex street

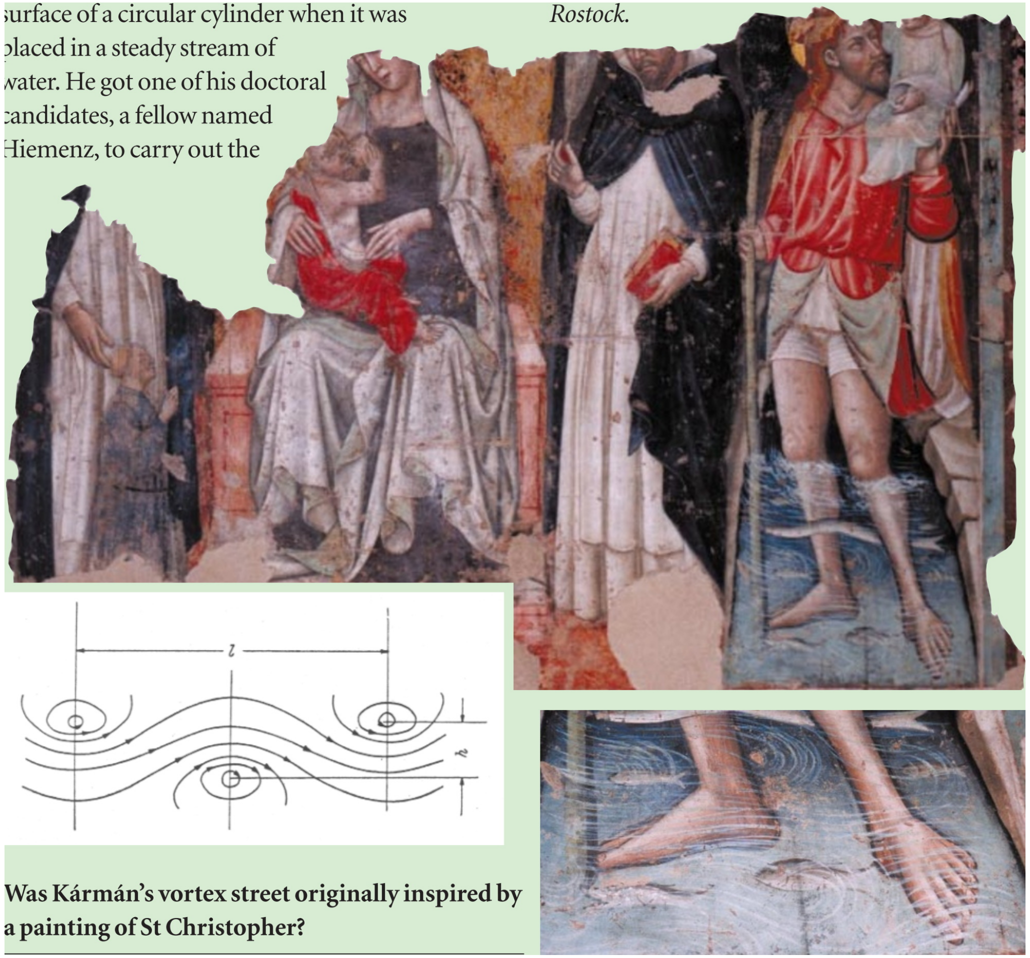


Alexander Selkirk Island in the southern Pacific Ocean.



These Karman vortices formed over the islands of Broutona, Chirpoy, and Brat Chirpoyev ("Chirpoy's Brother"), all part of the Kuril Island chain found between Russia's Kamchatka Peninsula and Japan.

Kármán's vortex street (1911)



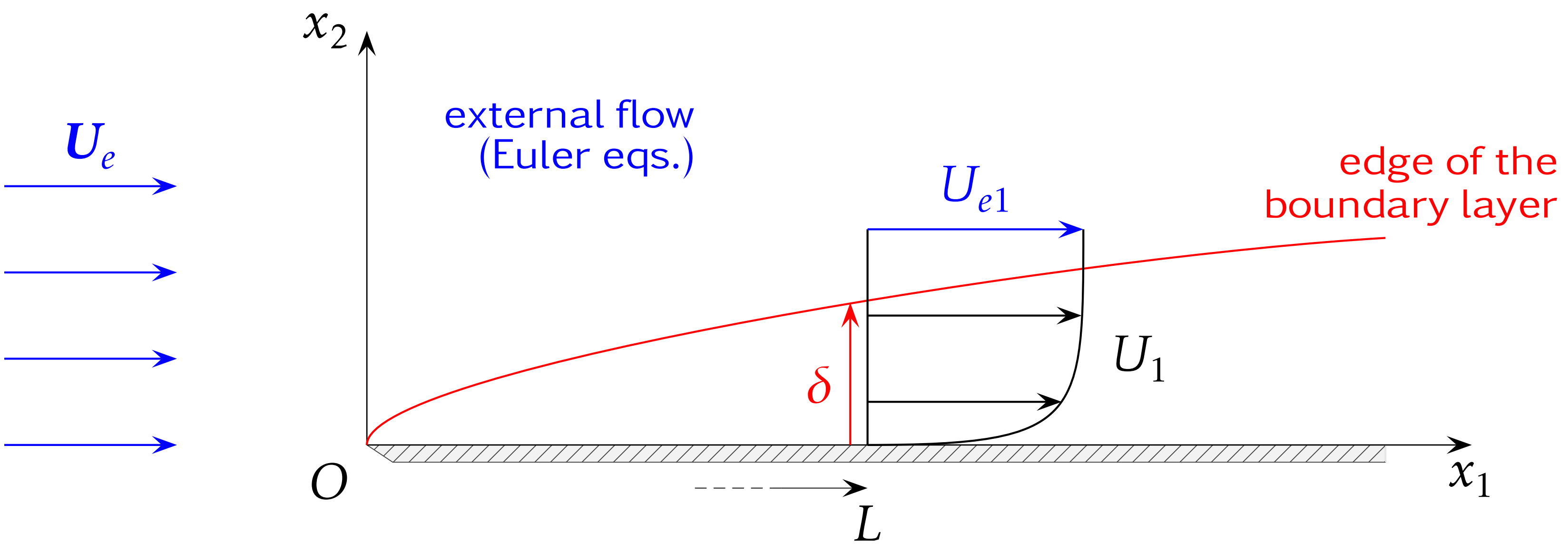
Mizota *et al.*, 2000, Nature, **404**

Legend has it that Theodore von Kármán was motivated by a painting of St. Christopher (Church of St. Dominic in Bologna, Italy) to study his vortex street ...

See also the vortex shedding frequency

Boundary layer developing on a flat plate

2-D laminar steady flow (fully turbulent for $\mathrm{Re}_\delta = U_{e1}\delta/\nu \geq 2800$ or equivalently $\mathrm{Re}_L \geq 3\times 10^5$)



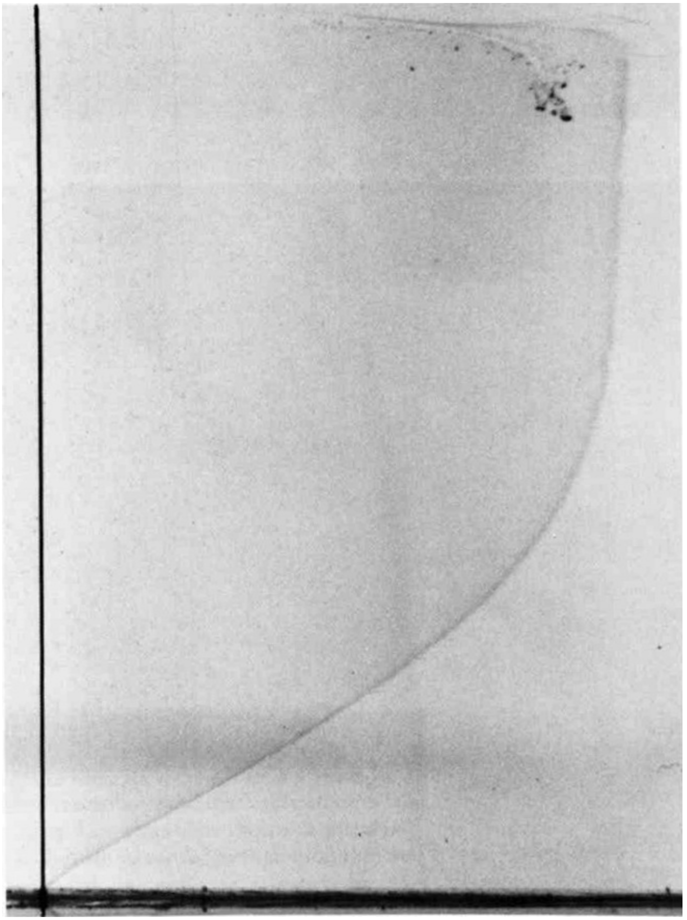
Characteristic scales

– boundary layer thickness δ
– longitudinal length L
– external velocity $(\mathcal{U}_1,\mathcal{U}_2)$

Assumption : $\delta \ll L$

(typically, $\delta \simeq 1$ cm for $L \simeq 1$ m along a commercial aircraft wing)

Boundary layer developing on a flat plate



Visualization of the laminar boundary layer, $U_{e1} = 9$ m.s^{-1}, $\mathrm{Re}_L \simeq 500$, $\delta_1 = 5$ mm

(L distance from the plate leading edge)

A fine tellurium wire perpendicular to the plate at the left is subjected to an electrical impulse of a few milliseconds duration. A chemical reaction produces a slender colloidal cloud, which drifts with the stream and is photographed a moment later to define the velocity profile.

(Van Dyke, 1982, fig. 30)

Boundary layer approximations

Navier-Stokes equations for a $2-D$ laminar flow

$$\begin{cases} \dfrac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U} = -\dfrac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{U} \\ \nabla\cdot\boldsymbol{U} = 0 \end{cases} \qquad \text{with} \qquad \begin{cases} U_1 = U_1(x_1, x_2) \\ U_2 = U_2(x_1, x_2) \\ U_3 = 0 \end{cases}$$

Conservation of mass

Both terms must be of the same order of magnitude,

$$\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} = 0$$

$$\sim \frac{\mathcal{U}_1}{L} \qquad \sim \frac{\mathcal{U}_2}{\delta} \qquad \Longrightarrow \qquad \boxed{\mathcal{U}_2 \sim \frac{\delta}{L}\mathcal{U}_1}$$

Boundary layer approximations (cont.)

Navier-Stokes equation along x_1

$$\begin{cases} U_1 \dfrac{\partial U_1}{\partial x_1} \; + \; U_2 \dfrac{\partial U_1}{\partial x_2} \; = \; -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_1} \; + \; \nu\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) U_1 \\ \sim \dfrac{\mathcal{U}_1^2}{L} \qquad \sim \dfrac{\mathcal{U}_1^2}{L} \qquad\qquad\qquad \sim \nu \dfrac{\mathcal{U}_1}{L^2} \quad \sim \nu \dfrac{\mathcal{U}_1}{\delta^2} \end{cases}$$

The first viscous term can be neglected with respect to the second one in $\sim \nu\, \mathcal{U}_1/\delta^2$.
The convective term on the left hand side must balance the viscous term on the right hand side : this term cannot be dominant, would lead to an inviscid solution ; cannot be negligible, would lead to a Stokes flow ($\mathrm{Re}_L \leq 1$)

$$\frac{\mathcal{U}_1^2}{L} \sim \nu \frac{\mathcal{U}_1}{\delta^2} \quad \Longrightarrow \quad \left(\frac{\delta}{L}\right)^2 \sim \frac{\nu}{\mathcal{U}_1 L} \quad \Longrightarrow \quad \boxed{\frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2}} \qquad \mathrm{Re}_L \equiv \frac{\mathcal{U}_1 L}{\nu}$$

The Reynolds number must therefore be high enough to ensure that $\delta \ll L$, and moreover, $\delta \sim \sqrt{L}$

Boundary layer approximations (cont.)

Navier-Stokes equation along x_2

$$\begin{cases} U_1 \dfrac{\partial U_2}{\partial x_1} & + & U_2 \dfrac{\partial U_2}{\partial x_2} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_2} & + & \nu\left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}\right) U_2 \\ \sim \left(\dfrac{\delta}{L}\right)^2 \dfrac{\mathcal{U}_1^2}{\delta} & & \sim \left(\dfrac{\delta}{L}\right)^2 \dfrac{\mathcal{U}_1^2}{\delta} & & & & \sim \nu \dfrac{\delta}{L}\dfrac{\mathcal{U}_1}{L^2} \quad \sim \nu \dfrac{\delta}{L}\dfrac{\mathcal{U}_1}{\delta^2} \\ \sim \dfrac{1}{\mathrm{Re}_L}\dfrac{\mathcal{U}_1^2}{\delta} & & \sim \dfrac{1}{\mathrm{Re}_L}\dfrac{\mathcal{U}_1^2}{\delta} & & & & \sim \dfrac{1}{\mathrm{Re}_L}\dfrac{\mathcal{U}_1^2}{\delta} \end{cases}$$

(All the terms are smaller by a factor $(\delta/L)^2 \sim 1/\mathrm{Re}_L$ wrt the projection along x_1, in particular for the pressure term)

The pressure term must have the same order of magnitude to ensure the balance. By integration in the transverse direction with $p \sim \rho\mathcal{U}_1^2$,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_2} \sim \frac{1}{\mathrm{Re}_L}\frac{\mathcal{U}_1^2}{\delta} \qquad \Longrightarrow \qquad [\,p\,]_0^\delta \sim \frac{1}{\mathrm{Re}_L}\,\rho\mathcal{U}_1^2$$

The pressure change across the boundary layer is thus very small, and can be neglected : $p(x_1,x_2) \simeq p(x_1)$ and $\partial p/\partial x_2 \simeq 0$. For a given location x_1, $p = p(x_2=\delta) = p_e$ and $p = p(x_2 = 0) = p_e$ also!

Boundary layer approximations (cont.)

Finally, the gouverning equations read

$$\begin{cases} U_1 \dfrac{\partial U_1}{\partial x_1} + U_2 \dfrac{\partial U_1}{\partial x_2} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_1} + \nu \dfrac{\partial^2 U_1}{\partial x_2^2} \\ \dfrac{\partial p}{\partial x_2} = 0 \\ \dfrac{\partial U_1}{\partial x_1} + \dfrac{\partial U_2}{\partial x_2} = 0 \end{cases}$$

associated with the following orders of magnitude

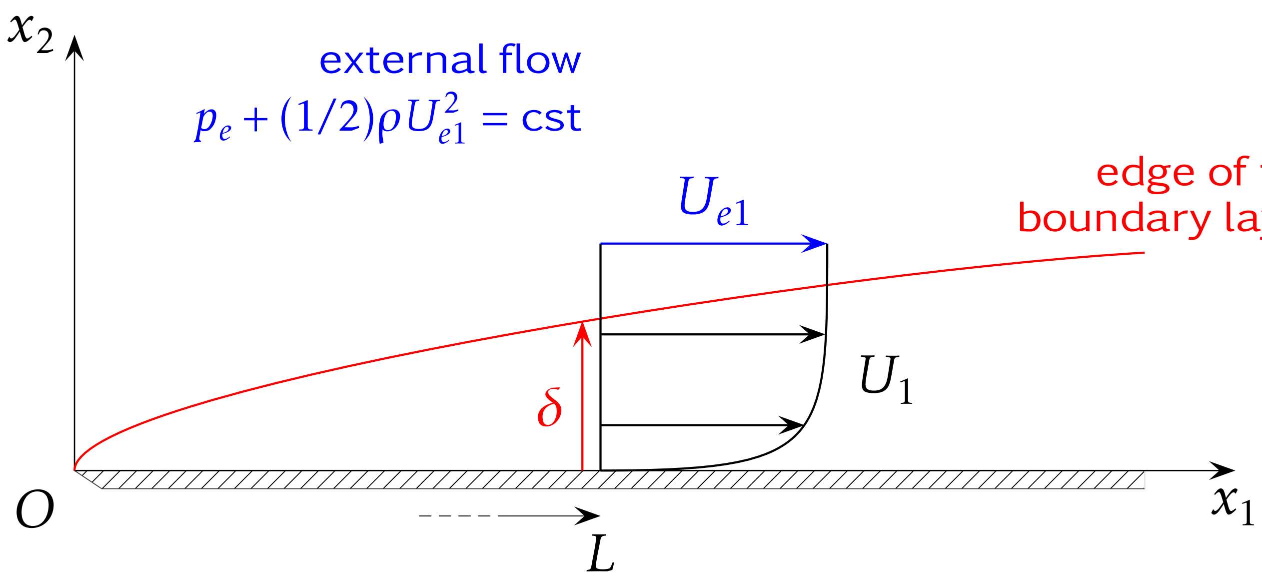
$$U_1 \sim \mathcal{U}_1 \qquad U_2 \sim \frac{\delta}{L}\mathcal{U}_1 \qquad \frac{\delta}{L} \sim \mathrm{Re}_L^{-1/2} \qquad \mathrm{Re}_\delta = \mathrm{Re}_L^{-1/2}\frac{\mathcal{U}_1 L}{\nu} \sim \mathrm{Re}_L^{1/2}$$

Re_δ is also an important parameter involved in the boundary-layer stability theory (transition between a laminar and a turbulent regime) ; By definition, $1 \ll \mathrm{Re}_\delta \ll \mathrm{Re}_L$

Boundary layer approximations (cont.)

The pressure is constant across the boundary layer, $p = p(x_1) = p_e(x_1)$, which means that the pressure is imposed by the external flow

$$\frac{\partial p}{\partial x_1} = \frac{dp_e}{dx_1} = -\rho U_{e1} \frac{dU_{e1}}{dx_1}$$



- **Prandtl's equations (1904)**

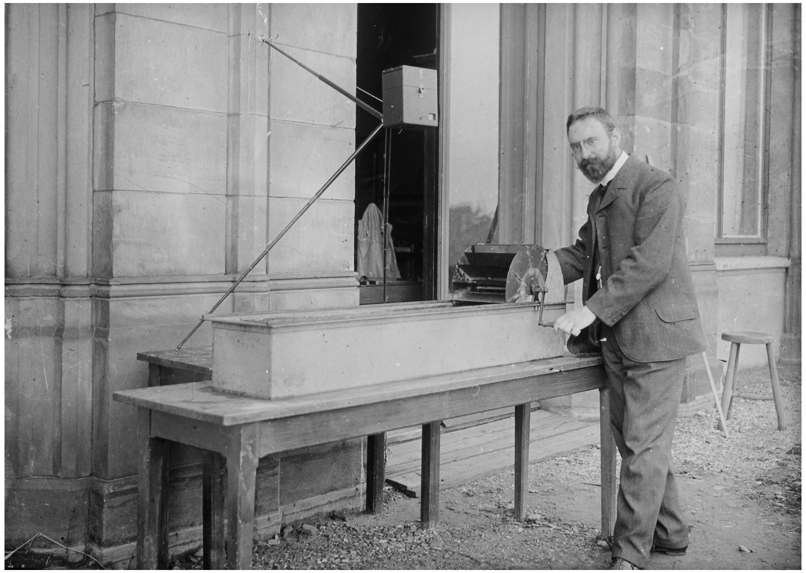
$$\left\{\begin{array}{l} U_1 \dfrac{\partial U_1}{\partial x_1} + U_2 \dfrac{\partial U_1}{\partial x_2} = U_{e1} \dfrac{dU_{e1}}{dx_1} + \nu \dfrac{\partial^2 U_1}{\partial x_2^2} \\ \dfrac{\partial U_1}{\partial x_1} + \dfrac{\partial U_2}{\partial x_2} = 0 \end{array}\right. \qquad \begin{array}{l} U_1 = U_2 = 0 \text{ at } x_2 = 0 \\ U_1 \rightarrow U_{e1} \text{ as } x_2 \rightarrow \infty \end{array} \tag{15}$$

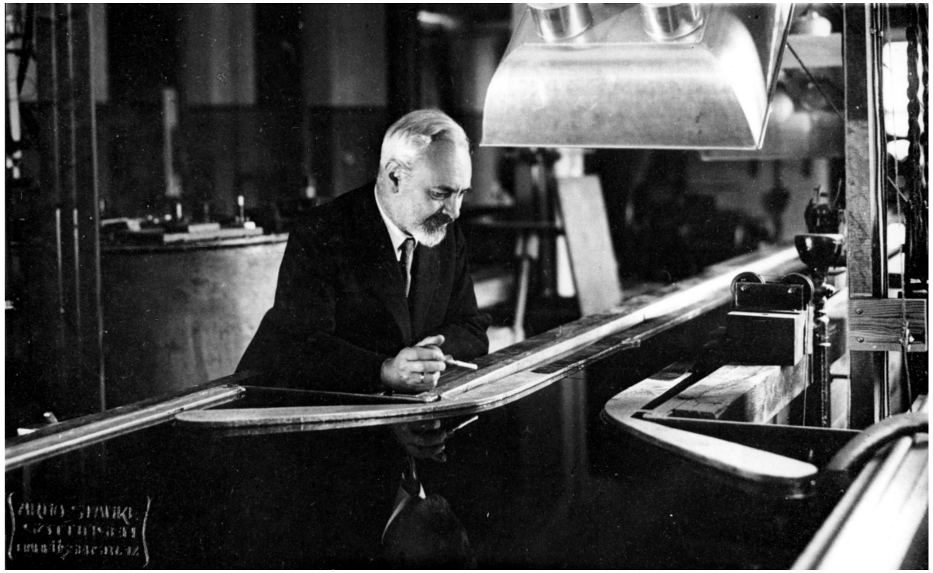
The Prandtl equations are a parabolized form of the Navier-Stokes Eqs.
Formally, one has

$$\frac{\partial U_1}{\partial x_1} = f\left(U_1, U_2, \frac{\partial U_1}{\partial x_2}, \frac{\partial^2 U_1}{\partial x_2^2}\right) \qquad U_2 = -\int_0^{x_2} \frac{\partial U_1}{\partial x_1} dx_2$$

where the external flow $(\boldsymbol{U}_e, p_e)$ is assumed to be known. An initial velocity profile $U_1(x_2)$ at a given location x_1 is required to start the integration along x_1.

Ludwig Prandtl (1875-1953)





Ludwig Prandtl with his water tunnel in 1903 (for flow visualization of large structures using particle tracers)

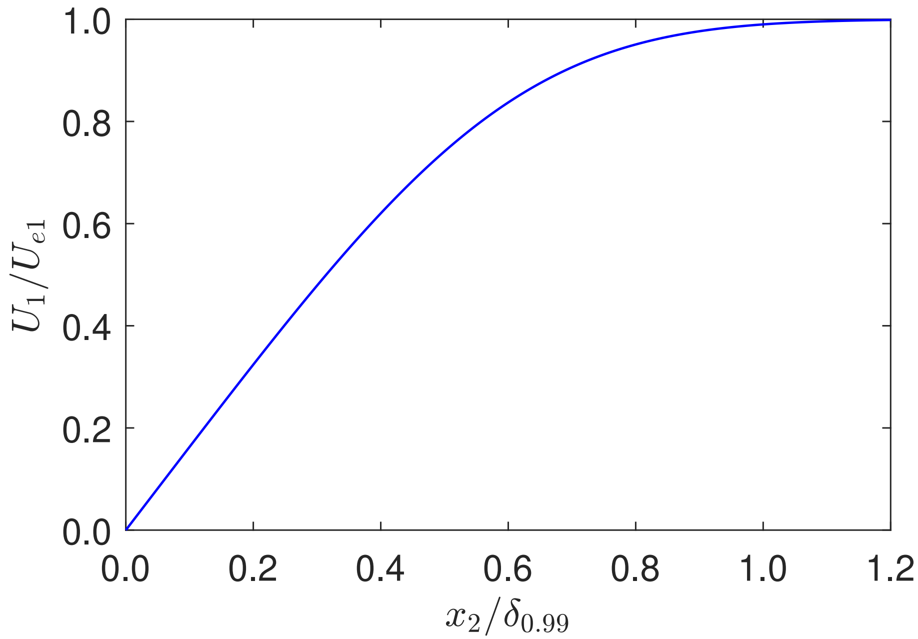
and in the mid to late 1930s

A voyage through Turbulence
edited by, P. A. Davidson, Y Kaneda, H.K. Moffatt & K.R. Sreenivasan
(Cambridge University Press, 2011)

Anderson Jr, D.J., 2005, *Physics Today*, **58**(12), 42–48.

The Blasius solution (PhD 1907)

Self-similar solution for laminar flow over a flat plate (U_{e1} = cst, no pressure gradient), the profile can be compared with the visualization slide 152



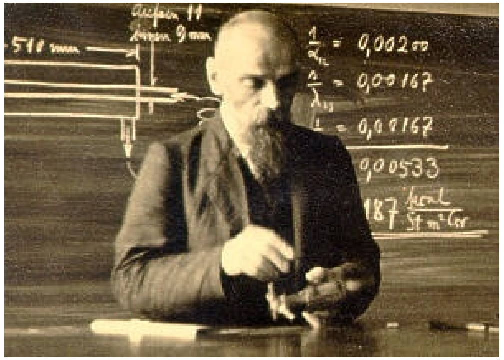
Boundary layer thickness δ

$$\frac{\delta}{x_1} \simeq 4.92 \frac{1}{\sqrt{\mathrm{Re}_{x_1}}} \qquad (\delta \sim x_1^{1/2})$$

Friction coefficient C_f (one face plate)

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U_{e1}^2} \sim \frac{\mu \frac{\mathcal{U}_1}{\delta}}{\rho \mathcal{U}_1^2} \sim \frac{\nu}{\mathcal{U}_1 \delta} \sim \frac{1}{\mathrm{Re}_l^{1/2}}$$

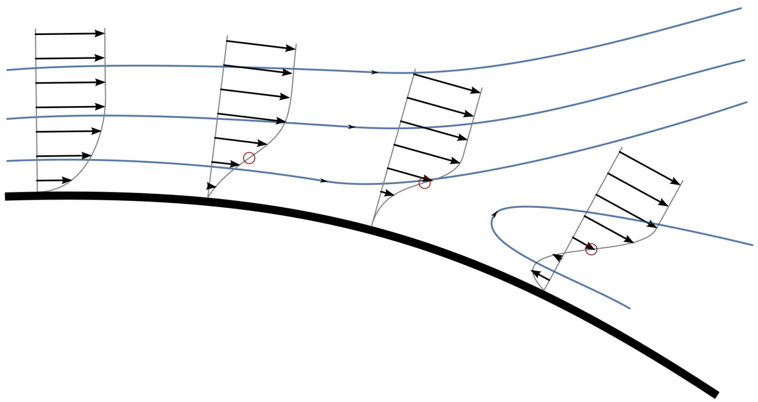
$$C_f \simeq \frac{0.664}{\sqrt{\mathrm{Re}_{x_1}}}$$



Heinrich Blasius (1883-1970), first PhD student of Prandtl

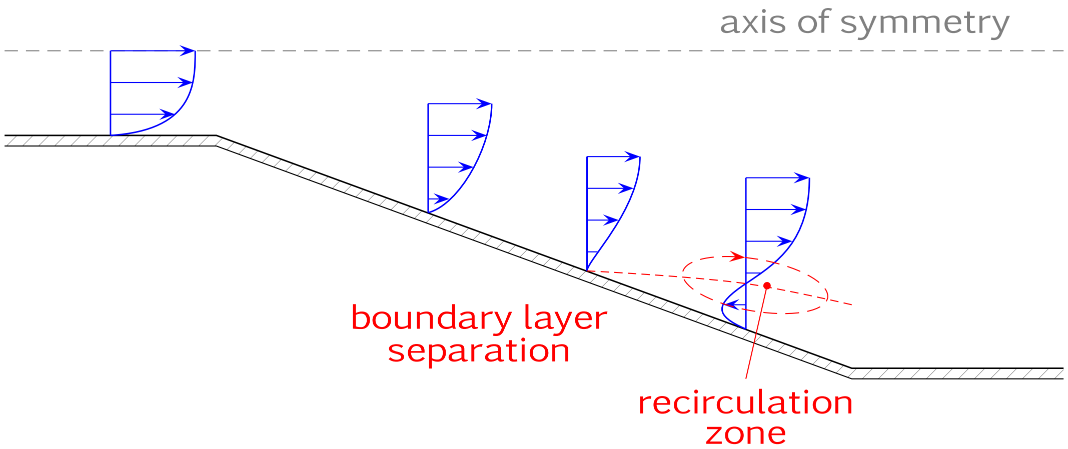
Role of the pressure gradient

Favorable pressure gradient $dp_e/dx_1 < 0$ (accelerated flow)
Adverse pressure gradient $dp_e/dx_1 > 0$ (decelerated flow)



Adverse pressure gradient induced by the profile curvature, leading to a flow separation

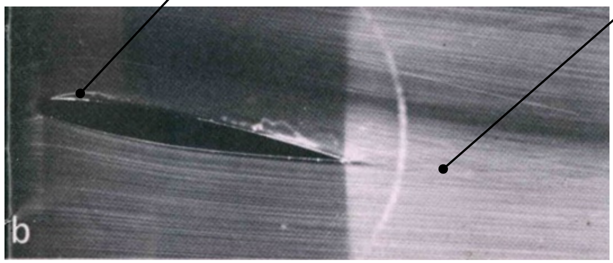
(from Wikipedia, Olivier Cleynen)



Sketch of the mean flow in a conical diffuser

- Creeping flow or Stokes flow ; Stokes force (drag for a smooth sphere) ; low Reynolds number flow
- High Reynolds flow around streamlined body and bluff body ; drag crisis ; boundary layer
- Laminar boundary layer : Prandt's equations

Streamlined and bluff bodies

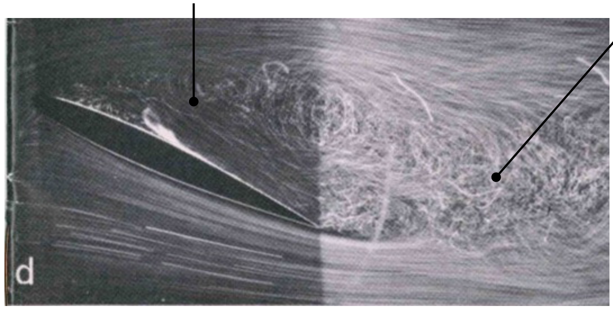


Unstalled aerofoil

Streamlined or slender body

Streamwise direction is often much larger than the cross-sectional dimension

No (small) drag crisis



Stalled aerofoil

Bluff body

Dimension along the incoming flow and in the transverse direction are similar

Drag crisis, see slide 143

Outline

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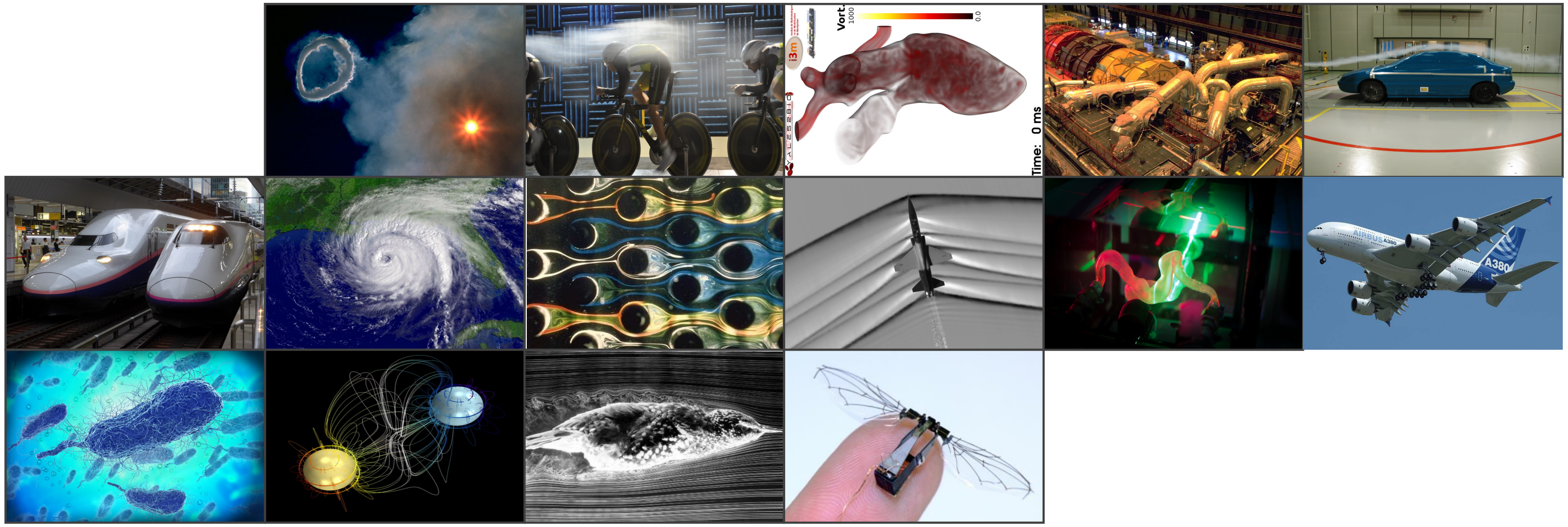
5 - Vorticity



5 - Vorticity

Vorticity

- Definition
- Circular vortex
- Circulation
- Momentum conservation
- Vorticity equation
- 2-D flows
- Boundary induced vorticity

High Reynolds number regime

- Flow around a bluff body
- Flow around a streamlined body

Key results

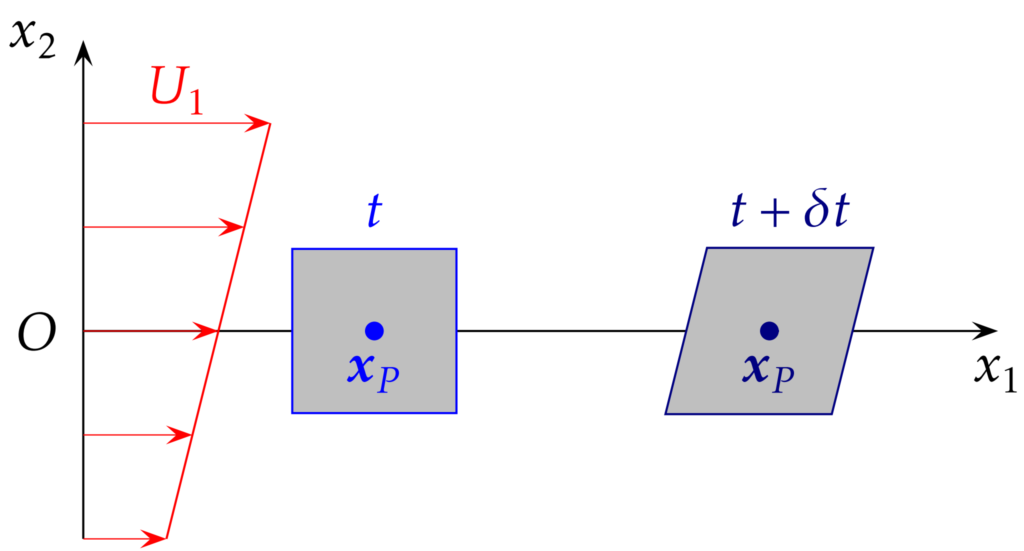
Definition

The vorticity vector is defined as $\boxed{\boldsymbol{\omega} = \nabla \times \boldsymbol{U}}$

When $\boldsymbol{\omega} = 0$, absence of vorticity, the flow is irrotational

The vorticity is twice the angular velocity $\boldsymbol{\Omega}$ of the solid-body rotation motion of the fluid particle (see slide 66)

$$\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{U}(\boldsymbol{x}_P) + \underbrace{\overline{\overline{\boldsymbol{D}}}(\boldsymbol{x}_P)\cdot(\boldsymbol{x}-\boldsymbol{x}_P)}_{\text{deformation}} + \underbrace{\boldsymbol{\Omega}(\boldsymbol{x}_P)\times(\boldsymbol{x}-\boldsymbol{x}_P)}_{\text{rotation}} \text{ as } \boldsymbol{x}\to\boldsymbol{x}_P \qquad \boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\omega}$$

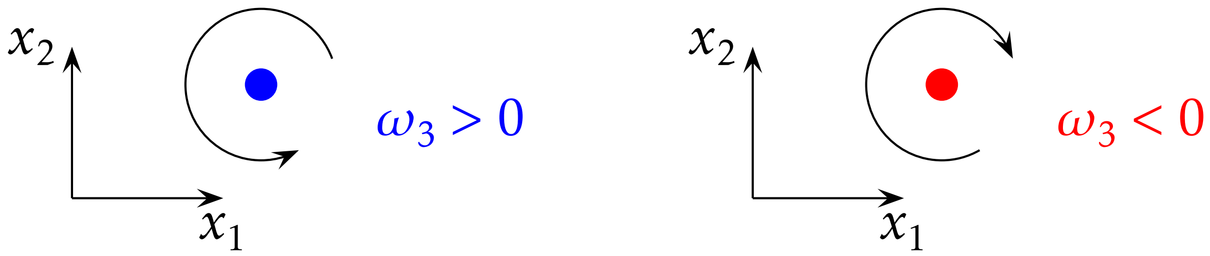


Decomposition of fluid-particle motion in a shear flow

Definition (cont.)

2-D flow $\boldsymbol{U} = (U_1(x_1, x_2, t), U_2(x_1, x_2, t), 0)$

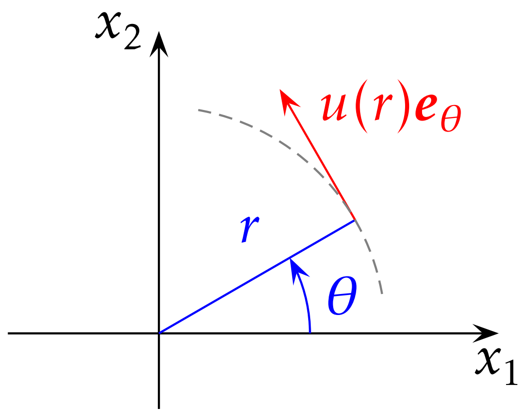
$$\boldsymbol{\omega} = (0, 0, \omega_3) \qquad \omega_3 = \frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2}$$



Circular vortex

Using cylindrical coordinates, the velocity of a circular vortex is defined as $\boldsymbol{U} = u(r)\boldsymbol{e}_\theta$

$U_\theta = u(r)$, $U_r = U_z = 0$



$$\omega_z(r) = \frac{1}{r}\frac{\partial}{\partial r}(ru) \quad \Longrightarrow \quad u(r) = \frac{1}{r}\int_0^r r'\omega_z(r')dr'$$

The vorticity can be determined from a given velocity field, and vice versa (but the latter requires a more tricky calculation in general, through the Biot & Savart law)

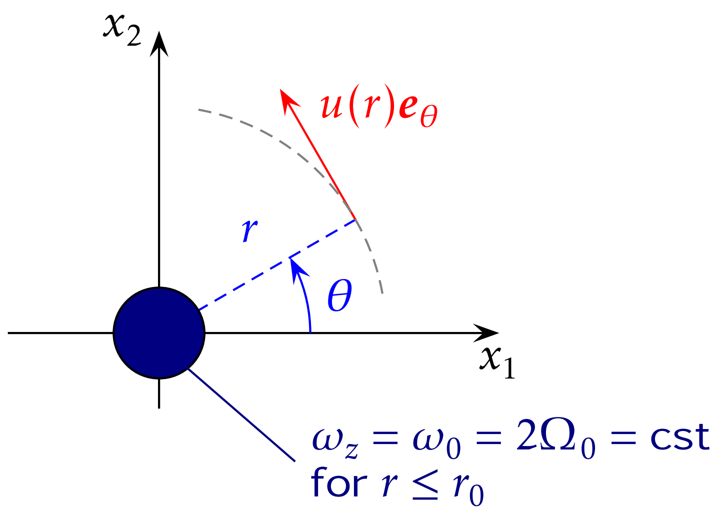
Spatially nonlocal relation between velocity and vorticity : the concentrated (localized) vorticity contributes to the velocity field *everywhere in space*

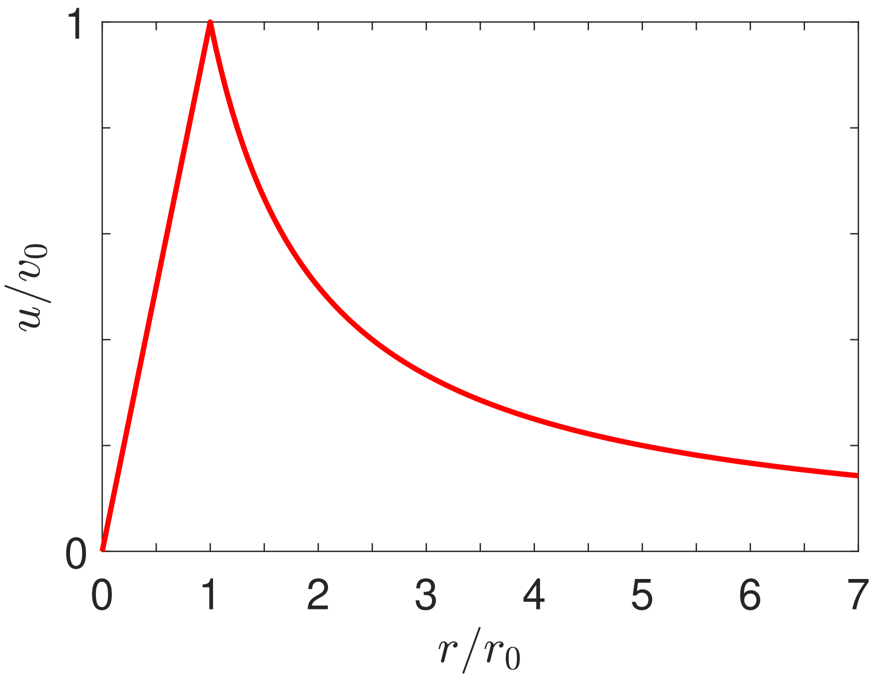
Example of the Rankine vortex (1858)

Rankine (1820-1872)

$$\begin{cases} u(r) = v_0 \dfrac{r}{r_0} = \Omega_0 r & r \leq r_0 \\ u(r) = v_0 \dfrac{r_0}{r} = \Omega_0 r_0 \dfrac{r_0}{r} & r > r_0 \end{cases}$$

$$(v_0 = \Omega_0 r_0 = \omega_0 r_0 / 2)$$



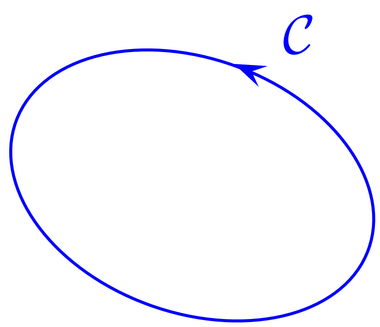


Solid body motion inside the vortex itself, *i.e.* for $r \leq r_0$ in the vortical region

Irrotational flow outside, for $r > r_0$: the localized circular patch of vorticity produces a velocity field away from the vortical region

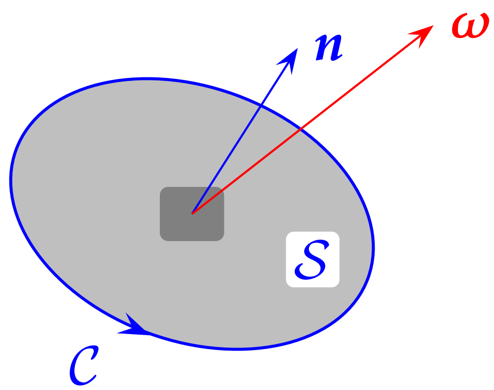
Concept of circulation

The line integral of the velocity along a closed curve $\mathcal{C}$ provides the circulation $\Gamma_{\mathcal{C}}$



$$\Gamma_{\mathcal{C}} \equiv \oint_{\mathcal{C}} \boldsymbol{U} \cdot d\boldsymbol{l} \qquad (\mathrm{m^2.s^{-1}})$$

Consider now the surface $\mathcal{S}$ bounded by the curve $\mathcal{C}$. According to the divergence theorem, the circulation represents also the vorticity flux crossing this surface



$$\Gamma_{\mathcal{C}} = \oint_{\mathcal{S}} \nabla \times \boldsymbol{U} \cdot \boldsymbol{n} \, d\mathcal{S} = \int_{\mathcal{S}} \boldsymbol{\omega} \cdot \boldsymbol{n} \, d\mathcal{S}$$

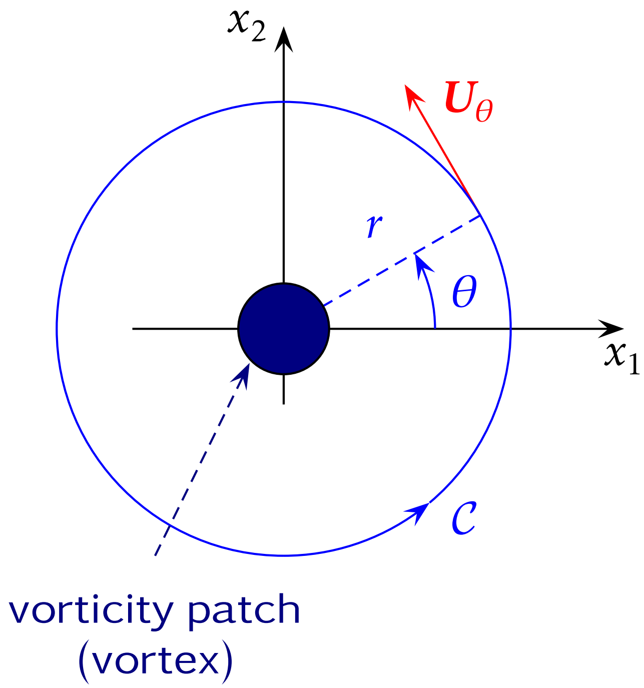
(The right-hand-screw convention is here applied, counter-clockwise sense)

Circulation of a circular vortex

If the curve $\mathcal{C}$ encloses all the vorticity, the circulation Γ is independent of the choice of $\mathcal{C}$, and thus characterize the vortex. Hence,

$$\Gamma = 2\pi r u(r) \quad \Longrightarrow \quad u(r) = \frac{\Gamma}{2\pi r}$$

In the irrotational region (outside the vortex), the velocity decreases as $1/r$



A point vortex can also be defined, when all the vorticity is concentrated in a single point, that is $\omega_3 = \Gamma\delta(x_1)\delta(x_2)$ in s^{-1}. The relation $u(r) = \Gamma/(2\pi r)$ is then valid everywhere except at the vortex point

Reformulation of the momentum conservation

The momentum conservation Eq. reads

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{U} + \boldsymbol{g}$$

Using the following vectorial identity

$$\boldsymbol{U} \cdot \nabla \boldsymbol{U} = \omega \times \boldsymbol{U} + \nabla\left(\frac{1}{2}\boldsymbol{U} \cdot \boldsymbol{U}\right)$$

an alternative form is obtained (by also reminding that $\boldsymbol{g} = -\nabla\Psi$),

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + \omega \times \boldsymbol{U} = -\frac{1}{\rho}\nabla \mathcal{H} + \nu \nabla^2 \boldsymbol{U} \qquad \mathcal{H} = p + \frac{1}{2}\rho U^2 + \Psi} \tag{16}$$

Assuming an inviscid and steady flow, Bernoulli's equation can again be established from Eq. (16) by taking the scalar product with $\boldsymbol{U}$, which leads to $\boldsymbol{U} \cdot \nabla \mathcal{H} = 0$ and thus, $\mathcal{H} = \text{cst}$ along a streamline.

For an irrotational flow (no vorticity), $\mathcal{H} = \text{cst}$ everywhere in the flow : stronger version of Bernoulli's theorem

Vorticity equation

By taking the curl of Eq. (16), and by using the following vectorial identities

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{U}) = \boldsymbol{U} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{U} + \boldsymbol{\omega} \nabla \cdot \boldsymbol{U} - \boldsymbol{U} \nabla \cdot \omega \qquad (\nabla \cdot \nabla \times \boldsymbol{U} = 0)$$

$$\nabla \times \nabla \mathcal{H} = 0 \qquad (\rho = \text{cst})$$

the transport equation for vorticity is finally obtained

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{\omega}}_{\substack{\text{convection} \\ = D\omega/Dt}} = \underbrace{\boldsymbol{\omega} \cdot \nabla \boldsymbol{U}}_{\text{3-D evolution}} + \underbrace{\nu \nabla^2 \boldsymbol{\omega}}_{\substack{\text{viscous} \\ \text{diffusion}}} \tag{17}$$

The evolution of the vorticity field is first linked to the presence of 3-D effects, especially for turbulent flow, and to viscous diffusion.

For an inviscid flow, $D\omega/Dt = \boldsymbol{\omega} \cdot \nabla \boldsymbol{U}$. A fluid particle remains irrotational during its motion (hence the interest in irrotational flows when it makes sense)

2-D vortical flows

For 2-D flows, $\boldsymbol{\omega} = (0, 0, \omega_3)$ and $\boldsymbol{\omega} \cdot \nabla \boldsymbol{U} = 0$ by assumption.
The vorticity (scalar) equation then reads as

$$\frac{\partial \omega_3}{\partial t} + \boldsymbol{U} \cdot \nabla \omega_3 = \nu \nabla^2 \omega_3$$

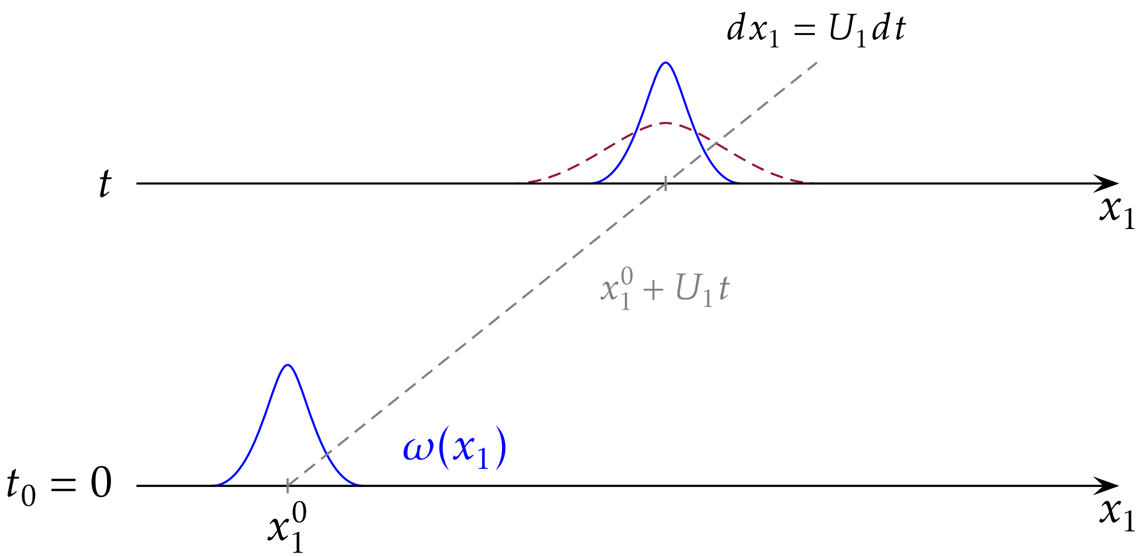
Only viscous diffusion is applied to a convected fluid particle (a special class of flows, again).

It can also be observed that for an inviscid 2-D flow, $D\omega_3/Dt = 0$. Vorticity is constant along the flow, and the vorticity field is simply convected by the flow

2-D vortical flows

Interpretation of the advection-diffusion transport equation. In terms of characteristic curves

$$\frac{\partial \omega}{\partial t} + U_1 \frac{\partial \omega}{\partial x_1} = 0 \quad \Longleftrightarrow \quad \frac{D\omega}{Dt} = 0 \quad \text{along} \quad \frac{dx_1}{dt} = U_1$$



The material derivative D/Dt is the total derivative of ω along the curve $dx_1 = U_1 dt$ (a straight line here by assuming $U_1 =$ cst)

— solution for $\dfrac{D\omega}{Dt} = 0$, that is $\omega(x_1 - U_1 t)$

- - - solution for $\dfrac{D\omega}{Dt} = \nu \nabla^2 \omega$

Viscous diffusion in 2-D

It is again considered a circular vortex $\boldsymbol{U} = (U_r, U_\theta, U_z) = (0, u(r), 0)$
$\boldsymbol{\omega} = (0, 0, \omega_z)$ with $\omega_z = \omega_z(r, t)$

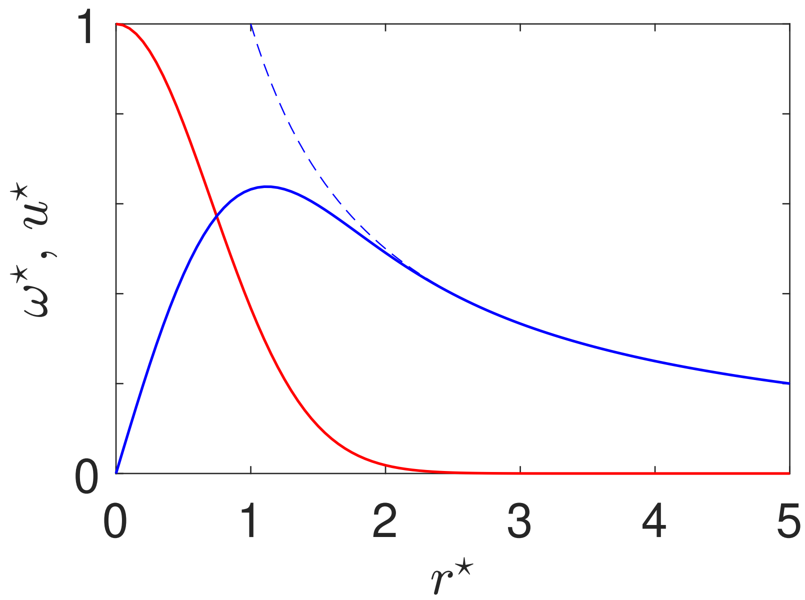
$\boldsymbol{U} \perp \nabla\omega_z \Longrightarrow \boldsymbol{U}\cdot\nabla\omega_z = 0$, no convection here

$$\frac{\partial \omega_z}{\partial t} = \nu\nabla^2\omega_z \quad \text{with} \quad \nabla^2\omega_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega_z}{\partial r}\right)$$

A particular solution to this problem is the Lamb-Oseen vortex

$$\omega_z = \frac{\Gamma}{4\pi\nu t}\exp\left(\frac{-r^2}{4\nu t}\right) \quad u = \frac{\Gamma}{2\pi r}\left[1 - \exp\left(\frac{-r^2}{4\nu t}\right)\right]$$

The core radius of the vortex varies as $r_c(t) = \sqrt{4\nu t} \propto \sqrt{\nu t}$, which is the signature of molecular diffusion



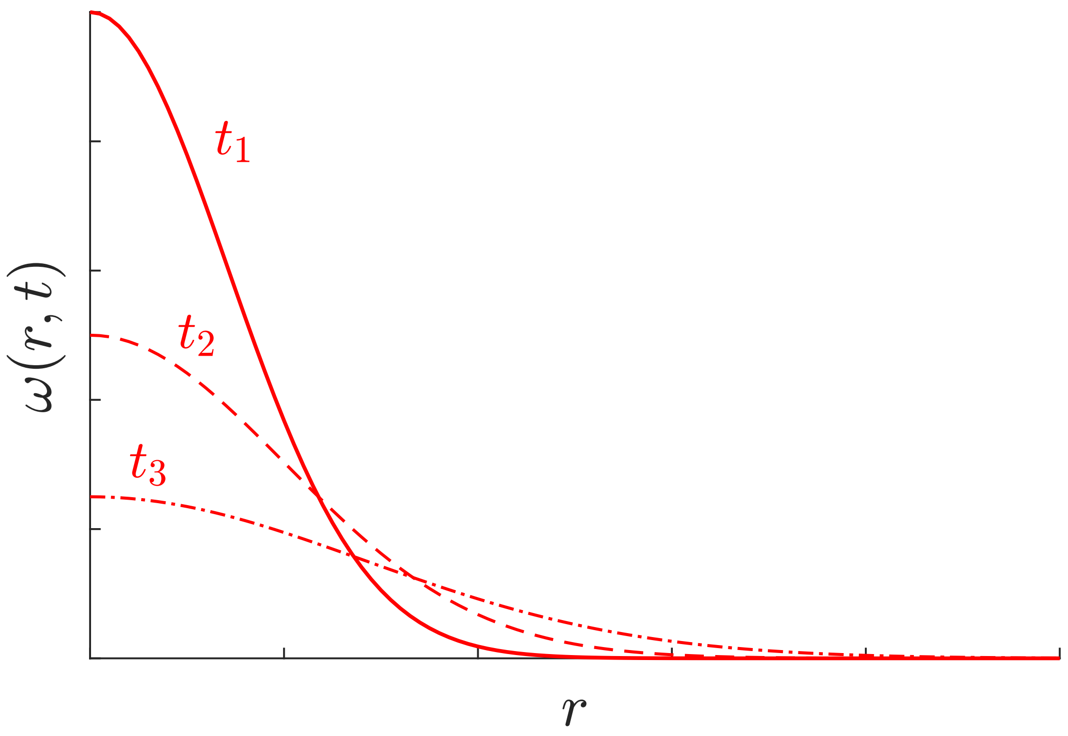
$r^\star = r/r_c$

$\omega_z^\star = (\pi r_c^2/\Gamma)\omega_z \quad u^\star = (2\pi r_c/\Gamma)u$

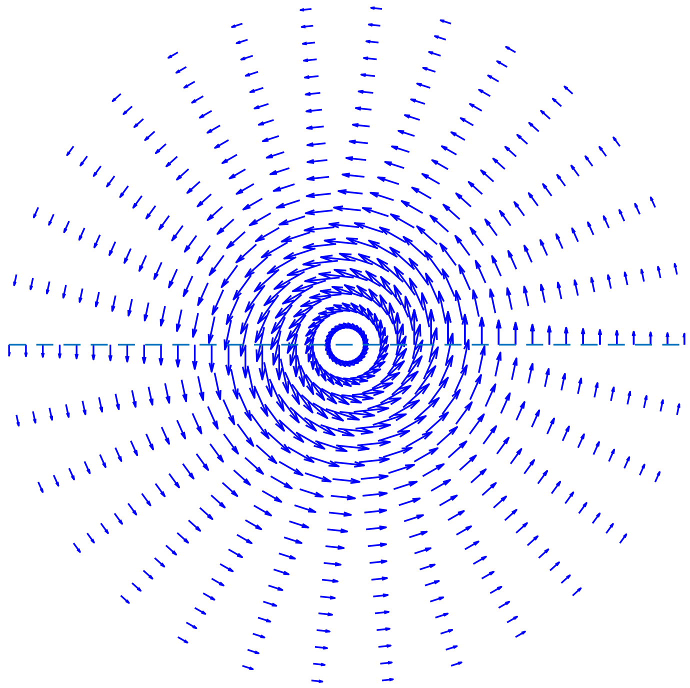
$u^\star = 1/r^\star$ (dashed line)

Viscous diffusion in 2-D (cont.)

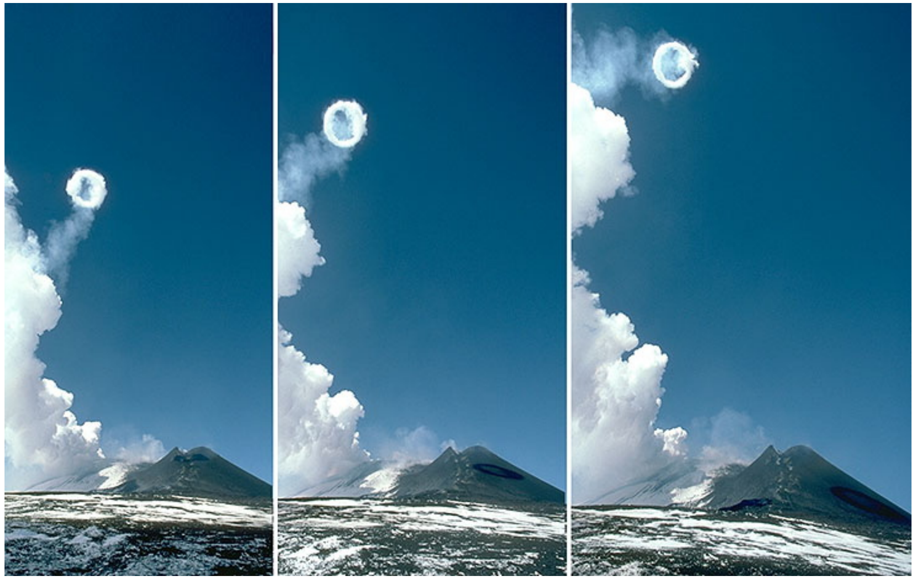
The total vorticity (and thus its circulation) is conserved, but a time decay is observed, from vorticity initially concentrated at a point $\omega_3 = \Gamma\delta(x_1)\delta(x_2)$ as shown below, due to viscous diffusion



Velocity field of the Lamb-Oseen vortex for $0 \leq r^\star \leq 5$



- **Steam vortex ring ejected by Etna**
  (Etna is the tallest and most active volcano in Europe)



Japanese smoke ring experiment : giant vortex box of 30 m ! (starting vortex)

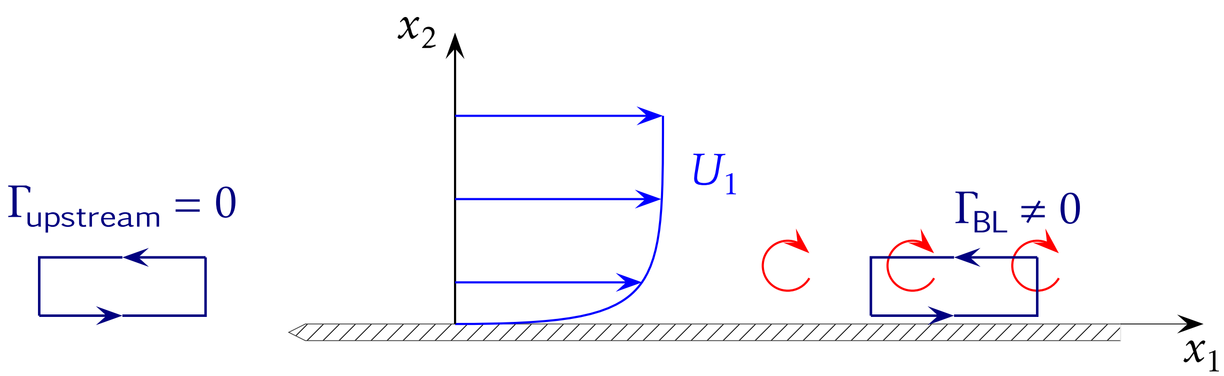
Vortex rings drift (why ?) across the blue sky with no points of reference. However, the volcanologists estimate the rings to be about 200 m across and up to 1000 m above the ground.

Image by Dr Jurg Alean and Dr Marco Fulle, Stromboli online

Boundary induced vorticity

Three mechanisms have been identified from the vorticity transport Eq. (17), *i.e.* convection, 3-D effects and viscous diffusion, but they cannot explain the production itself : the origin of vorticity always lies at the frontiers of flow

Vorticity production at walls (and then transverse diffusion and convection by the boundary layer), identified through the circulation :



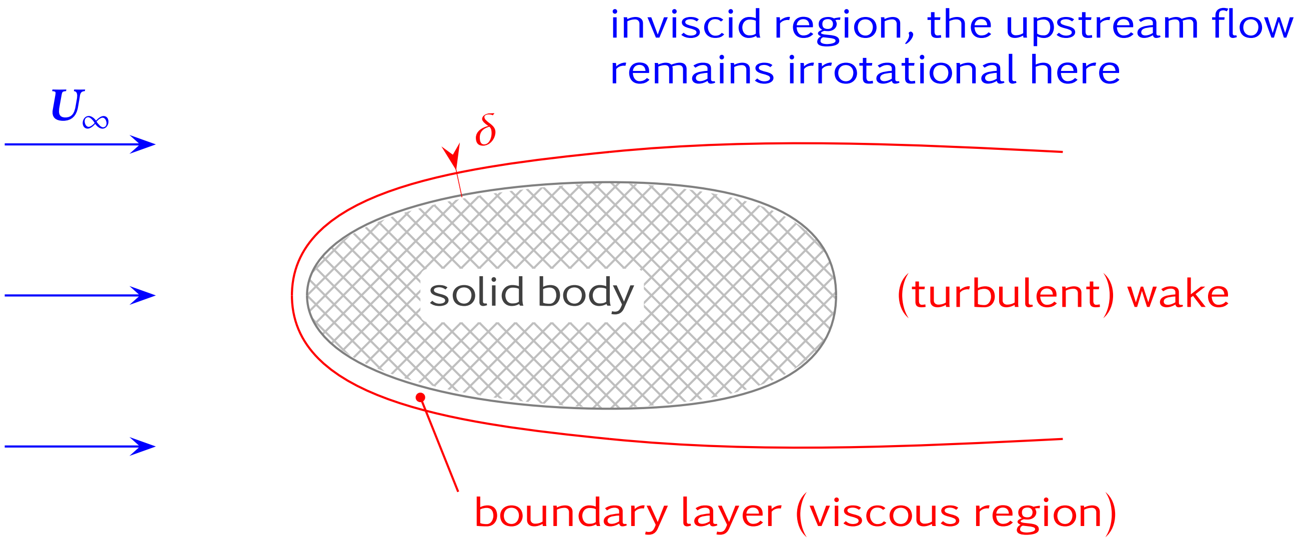
$$\overline{\overline{\boldsymbol{D}}} = \frac{\mu}{2}\begin{pmatrix} 0 & \frac{\partial U_1}{\partial x_2} \\ \frac{\partial U_1}{\partial x_2} & 0 \end{pmatrix}$$

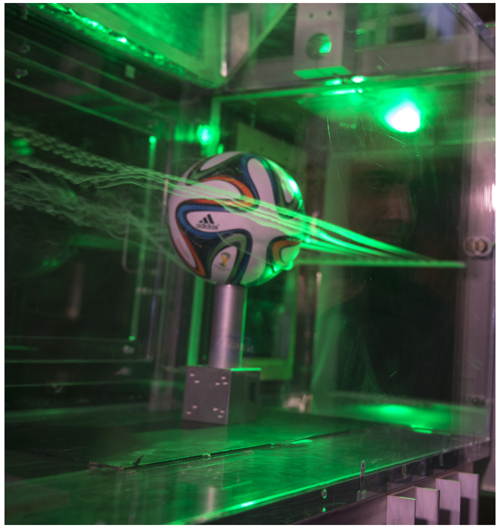
$$\omega_3 = \frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2} = -\frac{\partial U_1}{\partial x_2} \text{ at the wall}$$

The viscous force $d\boldsymbol{F} = \boldsymbol{T}\ ds$ applied to the fluid at the wall is

$$\boldsymbol{T} = -\mu\frac{\partial U_1}{\partial x_2}\boldsymbol{e}_1 = \mu\omega_3\boldsymbol{e}_1 \qquad \text{(with } \boldsymbol{T} = \overline{\overline{\boldsymbol{\tau}}}\cdot\boldsymbol{n} = \mu\boldsymbol{\omega}\times\boldsymbol{n} \text{ with } \boldsymbol{n} = -\boldsymbol{e}_2 \text{ here)}$$

Developed flow around a bluff body at high Reynolds number



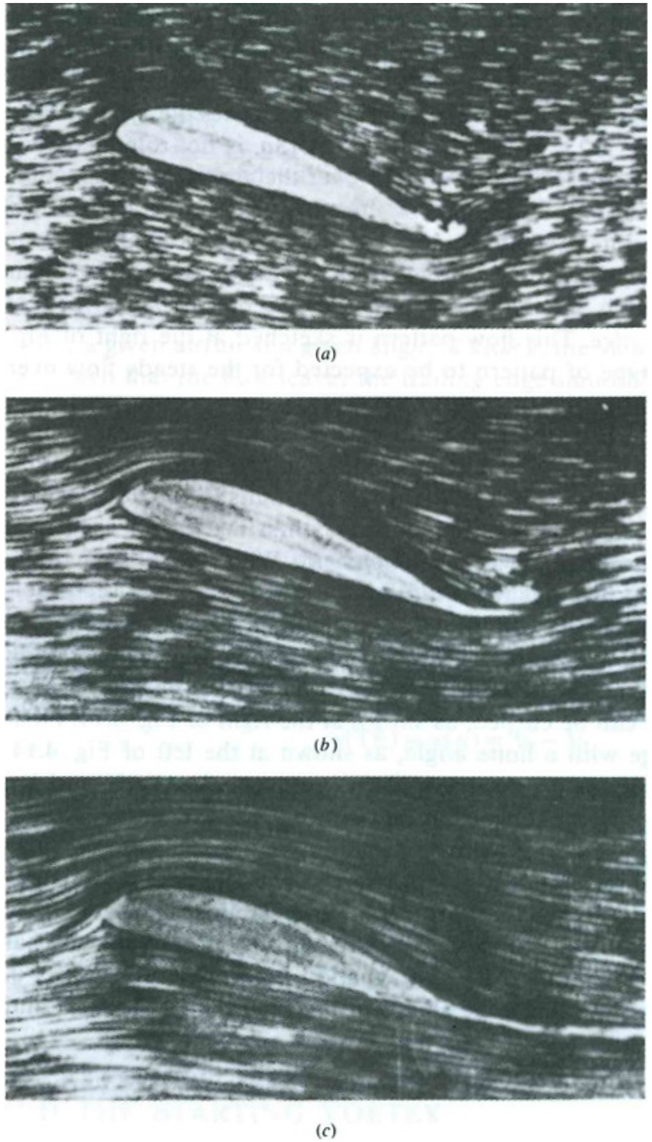


A close up of the Brazuca ball in Ames' Fluid Mechanics Laboratory's two-foot by two-foot wind tunnel. Smoke highlighted by lasers visualizes air flow around the ball.

$U_\infty \simeq 48$ km.h^{-1} (well below the typical kicking speed of a professional player, around critical Re)

(NASA's Ames Research Center, R. Mehta, 2014)

Development of a steady flow over an airfoil



The airfoil is impulsively started from rest and attains a steady velocity through the fluid.

(a) A moment just after starting
(b) An intermediate time
(c) The final steady flow

From Prandtl & Tietjens (1934), in the textbook by J.D. Anderson, Jr. (1991)

- Concepts of vorticity, irrotational flow, non-local relation between velocity and vorticity, circulation, vortex and point vortex, stream function
- Transport equation for vorticity
- Vorticity in wakes and boundary layers
- Permanence of irrotational motion for an inviscid flow

Short digression : velocity potential and stream function

Velocity potential $\phi(\boldsymbol{x}, t)$

The flow must be irrotational (but can be compressible and three-dimensional)

$$\boldsymbol{U} = \nabla\phi \quad \left(U_i = \frac{\partial\phi}{\partial x_i}\right) \quad \Longrightarrow \quad \boldsymbol{\omega} = \nabla\times\boldsymbol{U} = 0$$

For incompressible flow, $\nabla\cdot\boldsymbol{U} = 0$ and ϕ satisfies Laplace's equation $\nabla^2\phi = 0$

Stream function $\psi(\boldsymbol{x}, t)$

Relevant for two-dimensional (axisymmetric) incompressible flows only

$$U_1 = \frac{\partial\psi}{\partial x_2} \quad U_2 = -\frac{\partial\psi}{\partial x_1} \quad \Longrightarrow \quad \nabla\cdot\boldsymbol{U} = 0$$

$$\boldsymbol{\omega} = (0, 0, \omega_3) \qquad \omega_3 = \frac{\partial U_1}{\partial x_2} - \frac{\partial U_2}{\partial x_1} = \nabla^2\psi$$

$\psi = \text{cst}$ is constant on streamlines (at fixed t)

Outline

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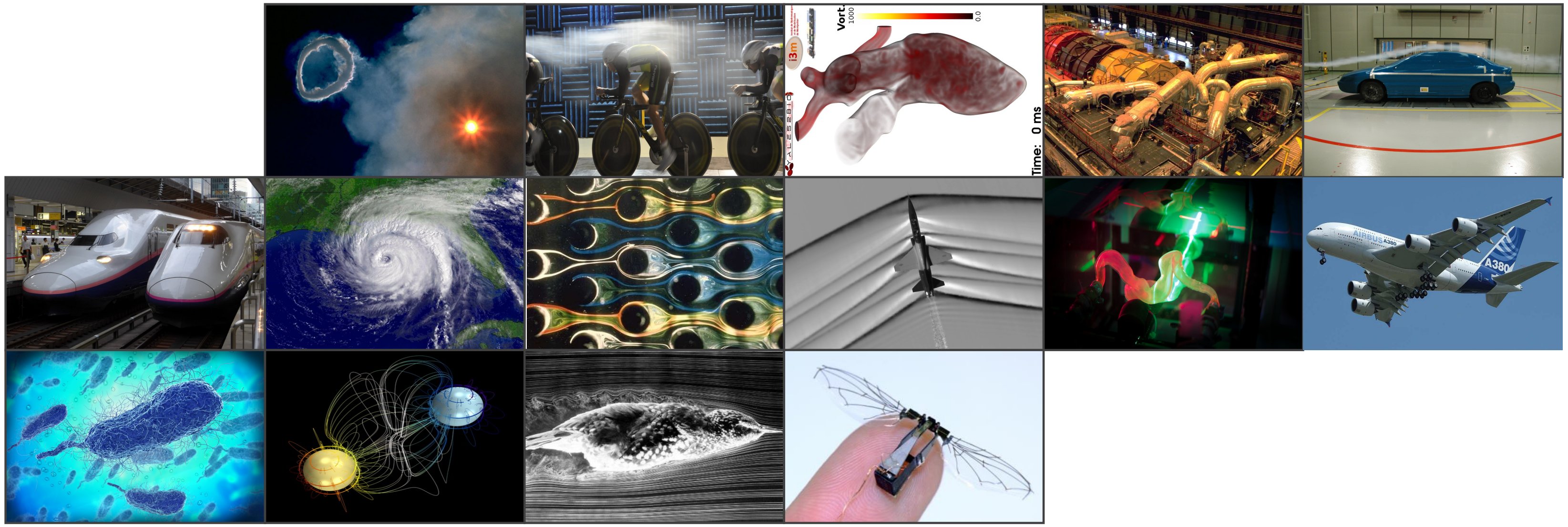
6 - Turbulent flow



6 - Turbulent flow

A short description of turbulent flow

Statistical description

- Reynolds decomposition
- Reynolds-Averaged Navier-Stokes (RANS) equations

Turbulence modeling

- Turbulent kinetic energy and dissipation
- Concept of turbulent viscosity
- Turbulent kinetic energy budget

Energy Cascade

- Scales
- Phenomenological description

Key results

Turbulent flow

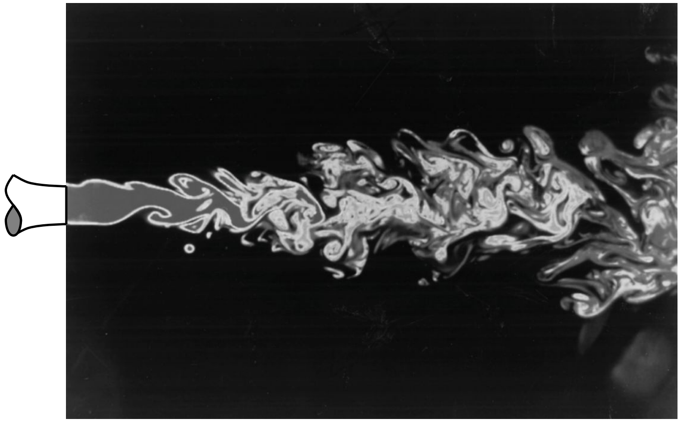
- unsteady aperiodic motion
- unpredictable behaviour over a long time horizon
- presence of a wide range of time and three-dimensional space scales

Turbulence appears when the source of the kinetic energy which drives the fluid motion is able to overcome viscosity effects. Reynolds number must be large enough to observe the turbulent state, refer to Reynolds′ experiment

- astrophysics, geophysical flows including ocean circulation, climate, weather forecast, hydrology, dispersion of aerosols
- external aerodynamics for aeronautics & ground transportation, internal flows in mechanical engineering, biomechanics, biological flows
- noise of turbulent flows (aeroacoustics), sound propagation (atmosphere, ocean), fluid-solid coupling and vibroacoustics

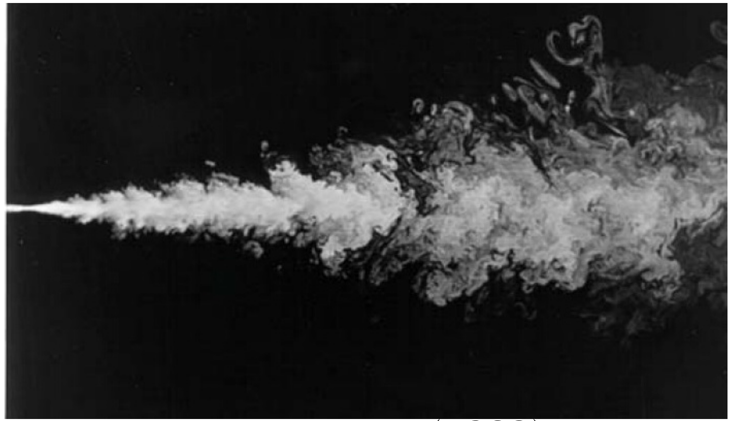
Turbulent subsonic (round) jet

$\mathrm{Re}_D = u_j D/\nu$



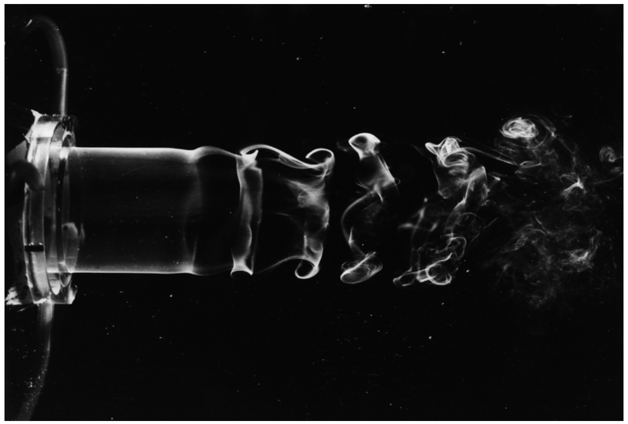
Prasad & Sreenivasan (1989)

$Re_D \simeq 4000$



Dimotakis *et al.* (1983)

$Re_D \simeq 10^4$

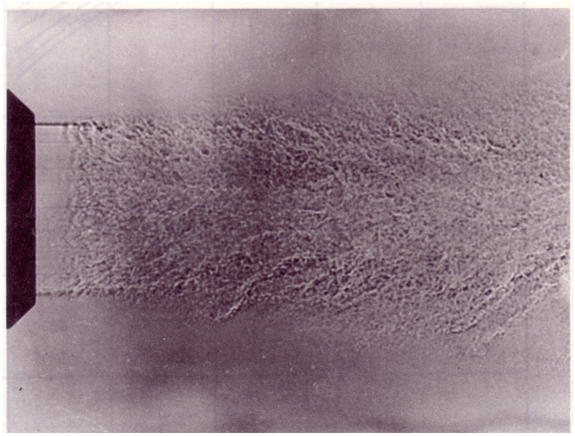


Kurima, Kasagi & Hirata (1983)

$Re_D \simeq 5.6 \times 10^3$

Ayrault, Balint & Schon (1981)

$Re_D \simeq 1.1 \times 10^4$

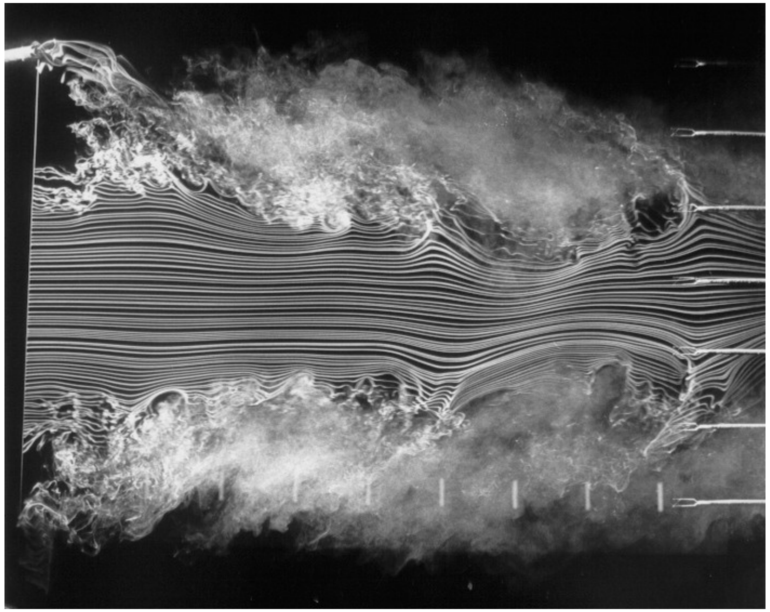


Mollo-Christensen (1963)

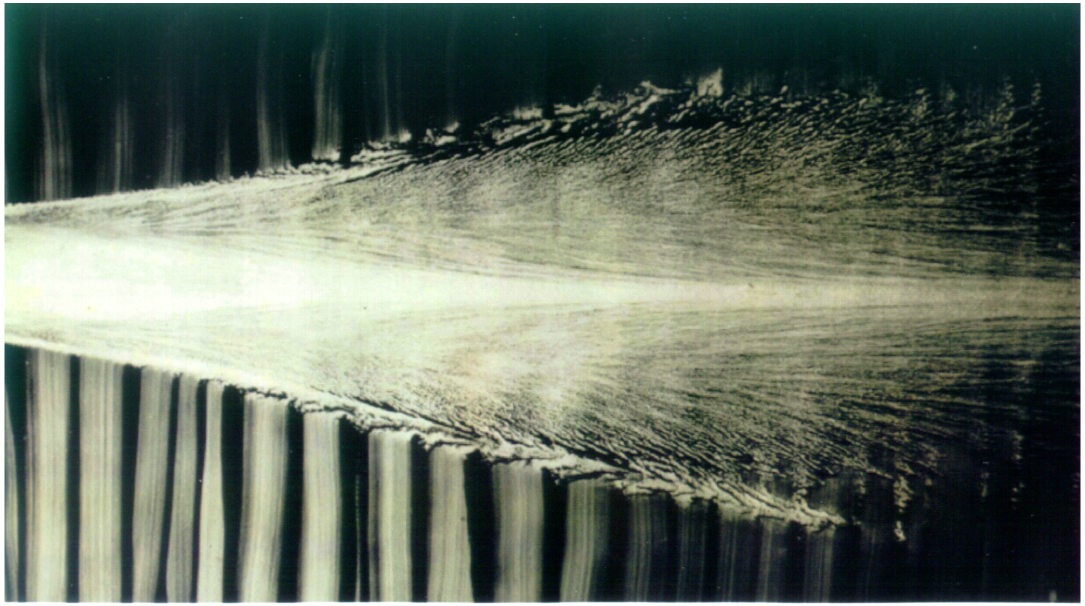
$Re_D = 4.6 \times 10^5$

Importance of entrainment for free shear flows

The flow divergence is induced by the entrainment of the surrounding fluid into the jet (turbulent mixing)



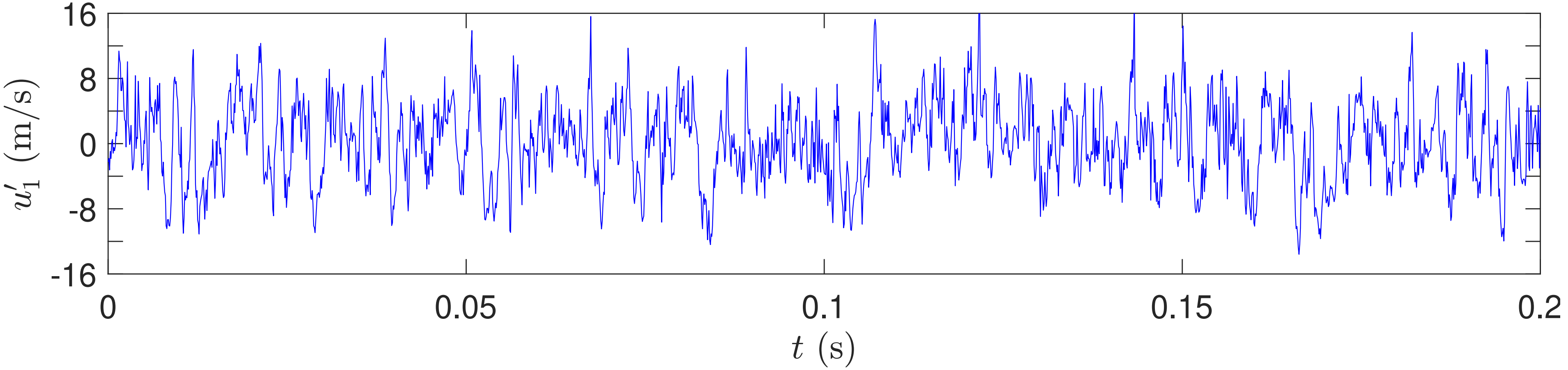
Visualization with smoke wires
$\mathrm{Re}_D \simeq 5.4 \times 10^4$
Courtesy of H. Fiedler (1987)



Entrainment by a turbulent round jet from a wall
$\mathrm{Re}_D = 10^6$
Florent, *J. Méc.* (1965)

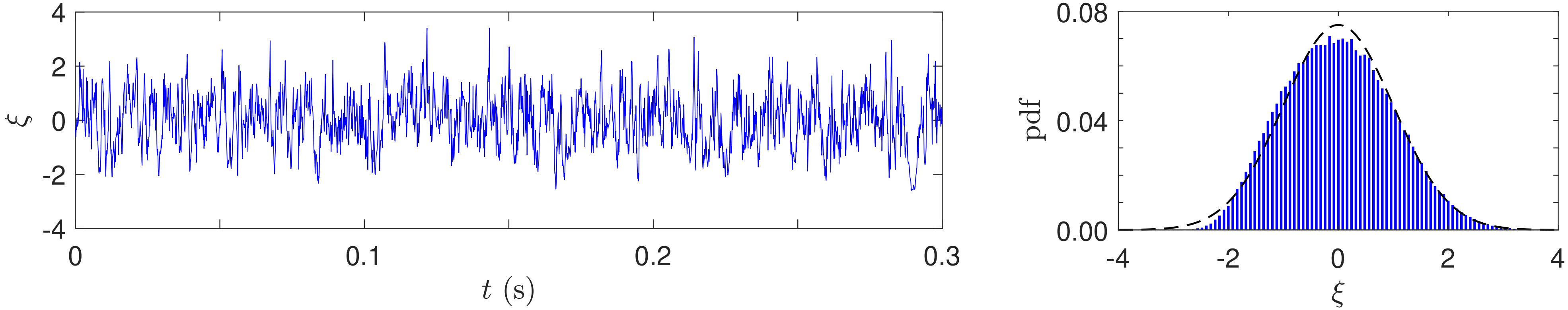
- **Fluctuating velocity signal in the shear layer of a subsonic round jet**
  (measured by crossed-wire probes at $x_1 = 2D$, $x_2 = D/2$, $x_3 = 0$)
  Nozzle diameter $D = 50$ mm, exit velocity $U_j = 30$ m.s^{-1}
  $\rightsquigarrow$ Reynolds number $\mathrm{Re}_D = 10^5$

$u_1'(t)$

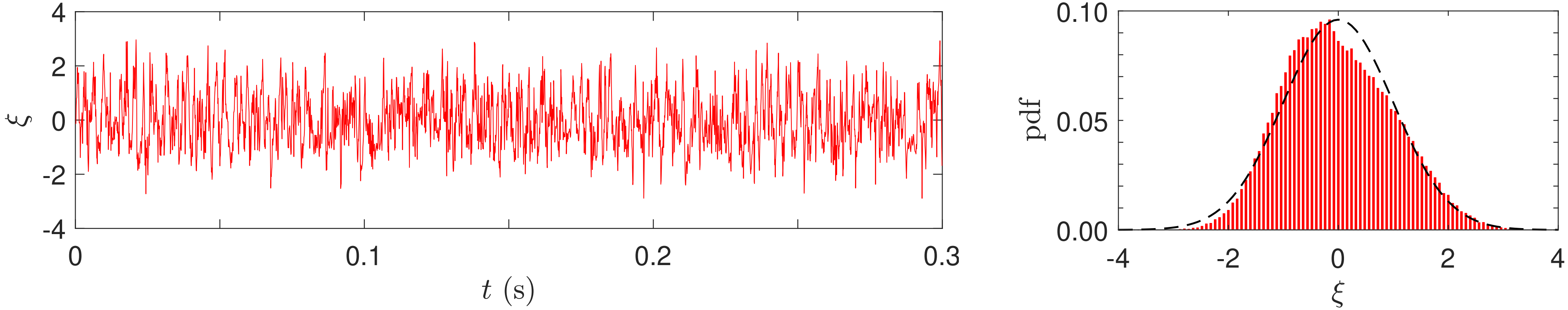


- **Fluctuating velocity signal in the shear layer of a jet (cont.)**
  $u'_1(t)$ and $u'_2(t)$ with $\xi = u'_\alpha(t)/u_{\alpha\text{rms}}$

$S_{u'_1} \simeq 0.21 \qquad T_{u'_1} \simeq 2.74$



$S_{u'_2} \simeq 0.24 \qquad T_{u'_2} \simeq 2.76$



What is the reason for this asymmetry of transverse velocity fluctuations ?

Fluctuating velocity signal in the shear layer of a jet (cont.)

For a centered variable $x'_i \equiv x_i - \bar{X}_i$ of root-mean-square deviation $x'_{i,\mathrm{rms}} \equiv \left(\overline{x'^2_i}\right)^{(1/2)} = \sigma_{x_i}$, the skewness S and the flatness or kurtosis T factors are defined by

$$S_{x_i} \equiv \frac{\overline{x'^3_i}}{x'^3_{i,\mathrm{rms}}} \qquad\qquad T_{x_i} \equiv \frac{\overline{x'^4_i}}{x'^4_{i,\mathrm{rms}}}$$

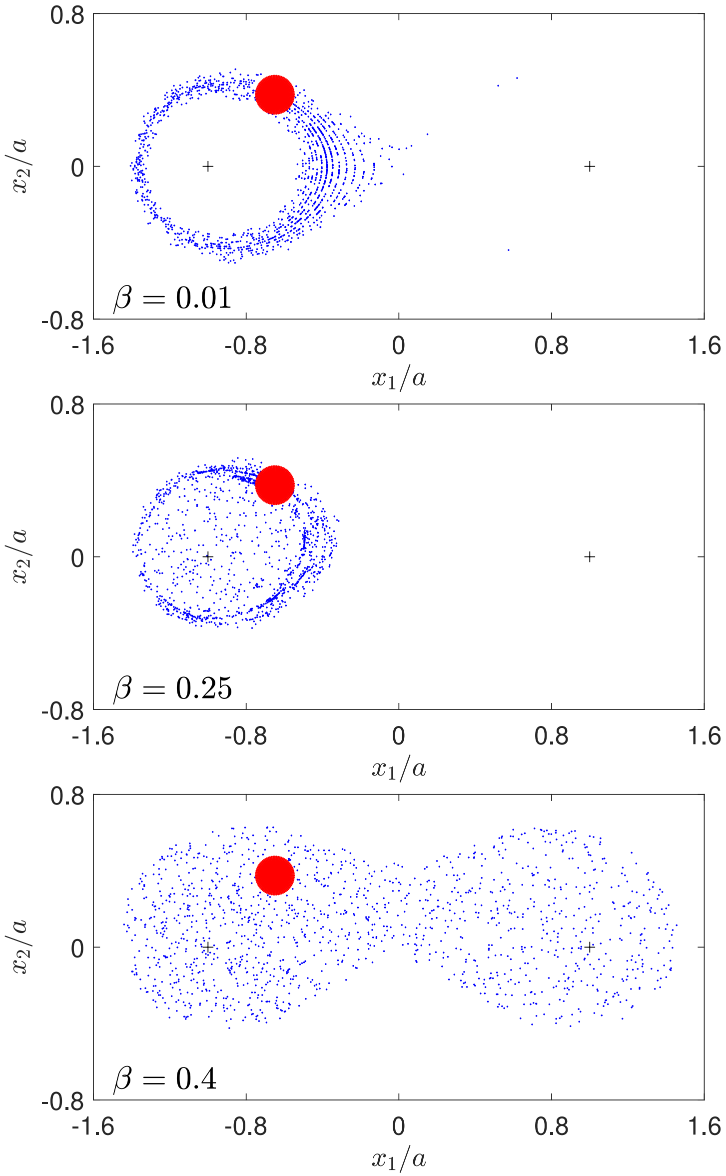
Skewness is a measure of the asymmetry of the probability density function (pdf) about its mean, and flatness is a measure of the tailedness of the pdf.

The Gaussian (normal) distribution is a usual reference pdf,

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma_{x_i}} \exp\left(-\frac{\xi^2}{2\sigma^2_{x_i}}\right) \qquad x'_{i,\mathrm{rms}} = \sigma_{x_i} \qquad S_{x_i} = 0 \qquad T_{x_i} = 3$$

$$1 = \int_{-\infty}^{+\infty} p(\xi)d\xi \qquad \overline{x'^n_i} = \int_{-\infty}^{+\infty} \xi^n p(\xi)d\xi$$

Chaotic mixing : blinking vortex flow of Aref (1984)

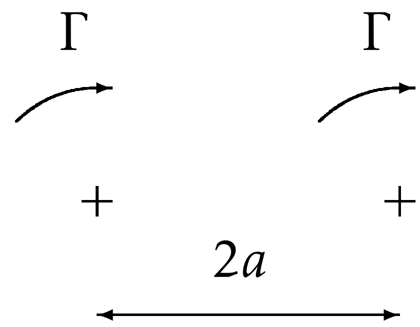


$\beta = 0.01$

$\beta = 0.25$

$\beta = 0.40$

$$\begin{cases} \text{for } 0 \leq t \leq T\text{, vortex } (-a, 0) \text{ on} \\ \text{for } T \leq t \leq 2T\text{, vortex } (a, 0) \text{ on} \end{cases}$$

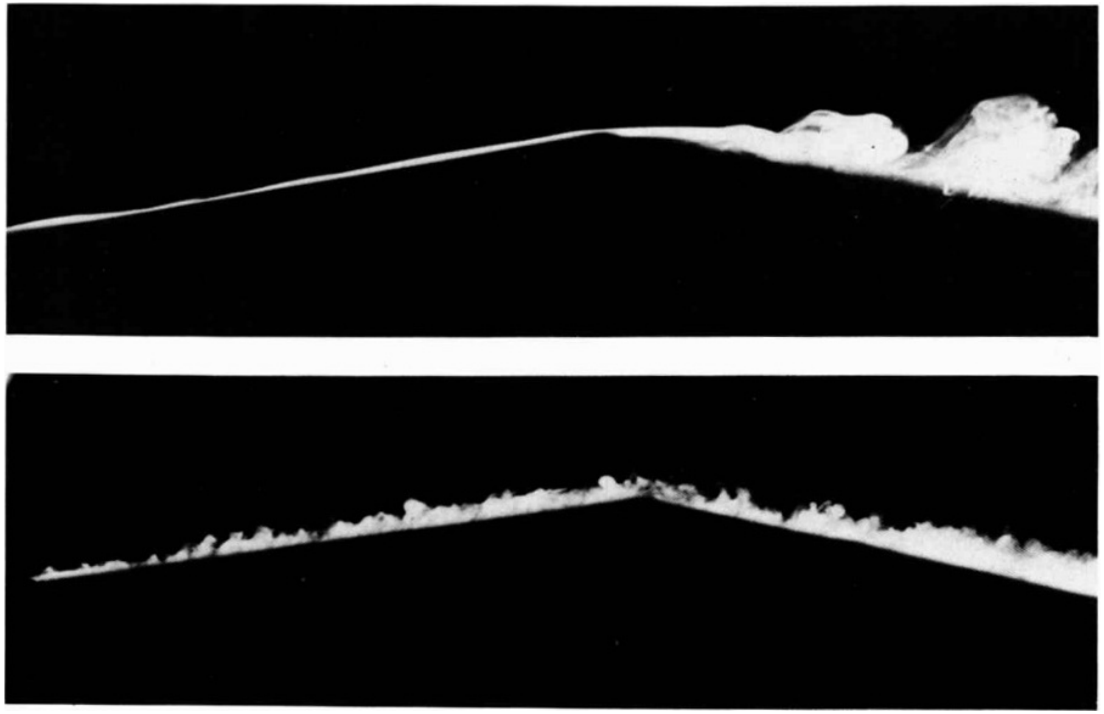


control parameter : $\beta = \dfrac{\Gamma T}{2\pi a^2}$

We observe the sensitivity of initial conditions, the transport efficiency and the velocity field induced by vorticity, see Chapter

Transport efficiency

Turbulence $\rightsquigarrow$ increase of diffusion, reduction of flow separation regions, decrease of pressure drag (versus increase in wall friction), increase in thermal exchanges



The laminar boundary layer in the upper photograph separates from the sharp corner whereas the turbulent boundary layer in the second photograph remains attached (from Van Dyke, fig. 156)

Refer also to illustrations in Chapters 3 & 4!

Mean and fluctuating quantities

The statistical mean $\bar{F}(\boldsymbol{x}, t)$ of a variable $f(\boldsymbol{x}, t)$ is defined as

$$\bar{F}(\boldsymbol{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f^{(i)}(\boldsymbol{x}, t)$$

where $f^{(i)}$ is the i-th realization : convenient when manipulating equations but difficult to implement in practice, or even impossible for geophysical flows !

We approximate the ensemble mean $\bar{F}$ of $f = \bar{F} + f'$ by a sufficiently long time average of one realization only :

$$\bar{F}(\boldsymbol{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\boldsymbol{x}, t')dt'$$

Time average makes sense only if turbulence is stationary, that is statistics are independent of time (refer to the concept of ergodicity in signal processing)

Reynolds decomposition

For a given variable f, Reynolds decomposition $f = \bar{F} + f'$ into mean and fluctuating (deviation) components is introduced.

- Centered fluctuating field

  $f \equiv \bar{F} + f'$ with $\overline{f'} = 0$ $\quad (f' = f - \bar{F}$ and $\overline{f'} = \bar{F} - \bar{F} = 0)$

- Rule for the product of two any variables (f and g here),

  $f\,g \equiv (\bar{F} + f')(\bar{G} + g') = \bar{F}\bar{G} + \bar{F}g' + f'\bar{G} + f'g'$

  and thus, $\overline{f\,g} = \bar{F}\,\bar{G} + \bar{F}\,\overline{g'} + \overline{f'}\,\bar{G} + \overline{f'g'} = \bar{F}\,\bar{G} + \overline{f'g'}$

$$\boxed{\overline{f\,g} = \bar{F}\,\bar{G} + \overline{f'g'}} \tag{18}$$

Reynolds decomposition for velocity : $U_i \equiv \bar{U}_i + u_i'$ with $\overline{u_i'} = 0$

$\bar{U}_i$ part which can be reasonably calculated

u_i' part which must be modelled (turbulent fluctuations)

Reynolds Averaged Navier-Stokes (RANS) equations

Assumptions (to simplify) : incompressible flow $\nabla \cdot \boldsymbol{U} = 0$
and homogeneous fluid, constant density ρ

How to determine the transport equation of the mean quantities ?
First, substitute Reynolds decomposition and then, average the equation !

$$\frac{\partial(\bar{U}_i + u_i')}{\partial x_i} = 0 \qquad \overline{\frac{\partial(\bar{U}_i + u_i')}{\partial x_i}} = 0 \qquad \Longrightarrow \qquad \frac{\partial \bar{U}_i}{\partial x_i} = 0 \qquad (19)$$

By substracting both equations,

$$\frac{\partial U_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial \bar{U}_i}{\partial x_i} = 0 \qquad \Longrightarrow \qquad \frac{\partial u_i'}{\partial x_i} = 0$$

The mean flow field $\bar{\boldsymbol{U}}$ is incompressible, and so is the fluctuating field $\boldsymbol{u}'$

- **Reynolds Averaged Navier-Stokes (RANS) equations**

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_i U_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \tau_{ij} = 2\mu D_{ij}$$

By introducing Reynolds decomposition, and taking the average

$U_i \equiv \bar{U}_i + u'_i \qquad p \equiv \bar{P} + p' \qquad \tau_{ij} \equiv \bar{\tau}_{ij} + \tau'_{ij}$

$$\frac{\partial(\rho\bar{U}_i)}{\partial t} + \frac{\partial(\rho\bar{U}_i\,\bar{U}_j)}{\partial x_j} = -\frac{\partial\bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho\overline{u'_i u'_j}\right) \tag{20}$$

The term $-\rho\overline{u'_i u'_j}$ is Reynolds stress tensor, unknown, thus leading to a closure problem for Eqs. (19) and (20). Generally this term is larger than the mean viscous stress tensor except for wall bounded flows, where the viscosity effects become preponderant close to the wall (no-slip boundary condition)

Turbulent kinetic energy k_t and dissipation ϵ

The turbulent kinetic energy and the turbulent dissipation are two key quantities to examine turbulent flows. By introducing Reynolds decomposition, using the rule (18), we obtain

for the kinetic energy,

$$\frac{\overline{U_i U_i}}{2} = \frac{\bar{U}_i \bar{U}_i}{2} + \frac{\overline{u_i' u_i'}}{2} \qquad k_t \equiv \frac{\overline{u_i' u_i'}}{2}$$

k_t is the turbulent kinetic energy

for the dissipation,

$$2\nu \overline{D_{ij} D_{ij}} = 2\nu \bar{D}_{ij} \bar{D}_{ij} + 2\nu \overline{d_{ij}' d_{ij}'} \qquad \epsilon \equiv 2\nu \overline{d_{ij}' d_{ij}'}$$

ϵ ($m^2.s^{-3}$) is the dissipation rate of k_t ($m^2.s^{-2}$) induced by the molecular viscosity

- **Concept of turbulent viscosity for turbulence models** introduced by Boussinesq (1877)

  Modeling of Reynolds stress tensor $-\rho\overline{u_i' u_j'}$. By analogy with the viscous stress tensor $\overline{\overline{\boldsymbol{\tau}}}$, one defines

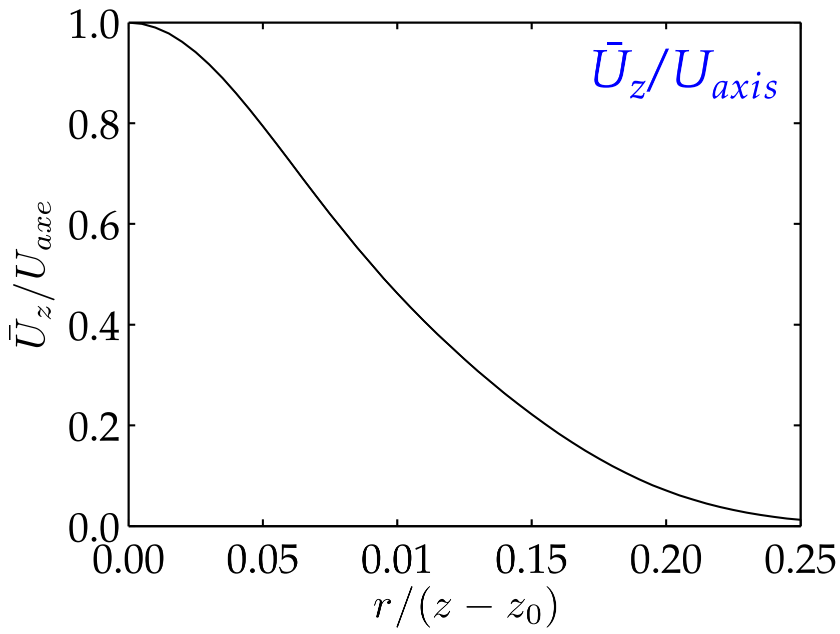
$$-\rho\overline{u_i' u_j'} = 2\mu_t \bar{D}_{ij} - \frac{2}{3}\rho k_t \delta_{ij} = \mu_t\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) - \frac{2}{3}\rho k_t \delta_{ij}$$

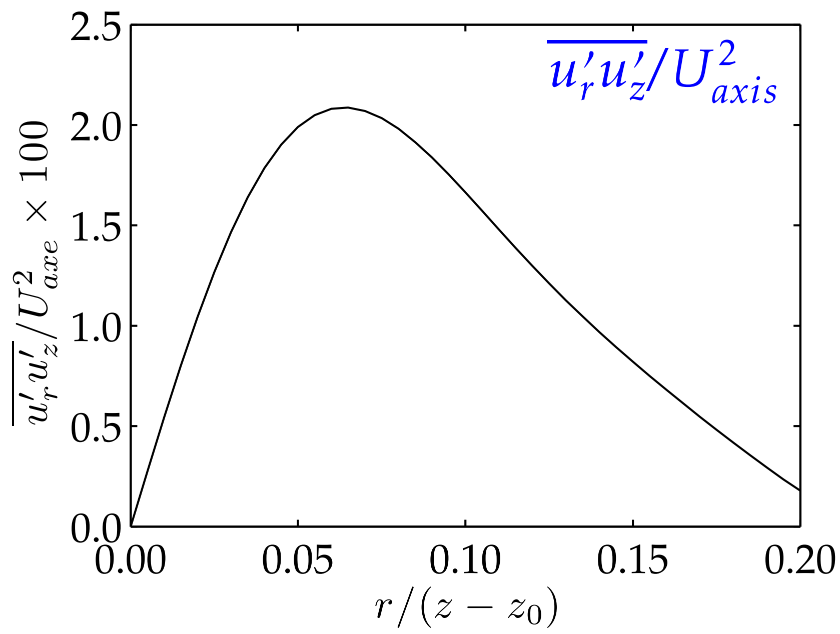
  where $\mu_t = \mu_t(\boldsymbol{x}, t)$ is the turbulent viscosity, a property of the flow field (and not of the fluid as for the molecular viscosity μ)

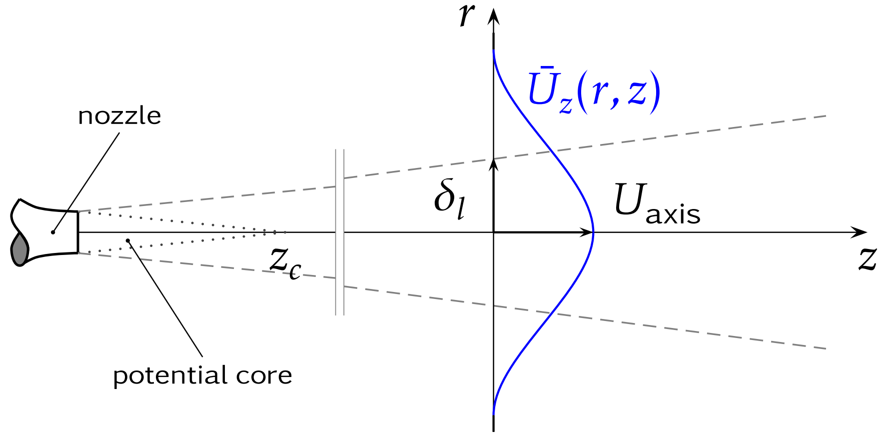
  The introduction of a turbulent viscosity for closing Reynolds stress tensor is an assumption, so not always verified by turbulent flows. In addition, it is also assumed that the turbulent viscosity remains positive (thus inducing specific behaviours in terms of energy transfer)

Illustration for a free subsonic round jet

M = 0.16 and $\mathrm{Re}_D = 9.5 \times 10^4$ (from Hussein, Capp & George, 1994)







$$\rho\overline{u_r'u_z'} \simeq -\mu_t \frac{\partial \bar{U}_z}{\partial r}$$

Concept of turbulent viscosity for turbulence models (cont.)

Reynolds Averaged Navier-Stokes (RANS) equations

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0$$

$$\frac{\partial(\rho \bar{U}_i)}{\partial t} + \frac{\partial(\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial(\bar{P} + \frac{2}{3}\rho k_t)}{\partial x_i} + \frac{\partial}{\partial x_j}\Big(2(\mu + \mu_t)\bar{D}_{ij}\Big)$$

How to compute the turbulent viscosity $\mu_t(\boldsymbol{x}, t) = \rho \nu_t$?

From dimensional arguments, the product of a velocity scale by a length scale, for example $\nu_t \sim k_t^{1/2} \times k_t^{3/2}/\epsilon \sim k_t^2/\epsilon$, and then write a transport equation for k_t and ϵ to obtain the famous $k_t - \epsilon$ model.

There are about 200 turbulent viscosity models published in the literature! (see textbooks by Wilcox and Durbin among others)

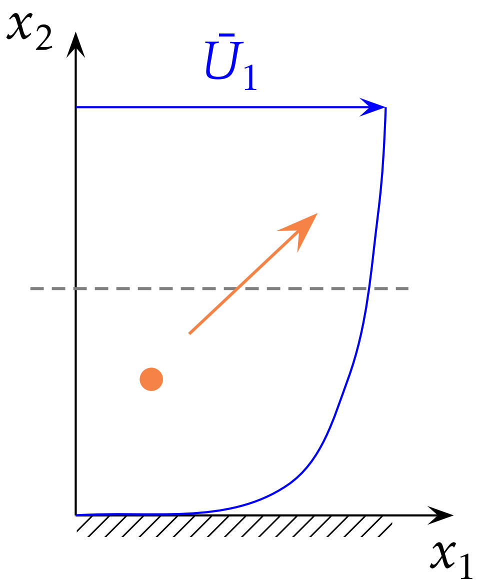
Turbulent kinetic energy budget

The transport equation for k_t is a key result, providing an overview for energy transfers between the mean and the turbulent flows (the demonstration can be found in textbooks). At first glance, production of k_t requires the presence of a mean velocity shear,

$$\underbrace{\frac{\partial(\rho k_t)}{\partial t} + \frac{\partial(\rho k_t \bar{U}_j)}{\partial x_j}}_{\text{advection by the mean flow}} = \underbrace{-\rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j}}_{\text{production } \mathcal{P}} \underbrace{- \rho \epsilon}_{\text{dissipation}}$$

$$+ \underbrace{\frac{\partial}{\partial x_k}\left(-\frac{1}{2}\rho \overline{u_i' u_i' u_k'} - \overline{p' u_k'} + \overline{u_i' \tau_{ik}'} \right)}_{\text{transport terms}}$$

Heuristic interpretation of the production term $\mathcal{P}$



$$\begin{cases} u_2' > 0 \\ u_1' < 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

$$\begin{cases} u_2' < 0 \\ u_1' > 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

$\mathcal{P} \simeq -\rho \overline{u_1' u_2'} \dfrac{d\bar{U}_1}{dx_2} > 0$ is thus expected!

The production term $\mathcal{P}$ is – in general – a transfer from the shear mean flow $\bar{\boldsymbol{U}}$ to the turbulent field $\boldsymbol{u}'$ (but becomes always a positive transfer term using a turbulent viscosity model, a drawback of turbulence models)

Scales

Large scales in $\mathcal{O}(L, u')$ – associated with the geometry (cavity, cylinder, jet, wake, car, airfoil, pipe, ...) and thus directly linked to the size of the flow itself

$\rightsquigarrow$ energy transfer between (basically from) large scale structures to small scale structures, but this transfer is stopped by the molecular viscosity for the smallest scales

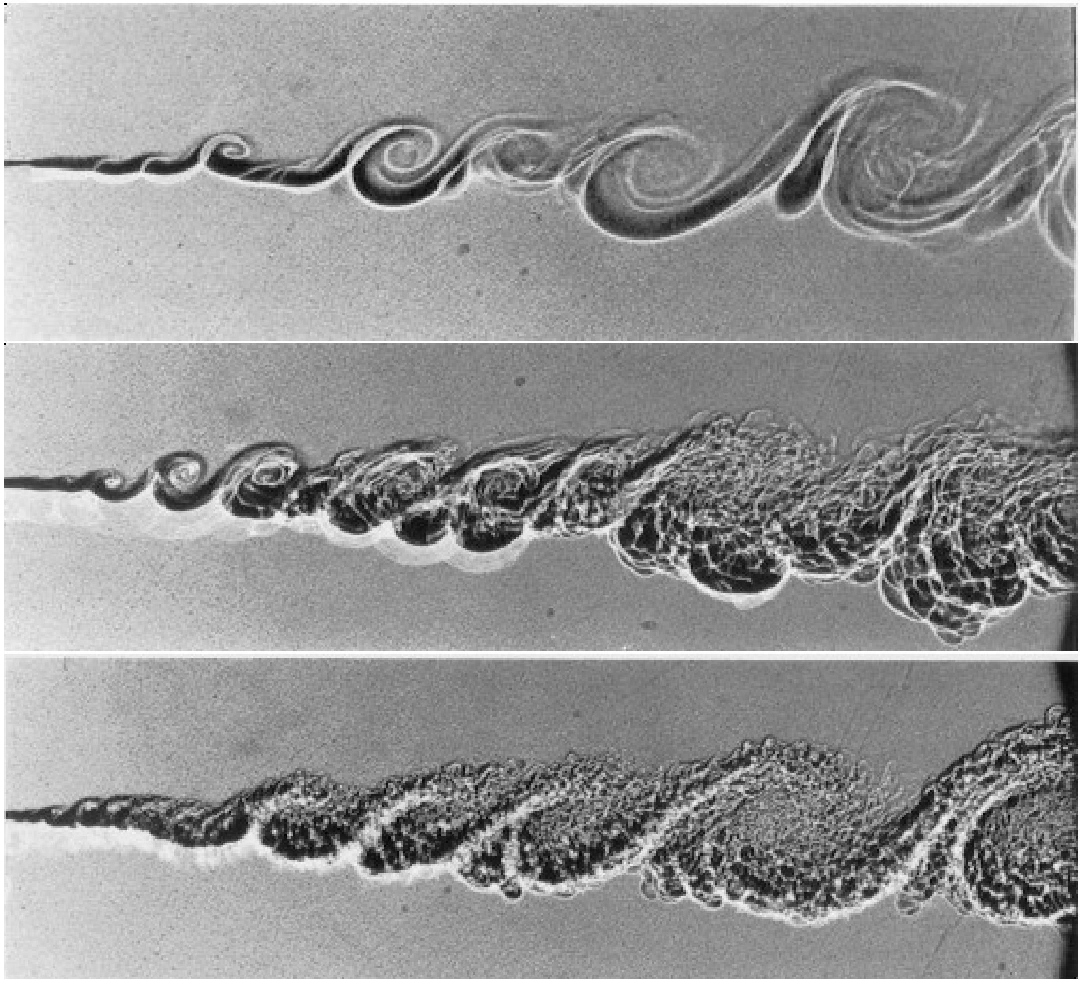
$$\frac{\partial \boldsymbol{u}'}{\partial t} \simeq \nu \nabla^2 \boldsymbol{u}'$$

Smallest scales in $\mathcal{O}(l_\eta, u_\eta)$ – known as the Kolmogorov scales ($\mathrm{Re} = u_\eta l_\eta / \nu = 1$). The Kolmogorov length scale l_η plays a fundamental role in experiments (sampling frequency) as well as in numerical simulations (grid size).

The ratio L/l_η is a function of Reynolds number $\mathrm{Re}_L = u' L/\nu$, where $u'^2 \simeq 2k_t/3$

$$\frac{L}{l_\eta} \sim \mathrm{Re}_L^{3/4}$$

- **Turbulent mixing layer (Brown & Roshko, 1974)**

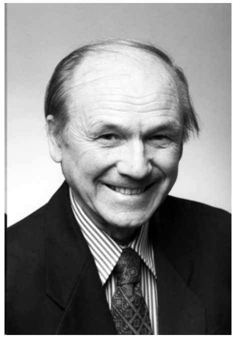


(Shadowgraphs with a spark source)

Energy cascade in a mixing layer by increasing Reynolds number (through pressure and velocity, ×2 for each view)

More small-scale structures are produced without basically altering the large-scale ones (linked to the transition process, as shown by Winant & Browand, 1974)

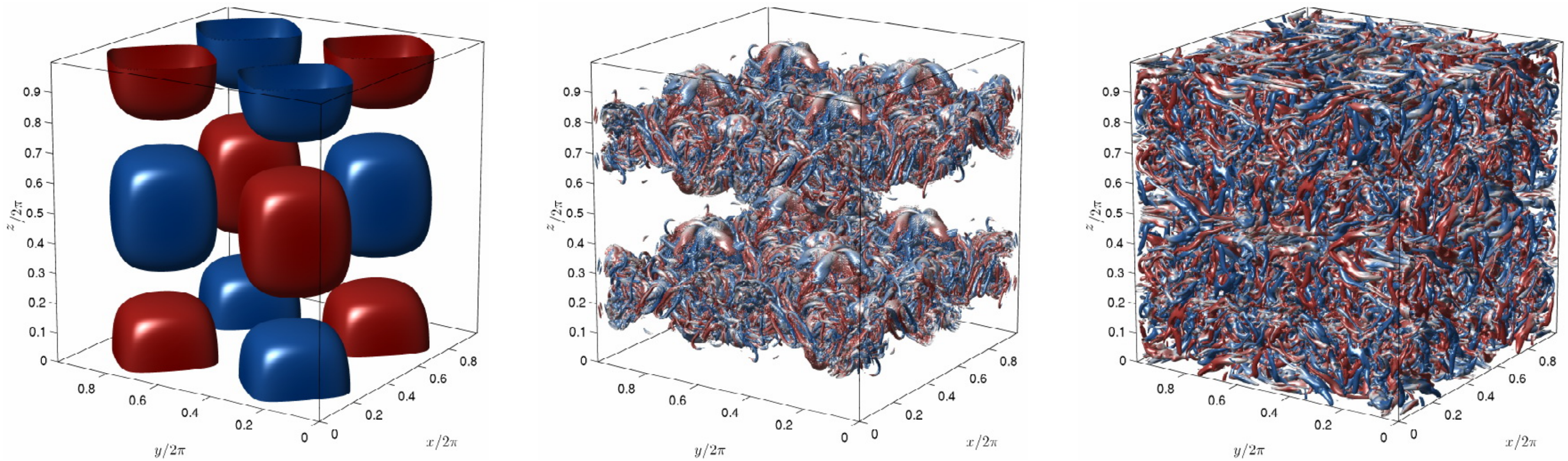
Anatol Roshko (1923-2017)



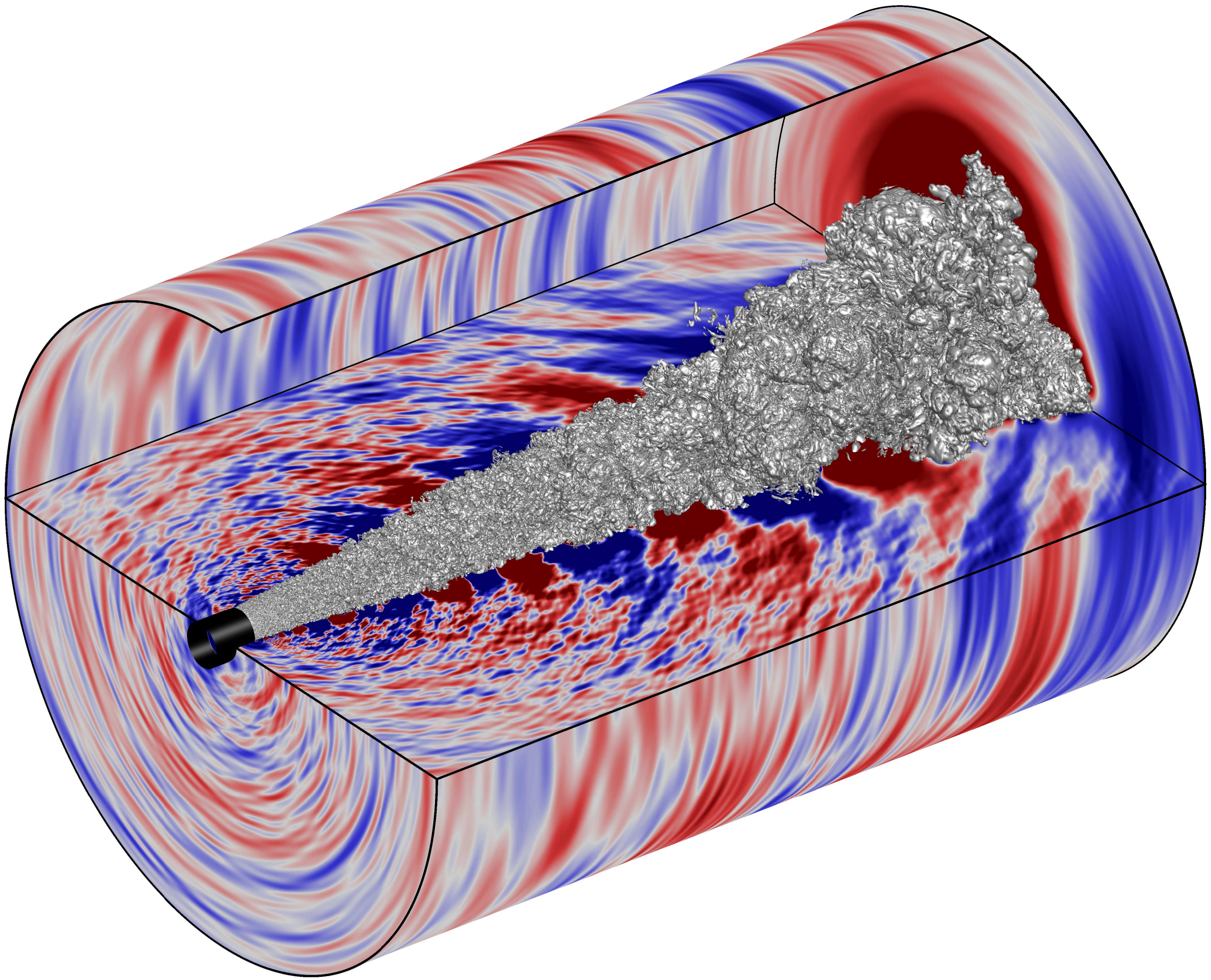
Taylor-Green vortex flow

Re $= 3000$ on a 384^3 grid at times $t = 0$, $t = 9$ and $t = 18$ (dimensionless variables)
From Fauconnier *et al.* (2013)

Vortex structures colored by z-vorticity, $\omega = \nabla \times \boldsymbol{u}$



- High-fidelity simulation of turbulent flow and its noise in a physically and numerically controlled environment



Isothermal turbulent jet at $M = 0.9$ and $Re_D = 10^5$, 1.1×10^9 points, $0 \leq r \leq 7.5D$ and $-D \leq z \leq 20D$

Pressure fluctuations (±55 Pa) and normalized vorticity ($|\omega| \times \delta_\theta / U_c$)

(Bogey, 2017)

- Features of turbulence
- Reynolds decomposition, RANS Eqs. (20)
  and the closure of Reynolds stress tensor $-\rho\overline{u_i' u_j'}$
- Concept of turbulent viscosity
- Key scales and energy cascade

Outline

General information . . . 2
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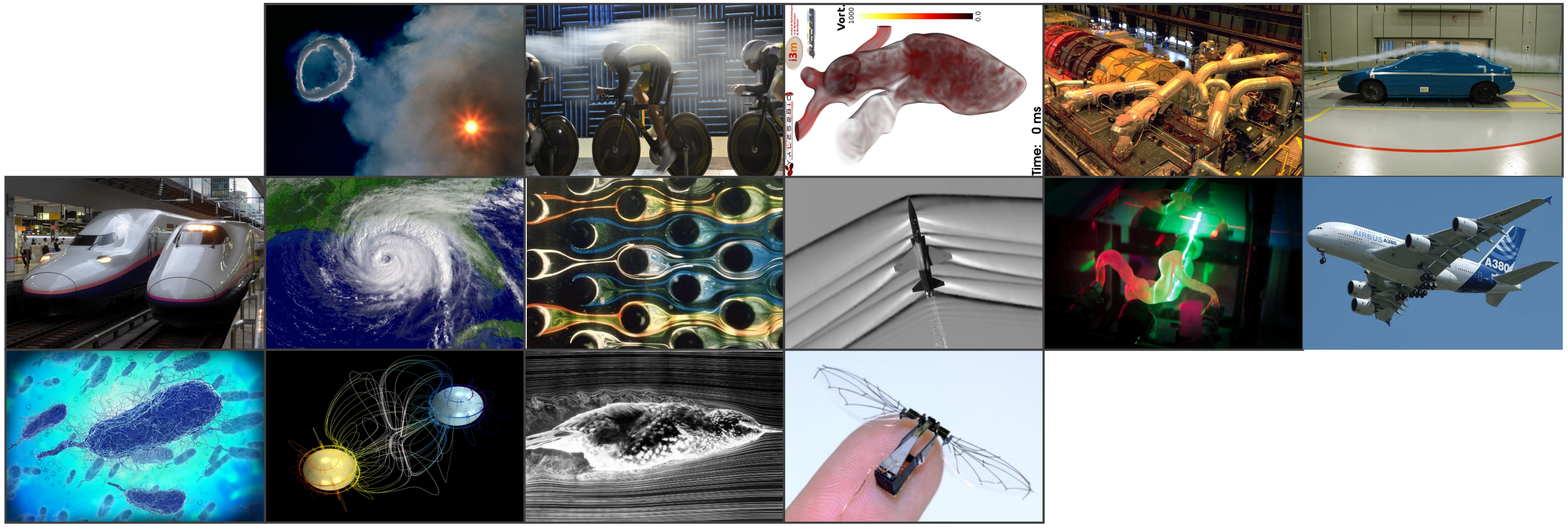
7 - Energy, thermodynamics and compressible flow



7 - Energy, thermodynamics and compressible flow

Conservation of total energy

- Compressible viscous stress tensor
- Internal energy
- First law of thermodynamics
- Heat input
- Local formulation of the energy conservation

Thermodynamics of equilibrium

- Thermodynamic variables
- First law of thermodynamics
- Gibbs identity
- Specific heats
- Perfect gas

Thermodynamics of motion

- Thermodynamic variables
- Constitutive laws
- Governing equations
- Boundary conditions

Entropy and the second law of thermodynamics

- Entropy's equation
- Conservation of internal energy
- Equation for temperature

Compressibility effects

- Mach number
- Selected historical milestones
- Supersonic flow

- **Flow is henceforth compressible :** $\nabla \cdot \boldsymbol{U} \neq 0$

  - As a result, the interactions between heat and dynamics must be now considered, with the introduction of thermodynamic relations

  - Newtonian law for the viscous stress tensor

    $\boldsymbol{\sigma} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}$ with $\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}} + \lambda(\nabla \cdot \boldsymbol{U})\overline{\overline{\boldsymbol{I}}}$

    where λ is the second viscosity

    It is straightforward to show that (left as an exercise)

    $\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} = 2\mu\overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}} + \lambda(\nabla \cdot \boldsymbol{U})^2 \geq 0$

  - Mean normal shear stress : $\sigma_{ii}/3 = \frac{1}{3}(-p + \lambda(\nabla \cdot \boldsymbol{U}))\delta_{ii} = -p + \lambda\nabla \cdot \boldsymbol{U}$

    $\leadsto$ the thermodynamic pressure now differs from the mean normal shear stress

Internal energy

$$E_t = \int_{\mathcal{D}} \rho \left(e + \frac{1}{2} U^2 \right) dv = \int_{\mathcal{D}} \rho e_t \, dv$$

The new variable $e = e(\boldsymbol{x}, t)$ is associated with the internal state description of the fluid (internal motion of molecules or thermal agitation)

e is the specific internal energy in J.kg^{-1}
(specific quantity = value per unit of mass of the fluid)

e_t is the specific total energy
E_t is the total energy contained in the fluid domain $\mathcal{D}$

First law of thermodynamics (conservation of energy)

$$\frac{dE_t}{dt} = P_{\text{ext}} + Q$$

$P_{\text{ext}} \equiv$ rate of work by body and surface (external) forces
$Q \equiv$ amount of heat provided to the fluid domain $\mathcal{D}$
(Remark : time rate of energy change - strictly speaking - power terms, in W)

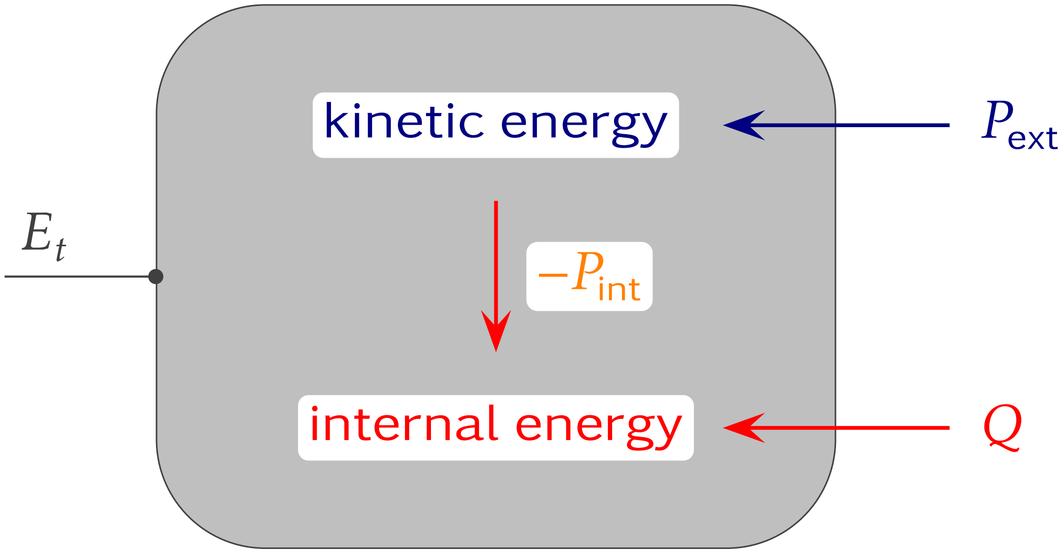
From the conservation of the kinetic energy, refer to Eq. (10) in slide 84, one has

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \frac{U^2}{2}\, d\nu = P_{\text{ext}} + P_{\text{int}}$$

from which it can be deduced that
(= heat addition + viscous effects/friction)

$$\frac{d}{dt}\int_{\mathcal{D}} \rho e\, d\nu = Q - P_{\text{int}}$$

First law of thermodynamics (cont.)



$$\frac{dE_t}{dt} = P_{\text{ext}} + Q$$

First law of thermodynamics (cont.)

Reminder of the expressions of P_{ext} and P_{int}, refer to Eq. (11) in slide 84,

$$P_{\text{ext}} \equiv \int_{\mathcal{S}} \boldsymbol{U} \cdot (\overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}) \, d\mathcal{s} + \int_{\mathcal{D}} \rho \boldsymbol{U} \cdot \boldsymbol{g} \, dv = \int_{\mathcal{D}} \left[\nabla \cdot (\boldsymbol{U} \cdot \overline{\overline{\boldsymbol{\sigma}}}) + \rho \boldsymbol{U} \cdot \boldsymbol{g} \right] dv$$

$$-P_{\text{int}} \equiv \int_{\mathcal{D}} \overline{\overline{\boldsymbol{\sigma}}} : \overline{\overline{\boldsymbol{D}}} \, dv = \underbrace{-\int_{\mathcal{D}} p \nabla \cdot \boldsymbol{U} \, dv}_{\text{compression / dilatation}} \quad + \underbrace{\int_{\mathcal{D}} \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} \, dv}_{\text{viscous dissipation} > 0} \tag{21}$$

The first term is associated with the work of pressure on fluid particles : this term is positive or negative (acts like a ressort), and is null for an incompressible flow

The second term represents the work of viscous forces acting on fluid particles

Heat input

$$Q \equiv Q_s + Q_v$$

Surface heat input

$$Q_s = -\int_{\mathcal{S}} q \, ds$$

Input heat flux accross the surface $\mathcal{S}$: thermal conduction, tranfer of energy through the motion / collisions of molecules and electrons (no mass transfer)

q in W.m^{-2}

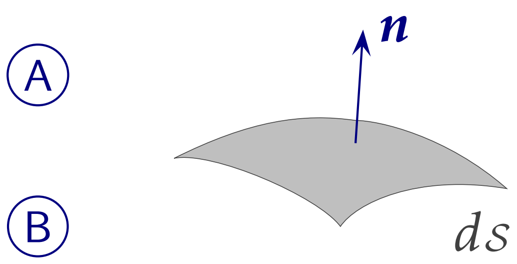
Volumetric heat input

$$Q_v = \int_{\mathcal{D}} \rho q_\star \, dv$$

Radiative heat transfer : direct input inside the domain $\mathcal{D}$, for instance by radiation absorption or emission (electromagnetic waves)

$q_\star$ in W.kg^{-1} (amount of heat addition per unit mass)

Heat flux vector



$q(\boldsymbol{x}, t, \boldsymbol{n})\, ds$ heat flux

heat flux vector $\boldsymbol{q}(\boldsymbol{x}, t)$

$$\boldsymbol{q}(\boldsymbol{x}, t) = \begin{bmatrix} q(\boldsymbol{n} = \boldsymbol{e}_1) \\ q(\boldsymbol{n} = \boldsymbol{e}_2) \\ q(\boldsymbol{n} = \boldsymbol{e}_3) \end{bmatrix}$$

Cauchy's relation : $q(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{q}(\boldsymbol{x}) \cdot \boldsymbol{n}$, linear relation in $\boldsymbol{n}$ (refer to a similar discussion in Chapter 1 for the stress tensor $\boldsymbol{T} = \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}$)

$\boldsymbol{q} \cdot \boldsymbol{n}\, ds$ is the heat flux from B to A accross the elementary surface ds

$$Q_s = -\int_{\mathcal{S}} \boldsymbol{q} \cdot \boldsymbol{n}\, ds = -\int_{\mathcal{D}} \nabla \cdot \boldsymbol{q}\, dv$$

Local formulation for the conservation of energy

Can be written in many different forms, all of which are useful!

$$
\begin{aligned}
&\text{total energy} && \rho\frac{De_t}{Dt} = \nabla\cdot(\boldsymbol{U}\cdot\overline{\overline{\boldsymbol{\sigma}}}) + \rho\boldsymbol{U}\cdot\boldsymbol{g} - \nabla\cdot\boldsymbol{q} + \rho q_\star \\
&\text{kinetic energy} && \rho\frac{D}{Dt}\left(\frac{1}{2}U^2\right) = \nabla\cdot(\boldsymbol{U}\cdot\overline{\overline{\boldsymbol{\sigma}}}) + \rho\boldsymbol{U}\cdot\boldsymbol{g} + p\nabla\cdot\boldsymbol{U} - \overline{\overline{\boldsymbol{\tau}}}:\overline{\overline{\boldsymbol{D}}} \\
&\text{internal energy} && \rho\frac{De}{Dt} = \underbrace{-p\nabla\cdot\boldsymbol{U}}_{\text{compression}} + \underbrace{\overline{\overline{\boldsymbol{\tau}}}:\overline{\overline{\boldsymbol{D}}}}_{\text{dissipation}} \underbrace{-\nabla\cdot\boldsymbol{q} + \rho q_\star}_{\text{heat input}}
\end{aligned}
\tag{22}
$$

$\nabla\cdot(\boldsymbol{U}\cdot\overline{\overline{\boldsymbol{\sigma}}}) + \rho\boldsymbol{U}\cdot\boldsymbol{g}$ local rate of external work ($\mathcal{P}_{\text{ext}}$)

$-\nabla\cdot\boldsymbol{q} + \rho q_\star$ local heat addition ($\mathcal{Q}$)

$p\nabla\cdot\boldsymbol{U} - \overline{\overline{\boldsymbol{\tau}}}:\overline{\overline{\boldsymbol{D}}}$ local rate of internal work ($\mathcal{P}_{\text{int}}$)

Integral formulations can also be established by using the Reynolds theorem (4) introduced in Chapter 2, see slide 34

Thermodynamic variables

Fluid in equilibrium

$$\begin{cases} \text{steady state, no flow (no motion), } \boldsymbol{U} = 0 \\ \overline{\overline{\boldsymbol{\sigma}}} = -p\overline{\overline{\boldsymbol{I}}} \text{ (only pressure)} \Longrightarrow \text{pressure is then defined} \\ \text{fluid properties such as } \rho,\, p \text{ and } e \text{ are independent of } \boldsymbol{x} \text{ and } t \end{cases}$$

Thermodynamics introduces new state variables : the absolute temperature T, the specific entropy s and the specific enthalpy $h \equiv e + p/\rho$

Among all these thermodynamic variables, that is ρ, e, p, T, s, and h, only ρ and e are defined independently of any thermodynamic equilibrium (*i.e.* defined independently of any thermodynamic analysis)

For a simple thermodynamic system, any variable can be calculated from two other independent variables. A local thermodynamic state in fluid mechanics, that is with the presence of flow, can be generalized by first considering $\rho(\boldsymbol{x}, t)$ and $e(\boldsymbol{x}, t)$, to thus determine other variables, $s = s(\rho, p)$ or $p = (\rho, s)$ for instance

Link with the first law of thermodynamics for a closed system

Both heat and work are forms of energy, and energy is conserved

James Prescott Joule (1818-1889)

For a closed system,

$$\delta q = de + pd\left(\frac{1}{\rho}\right) \qquad (23)$$

δq heat supplied to the system (positive toward the system)

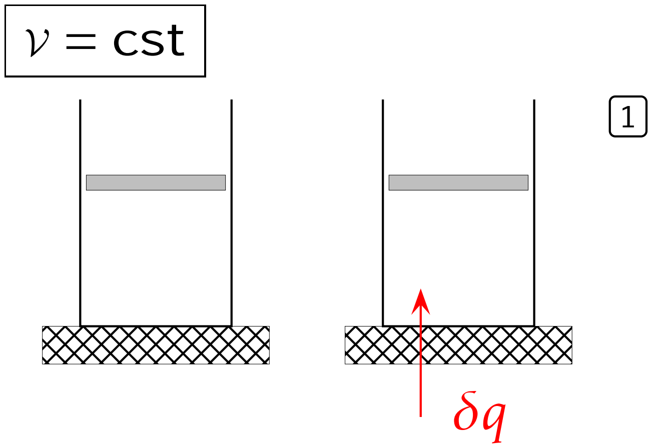
$-pdv_s$ work done on the system by the surroundings, positive toward the system, $v_s = 1/\rho$ is the specific volume

$d \equiv$ depends only upon the change of state (applied to a state variable)
$\delta \equiv$ depends upon the process used to change the state

When δq is obtained by a reversible process, that is a succession of equilibrium states in (23), the internal energy conservation (22) for an ideal fluid is retrieved,

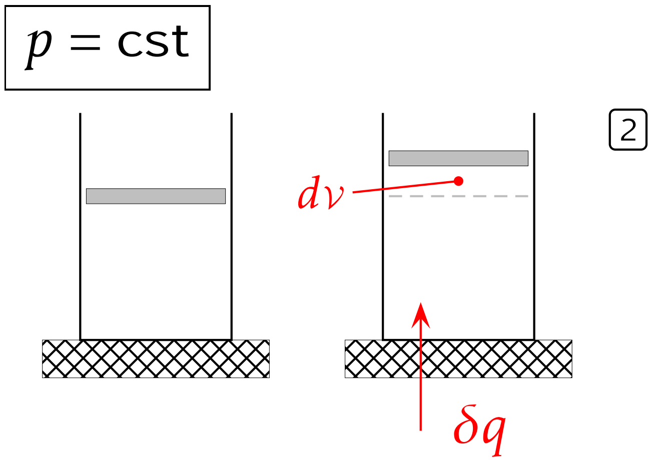
$$\text{Eq. (22)} \Longrightarrow \frac{De}{Dt} = -\frac{p}{\rho}\nabla\cdot\boldsymbol{u} + q_\star = \frac{p}{\rho^2}\frac{D\rho}{Dt} + q_\star = -p\frac{D}{Dt}\left(\frac{1}{\rho}\right) + q_\star$$

Specific heats



Heat δq is supplied to the system at constant volume, the internal energy increases by $\delta q = de = c_v dT$

$$c_v = \left.\frac{\partial e}{\partial T}\right|_\rho \quad \text{specific heat at constant volume}$$



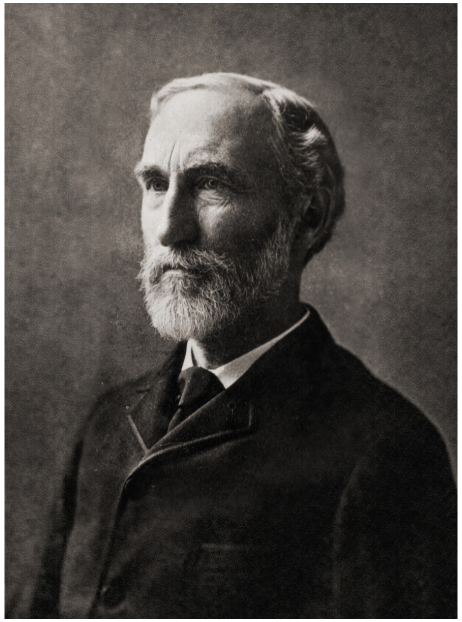
At constant pressure $\delta q = dh = c_p dT$

$$c_p = \left.\frac{\partial h}{\partial T}\right|_p \quad \text{specific heat at constant pressure}$$

With respect to the previous configuration at constant volume, the volume must now expand which requires work, $\delta q = de + p\, dv$

Ideal (perfect) gas $c_p = c_v + r$ Liquid (incompressible) : $c = c_p = c_v$

Second law of thermodynamics and Gibbs relation



Josiah Willard Gibbs
(1839 - 1903)

For a reversible process, $\delta q \equiv T ds$ where the entropy s is a new state variable (Rudolf Clausius, 1822-1888)

and combining with the first law of thermodynamics (23), the following fundamental identity is derived

$$T ds = de + p d\left(\frac{1}{\rho}\right) \qquad \text{Gibbs relation}$$

where for an arbitrary process,

$$ds = \left(\frac{\delta q}{T}\right)_{\text{rev}} + ds_{\text{irr}} \qquad ds_{\text{irr}} > 0$$

ds_{irr} is induced by thermal conduction, viscosity, species diffusion or shocks in supersonic flow

Adiabatic transformation $\delta q = 0$ and $ds \geq 0$
Adiabatic and reversible transformation $\delta q = 0$ and $ds = 0$: isentropic process

About thermodynamics !

"Thermodynamics is a funny subject. The first time you go through it, you don't understand it at all. The second time you go through it, you think you understand it, except for one or two points. The third time you go through it, you know you don't understand it, but by that time you are so used to the subject, it doesn't bother you anymore"

attributed to Arnold Sommerfeld around 1940
(1868-1951, German theoretical physicist)

Perfect (ideal) gas

Equation of State : $p = \rho r T$,
simple thermodynamic system $p = p(\rho, T)$

$$r = \frac{R}{M} \quad \text{with} \quad \begin{cases} R \simeq 8.314472 \text{ J mol}^{-1}\text{K}^{-1} & \text{ideal (universal) gas constant} \\ M \text{ molecular weight of the gas} \end{cases}$$

$c_p - c_v = r$ (Mayer's relation)

$$de = c_v dT \qquad dh = c_p dT \qquad \gamma = \frac{c_p}{c_v} > 1 \qquad s = c_v \ln(p/\rho^\gamma) + \text{cst}$$

Air $r = 287.06$ J.kg^{-1}.K^{-1}
$\gamma = 1.4$ $c_p = 1000$ J.kg^{-1}.K^{-1} $c_v = 720$ J.kg^{-1}.K^{-1}
(standard conditions)

Constitutive laws

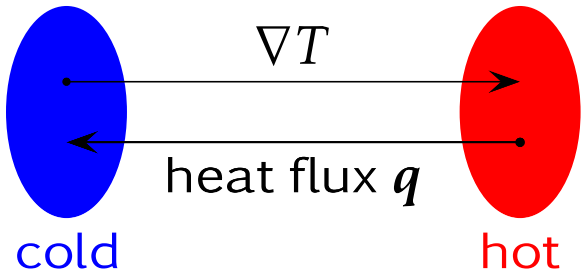
Newtonian law for the viscous stress tensor (see slide 214)

$\boldsymbol{\sigma} = -p\overline{\overline{\boldsymbol{I}}} + \overline{\overline{\boldsymbol{\tau}}}$ with $\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}} + \lambda(\nabla \cdot \boldsymbol{U})\overline{\overline{\boldsymbol{I}}}$

Fourier's law $\boldsymbol{q} = -k\nabla T$

where k is the thermal conductivity

(the minus sign expresses that heat flows from hot to cold)



The coefficients λ, μ and k are also functions of the local thermodynamic state, for instance with the thermal conductivity $k = k(\rho, e)$

Governing equations for a compressible flow

$\dfrac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{U} = 0$	mass
$\rho\dfrac{D\boldsymbol{U}}{Dt} = -\nabla p + \nabla\cdot\overline{\overline{\boldsymbol{\tau}}} + \rho\boldsymbol{g}$	momentum
$\rho\dfrac{De}{Dt} = -p\nabla\cdot\boldsymbol{U} + \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} - \nabla\cdot\boldsymbol{q} + \rho q_\star$	internal energy

with constitutive laws

$$\overline{\overline{\boldsymbol{\tau}}} = 2\mu\overline{\overline{\boldsymbol{D}}} + \lambda(\nabla\cdot\boldsymbol{U})\overline{\overline{\boldsymbol{I}}}$$
$$\boldsymbol{q} = -k\nabla T$$

and equations of state (by choosing ρ and T here to express the other thermodynamic variables)

$$p = p(\rho, T) \quad e = e(\rho, T) \qquad \mu = \mu(\rho, T) \quad \lambda = \lambda(\rho, T) \quad k = k(\rho, T)$$

Boundary conditions at a solid wall

For the velocity,
viscous fluid, $\boldsymbol{U} = \boldsymbol{U}_{\text{wall}}$
inviscid fluid, $\boldsymbol{U} \cdot \boldsymbol{n} = \boldsymbol{U}_{\text{wall}} \cdot \boldsymbol{n}$

For the temperature, $T = T_{\text{wall}}$ for a wall.

The fluid problem should be coupled with the thermal conduction inside the solid, by imposing the continuity of the temperature T and of the heat flux $q = -k\boldsymbol{n} \cdot \nabla T$ between the solid and the fluid.

As for an inviscid flow (no viscosity), the case $k = 0$ (thermal conductivity neglected) must be interpreted with some caution, see slide 264 about the thermal boundary layer

An ideal flow is a flow model for which $\nu = 0$ and $k = 0$
(no viscosity, no thermal conductivity)

Entropy's equation

By applying the Gibbs relation (see slide 225) to a fluid particle, in order to introduce entropy

$$T\,ds = de + p\,d\left(\frac{1}{\rho}\right) \quad \Longrightarrow \quad T\frac{Ds}{Dt} = \frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}$$

the last term can be recast by using the conservation of mass

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \nabla\cdot\boldsymbol{U} \quad \Longrightarrow \quad \boxed{\rho T\frac{Ds}{Dt} = \rho\frac{De}{Dt} + p\nabla\cdot\boldsymbol{U}}$$

Then, the conservation of the internal energy can be recast as

$$\rho\frac{De}{Dt} = -p\nabla\cdot\boldsymbol{U} + \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} - \nabla\cdot\boldsymbol{q} + \rho q_\star \quad \Longrightarrow \quad \rho\frac{Ds}{Dt} = \frac{\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}}{T} - \frac{\nabla\cdot\boldsymbol{q}}{T} + \frac{\rho q_\star}{T}$$

and finally, with the help of the following vectorial identity

$$\frac{\nabla\cdot\boldsymbol{q}}{T} = \frac{\boldsymbol{q}\cdot\nabla T}{T^2} + \nabla\cdot\left(\frac{\boldsymbol{q}}{T}\right)$$

Entropy's equation (cont.)

... the following transport equation for entropy is obtained,

$$\rho \frac{Ds}{Dt} = \underbrace{\frac{\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}}{T} - \frac{\boldsymbol{q} \cdot \nabla T}{T^2}}_{\text{production of entropy}} + \underbrace{\nabla \cdot \left(\frac{\boldsymbol{q}}{T} \right)}_{\text{entropy's flux}} + \underbrace{\frac{\rho q_\star}{T}}_{\text{volumetric heat input}}$$

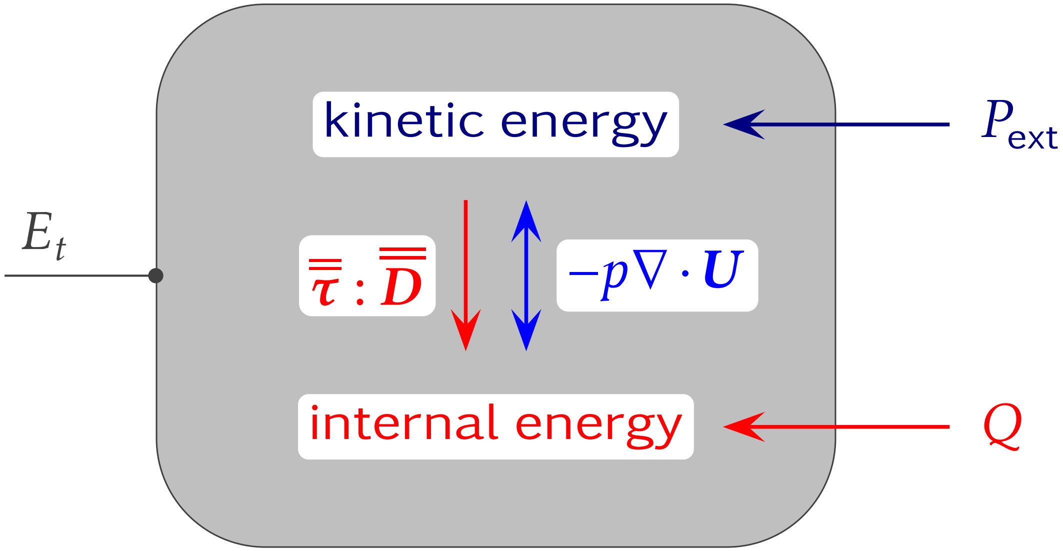
The production of entropy in a control domain $\mathcal{D}$ is thus given by (refer to the definition of dissipation for the first term)

$$\mathcal{P}_s \equiv \int_{\mathcal{D}} \left(\frac{\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}}{T} - \frac{\boldsymbol{q} \cdot \nabla T}{T^2} \right) dv = \begin{array}{c}\text{mechanical dissipation}\\ \text{induced by friction}\end{array} + \begin{array}{c}\text{thermal "dissipation"}\\ \text{induced by conduction}\end{array}$$

According to the second law of thermodynamics, entropy never decreases in an isolated system ($\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on $\mathcal{S}$, $q_\star = 0$), and increases in presence of an irreversible process. It can be mathematically demonstrated that

$$\mathcal{P}_s \geq 0 \quad \Longrightarrow \quad \begin{cases} \mu \geq 0 \\ \lambda \geq -2\mu/3 \\ k \geq 0 \end{cases}$$

Conservation of the internal energy



$$-P_{\text{int}} \equiv \int_{\mathcal{D}} \overline{\overline{\boldsymbol{\sigma}}} : \overline{\overline{\boldsymbol{D}}} \, dv = \underbrace{-\int_{\mathcal{D}} p \nabla \cdot \boldsymbol{U} \; dv}_{\text{reversible process}} + \underbrace{\int_{\mathcal{D}} \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} \, dv}_{\text{irreversible process}}$$

$$\begin{cases} \nabla \cdot \boldsymbol{U} < 0 & \text{compression} \qquad \text{kinetic energy} \rightarrow \text{internal energy} \\ \nabla \cdot \boldsymbol{U} > 0 & \text{expansion} \qquad \text{internal energy} \rightarrow \text{kinetic energy} \end{cases}$$

Transport equation for temperature

From $s = s(p, T)$, it can be written that

$$\frac{Ds}{Dt} = \left.\frac{\partial s}{\partial p}\right|_T \frac{Dp}{Dt} + \left.\frac{\partial s}{\partial T}\right|_p \frac{DT}{Dt} = -\frac{\beta}{\rho}\frac{Dp}{Dt} + \frac{c_p}{T}\frac{DT}{Dt}$$

(using Maxwell relations)

and by using the entropy conservation equation

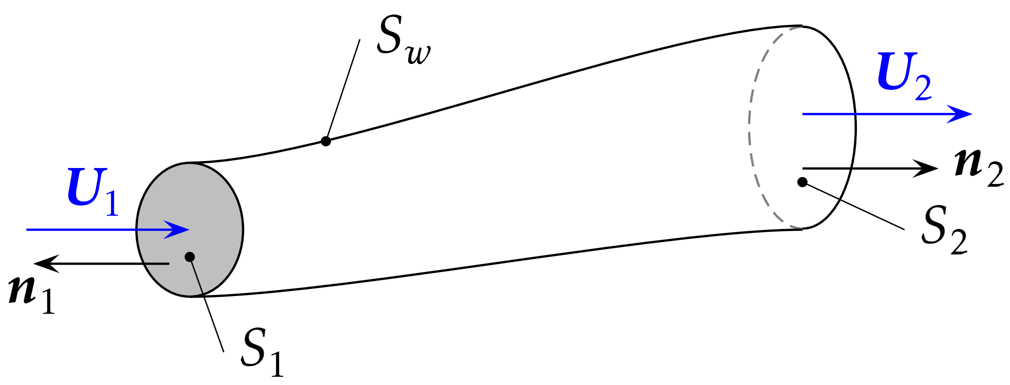
$$\rho T \frac{Ds}{Dt} = \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} - \nabla \cdot \boldsymbol{q} + \rho q_\star$$

the transport equation for the temperature can be finally derived (general formulation for gas and liquid)

$$\rho c_p \frac{DT}{Dt} = \overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} + \beta T \frac{Dp}{Dt} - \nabla \cdot \boldsymbol{q} + \rho q_\star \qquad (24)$$

$\beta \equiv -\dfrac{1}{\rho}\left.\dfrac{\partial \rho}{\partial T}\right|_p = \dfrac{1}{T}$ for a perfect gas (β thermal expansion coefficient)

1-D compressible ideal flow model



steady inviscid flow

no gravity

1-D duct model (flow uniform in a cross section)

Application of the Reynolds theorem (4) for a fixed domain $\mathcal{D}$, using Eq. (22) for the total energy e_t

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{\mathcal{D}} \rho \, dv = -\int_{\mathcal{S}} \rho \boldsymbol{U} \cdot \boldsymbol{n} \, ds \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{D}} \rho \boldsymbol{U} \, dv = -\int_{\mathcal{D}} \nabla p \, dv - \int_{\mathcal{S}} \rho \boldsymbol{U} \boldsymbol{U} \cdot \boldsymbol{n} \, ds = -\int_{\mathcal{S}} [p\boldsymbol{n} + \rho \boldsymbol{U}(\boldsymbol{U} \cdot \boldsymbol{n})] \, ds \\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{D}} \rho e_t \, dv = -\int_{\mathcal{D}} \nabla \cdot (p\boldsymbol{U}) \, dv - \int_{\mathcal{S}} \rho e_t \boldsymbol{U} \cdot \boldsymbol{n} \, ds = -\int_{\mathcal{S}} (p + \rho e_t) \boldsymbol{U} \cdot \boldsymbol{n} \, ds \end{cases}$$

1-D compressible inviscid flow model (cont.)

Introducing the total enthalpy $h_t = h + U^2/2$, with $h = e + p/\rho$ and consequently $\rho e_t + p = \rho h_t$, the last previous Eq. reads as

$$\frac{d}{dt}\int_{\mathcal{D}} \rho e_t \, dv = -\int_{\mathcal{S}} \rho h_t \boldsymbol{U}\cdot\boldsymbol{n}\, ds$$

The conservation of mass, momentum and total energy then provides

$$\begin{cases} [\rho U S]_1^2 = 0 \\ \left[(\rho U^2 + p)S\right]_1^2 = 0 \\ [\rho h_t U S]_1^2 = 0 \end{cases} \Longrightarrow \begin{cases} \rho_1 U_1 S_1 = \rho_2 U_2 S_2 = \dot{Q}_m \quad \text{mass flow rate} \\ (\rho_1 U_1^2 + p_1)S_1 = (\rho_2 U_2^2 + p_2)S_2 \\ h_{t1} = h_{t2} \end{cases}$$

The Pitot tube revisited, see slide 56

The total (or stagnation) enthapy h_t is thus preserved. Using the same notations as for the incompressible case,

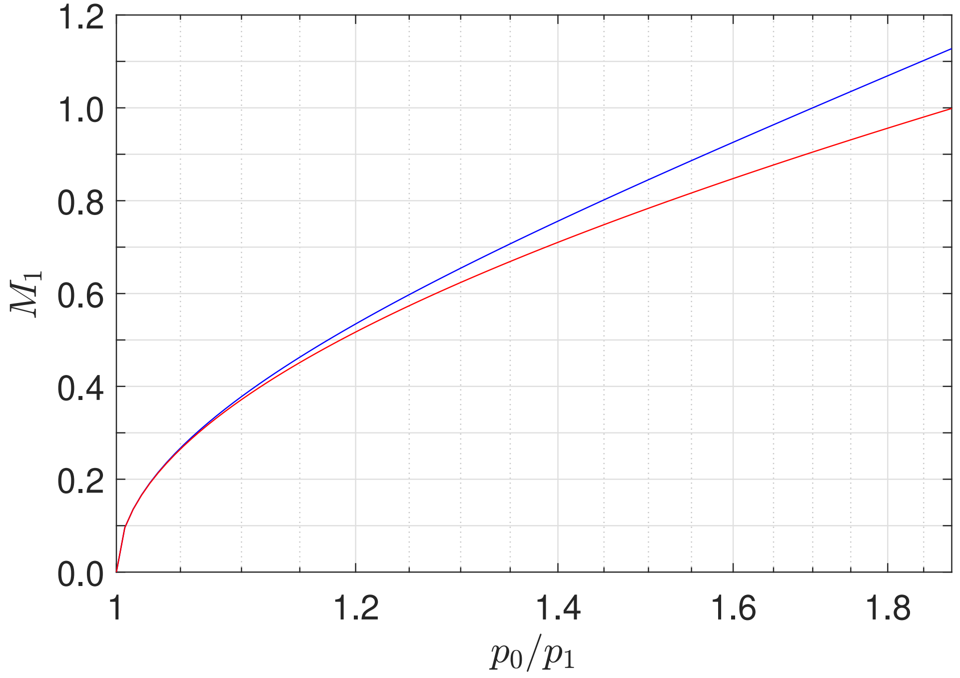
$$h_{t1} = h_{t0} \qquad c_p T_1 + \frac{U_1^2}{2} = c_p T_0 \qquad c_p = \frac{\gamma r}{\gamma - 1} \qquad \text{(perfect gas)}$$

$$T_0 = T_1 + \frac{\gamma - 1}{2}\frac{1}{\gamma r}U_1^2 \quad \Longrightarrow \quad T_0 = T_1\left(1 + \frac{\gamma - 1}{2}M_1^2\right) \qquad M_1 = \frac{U_1}{c_1} \qquad (c_1^2 = \gamma r T_1)$$

From the equation of state for ideal gas $p = \rho r T$ and the isentropic relation $p/\rho^\gamma =$ cst, refer to slide 227, it is then straightforward to obtain a similar relation for pressure

$$\frac{p_0}{p_1} = \left(1 + \frac{\gamma - 1}{2}M_1^2\right)^{\frac{\gamma}{\gamma - 1}}$$

The Pitot tube revisited (cont.)



Compressible flow model

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

Incompressible flow model

$$M_1^2 = \frac{2}{\gamma}\left(\frac{p_0}{p_1} - 1\right)$$

$M_1 \rightarrow M_1$ as $\Delta p = p_0 - p_1 \rightarrow 0$

Deviation by 5% of the incompressible model from the compressible one for $M_1 \simeq 0.3$, which provides the classical condition for applying incompressible flow model

Some historical milestones

Ernst Mach (1838-1916, Austrian physicist)
(The origins of the Mach number 1887)

Karl Gustaf Patrik de Laval (1845-1913, Sweden)
Supersonic convergent-divergent nozzle (1893)

Some historical milestones

14 oct. 1947 - First supersonic aircraft
Bell X-1 flying at $M \simeq 1.06$
Major Chuck Yeager (USAF, 1923-2020)

(US Air Force, 1947)

14 oct. 2012 - First person to break the sound barrier $M \simeq 1.25$ (freefall)
Felix Baumgartner
(Austrian, living in Germany)

(Red Bull Stratos project)

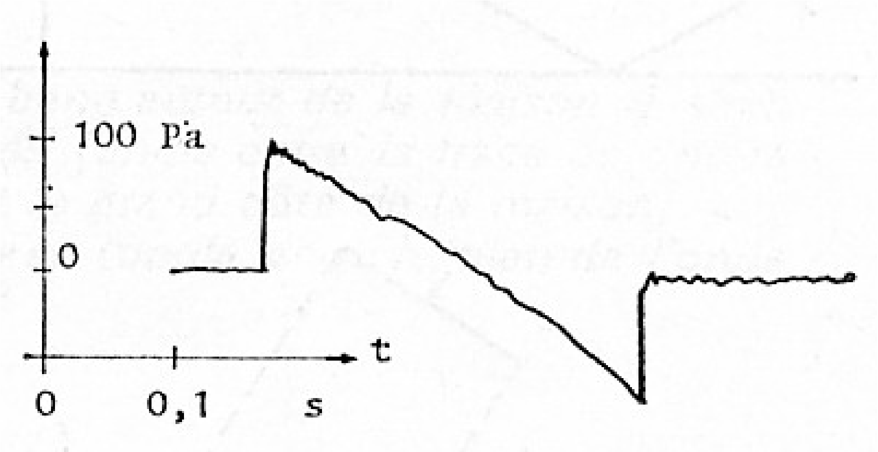
Sonic boom

F/A-18 Hornet breaks the sound barrier
Off the coast of Pusan, South Korea, July 7, 1999
(U.S. Navy/Ensign John Gay)

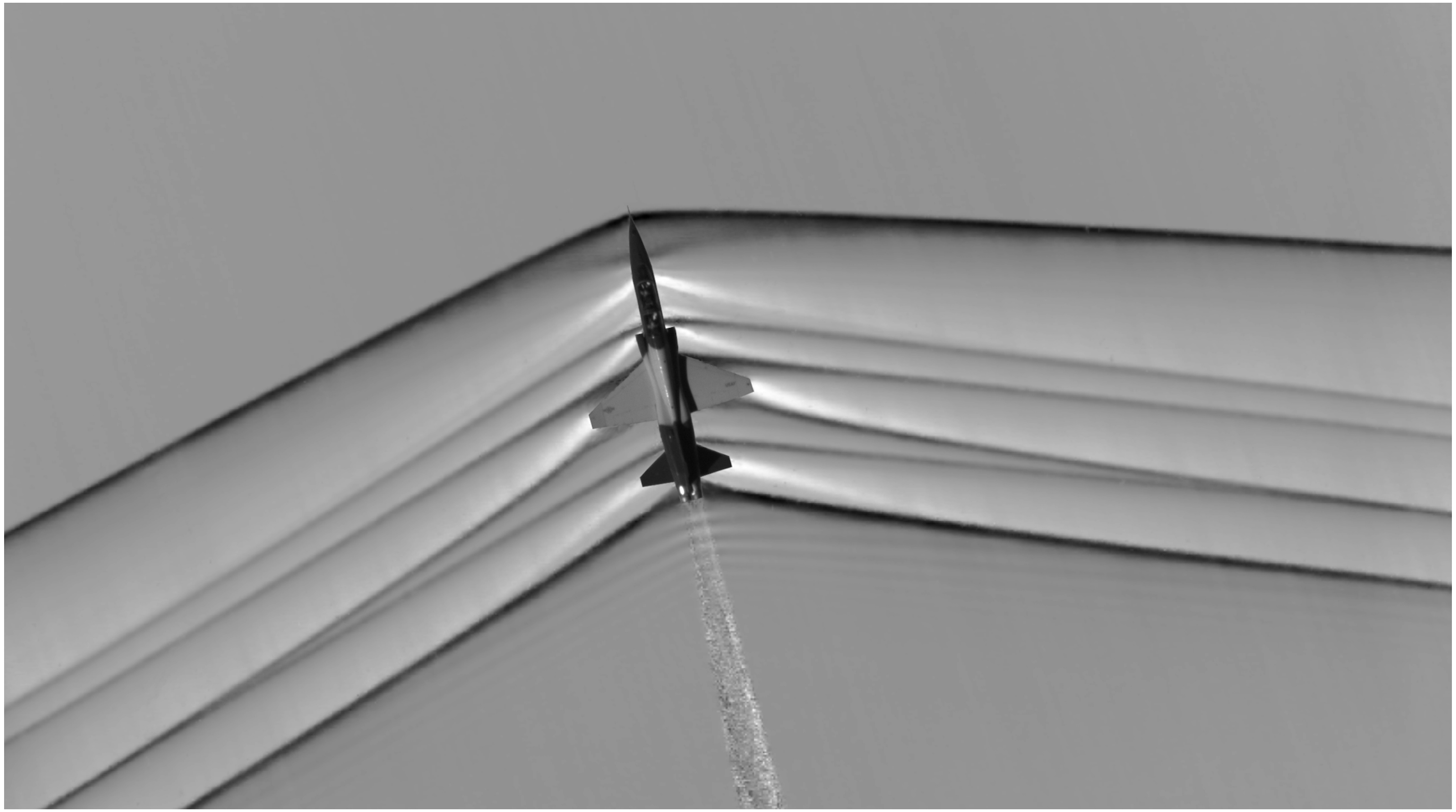
Concorde - Shock waves at Mach 2.2
in wind tunnel (ONERA)

The cloud forms as a result of the decrease in pressure and temperature behind the shock

N-wave pattern measured close to the ground from Concorde



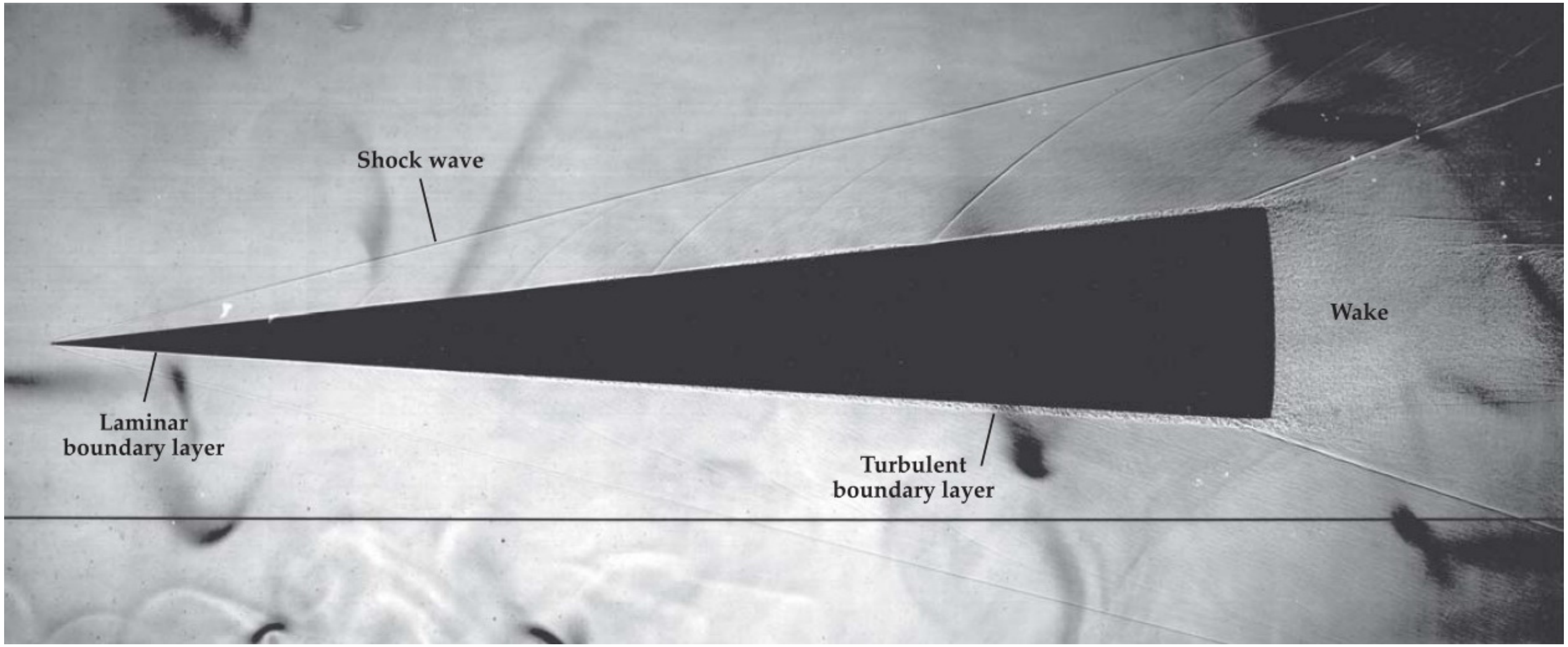
- **Shock waves of a supersonic jet flying above the Mojave desert (NASA, 2015)**



Shadowgraph of transition on a sharp cone at Mach 4.31

(Schneider, *Prog. Aero. Sci.*, 2004, from Naval Ordnance Lab ballistics range)

A shock wave emanating from the nose of a cone travelling at Mach 4 in a ballistic range shows up as a thin dark line in this Schlieren image ; the sharp jump in density across the shock produces a steep refractive-index gradient, which in turn deflects transmitted light, thereby producing the contrast that we observe in the figure. Also visible are laminar and turbulent boundary layers and the wake. Re $\simeq 6.2 \times 10^5$



- Compressible flow
- Conservation of energy, Eqs. (22)
- Governing equations for a compressible flow
- 1-D compressible inviscid flow model
- General transport equation for temperature, see Eq. (24) in slide 234

Outline

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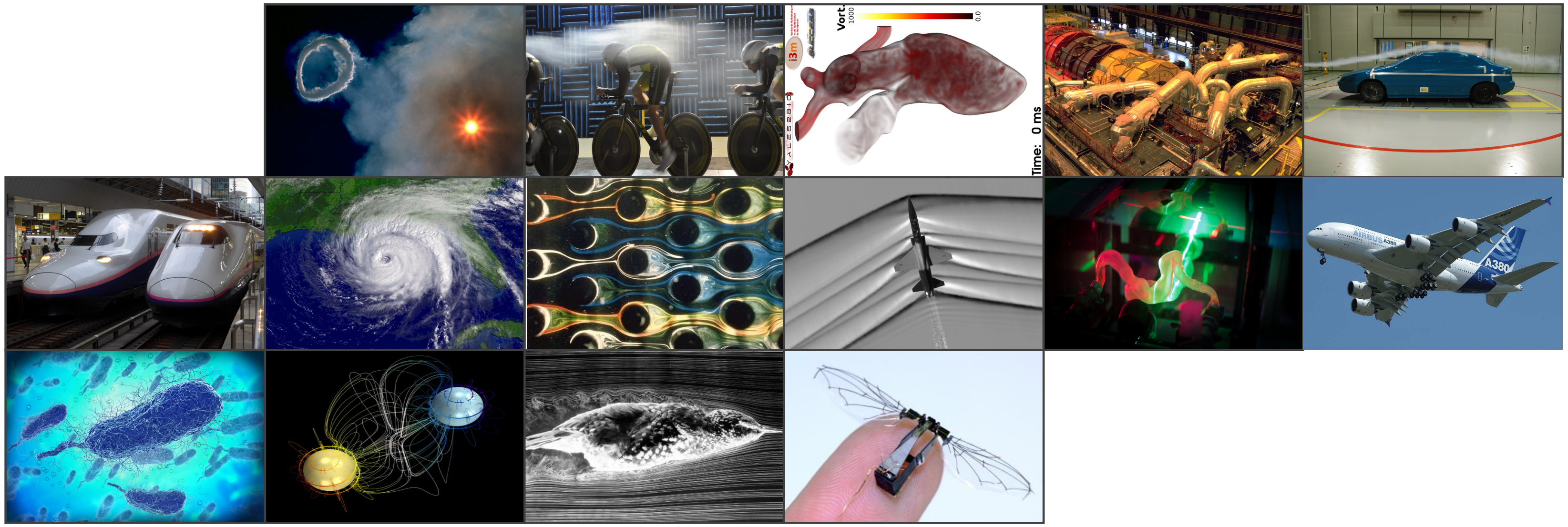
8 - Heat transfer



8 - Heat transfer

Heat transfer

- Assumptions and Boussinesq's approximation
- Transport Eq. for temperature
- Prandtl number

Heat conduction

- Some classical solutions

Advection - diffusion

- Péclet number
- High Péclet number flow
- Thermal boundary layer

Convection heat transfer

- Transfert coefficient and Nusselt number
- Biot number
- Solid-fluid heat transfer model
- Heat transfer to flow in pipe

Natural convection

- Forced and natural convection
- Grashof number

Key results

If you can't take the heat, don't tickle the dragon!

Assumptions

The assumption of low Mach number flow is relevant in many heat transfer problems, and is useful as it makes the problem much simpler to solve. It will be hold throughout this chapter.

Transport equation for temperature, see Eq. (24) in slide 234

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \boldsymbol{q} \; + \underbrace{\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}} + \beta T \frac{Dp}{Dt}}_{\text{neglected source terms}} \; + \underbrace{\rho q_\star}_{\text{not considered here}}$$

$-\nabla \cdot \boldsymbol{q}$	heat flux by thermal conduction $\boldsymbol{q} = -k \nabla T$ Fourier's law, k (constant) thermal conductivity
$\overline{\overline{\boldsymbol{\tau}}} : \overline{\overline{\boldsymbol{D}}}$	heat generated through deformation by viscous stresses ($\sim \mathrm{M}_a^2$) (frictional dissipation)
$\beta T \dfrac{Dp}{Dt}$	influence of pressure ($\sim \mathrm{M}_a^2$ for a gas), β thermal expansion coefficient $\beta \equiv -\dfrac{1}{\rho} \dfrac{\partial \rho}{\partial T}\Big\|_p = \dfrac{1}{T}$ for perfect gas (and $\beta \simeq 0$ for a liquid)
$\rho q_\star$	heat input (*e.g.* radiation), not considered here

Boussinesq's approximation

For the equation of state, in accordance with the assumption of low Mach number flow (incompressible), density is assumed to be a function of the temperature only, that is $\rho = \rho(T, P) \simeq \rho(T)$.

Density is assumed to be a linear function of the temperature for small temperature variations

$$\rho/\rho_0 \simeq 1 - \beta(T - T_0)$$

where $\rho_0 = \rho_0(T_0)$ stands for the fluid at rest. Finally, one usually may neglect density variations, $\beta(T - T_0)$ is a small term, except in the buoyancy force for natural convection. This leads to the Boussinesq approximation for the momentum conservation equation,

$$\rho_0\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{U} + \rho \boldsymbol{g} \qquad (25)$$

Transport equation for temperature

$$\boxed{\frac{\partial T}{\partial t} + \boldsymbol{U} \cdot \nabla T = a\nabla^2 T} \qquad \text{where} \qquad a \equiv \frac{k}{\rho c_p}$$

a is the thermal diffusivity, has units $\mathrm{m}^2.\mathrm{s}^{-1}$, and measures the ability of the molecular transport to remove temperature gradient (similar to the dynamic viscosity ν for velocity gradient)

This transport Eq. must be completed by appropriate boundary conditions.

The velocity is assumed to be known here, in order to determine $T(\boldsymbol{x}, t)$
(this point will be discussed in the last section)

Joseph Fourier

Baron Jean Baptiste Joseph Fourier (1768-1830)

Joseph Fourier lived a remarkable double life. He served as a high government official in Napoleonic France and he was also an applied mathematician of great importance. He was with Napoleon in Egypt between 1798 and 1801, and he was subsequently prefect of the administrative area ("Département") of Isère in France until Napoleon's first fall in 1814. During the later period he worked on the theory of heat flow and in 1807 submitted a 234-page monograph on the subject. It was given to such luminaries as Lagrange and Laplace for review. They found fault with his adaptation of a series expansion suggested by Daniel Bernoulli in the eighteenth century. Fourier's theory of heat flow, his governing differential equation, and the now-famous "Fourier series" solution of that equation did not emerge in print from the ensuing controversy until 1822.

(from Lienhard IV & Lienhard V, *A Heat Transfer Textbook*, 2017)

Prandt number

$$\boxed{\mathrm{Pr} \equiv \frac{\nu}{a} = \frac{\mu c_p}{k}} = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{L^2/a = \tau_{\text{thermal}}}{L^2/\nu = \tau_{\text{viscous}}}$$

Prandtl number of various fluids at 20° C (White, *Viscous Fluids*, 1991)

thermal diffusivity dominates — momentum diffusivity dominates

Pr: 0.01 — 1 — 100 — 10000

liquid metals — gas — oil

Mercury 0.024
Sodium 0.004

Air 0.72
Helium 0.70

Water 7
Freon-12 3.7

Castor oil 10000
Glycerin 12000

Heat transfer by conduction, *i.e.* without motion of the medium

The heat diffusion equation reads

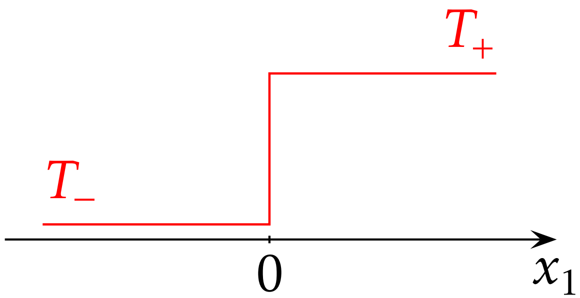
$$\frac{\partial T}{\partial t} = a\nabla^2 T$$

1-D elementary solution : initial temperature jump

$T = T_-$ for $x_1 < 0$ and $T = T_+$ for $x_1 > 0$ at $t = 0$

The temperature evolution is governed by

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x_1^2} \qquad t > 0$$



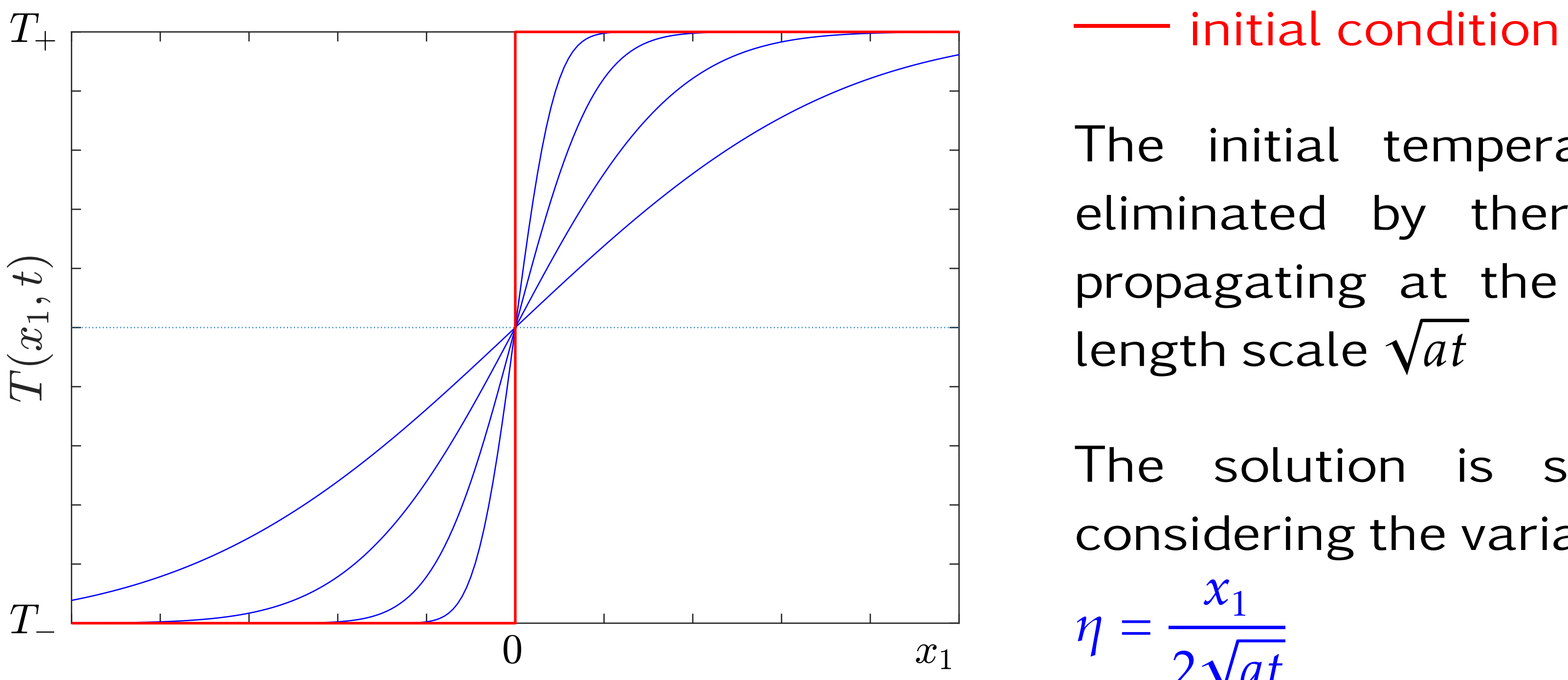
and the solution is given by

$$T(x_1,t) = \frac{T_- + T_+}{2} + \frac{T_+ - T_-}{2}\,\mathrm{erf}\left(\frac{x_1}{2\sqrt{at}}\right) \qquad \mathrm{erf}(\eta) \equiv \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\xi^2}d\xi$$

where erf is the error function.

1-D elementary solution : initial temperature jump (cont.)

Solution $T(x_1)$ plotted for different times t



The initial temperature jump is eliminated by thermal diffusion, propagating at the characteristic length scale $\sqrt{at}$

The solution is self-similar, by considering the variable η

$$\eta = \frac{x_1}{2\sqrt{at}}$$

General solution to the heat diffusion equation

Diffusion of a point heat source

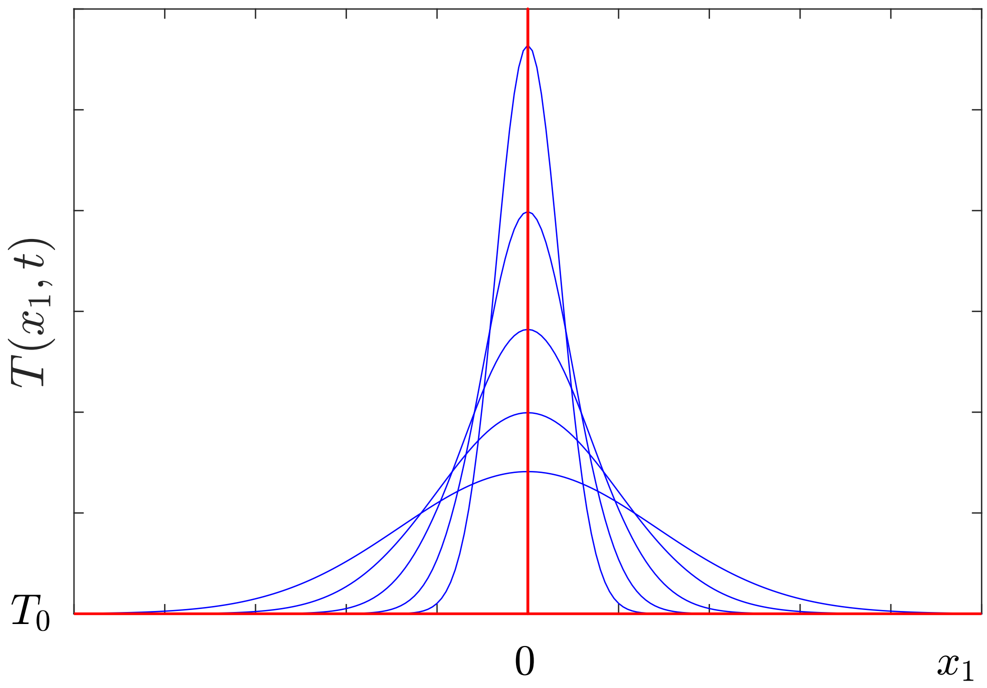
$$T(x_1,t) = T_0 + \frac{C}{\sqrt{4\pi a t}} \exp\left(-\frac{x_1^2}{4at}\right)$$

Gaussian shape,
width $\propto \sqrt{at}$ and peak $\propto \sqrt{t}$

$$\int_{-\infty}^{+\infty} (T - T_0)\, dx_1 = C$$

$T \longrightarrow T_0 + C\delta(x_1)$ as $t \longrightarrow 0$

—— initial condition



Thermal shock in a semi-infinite medium

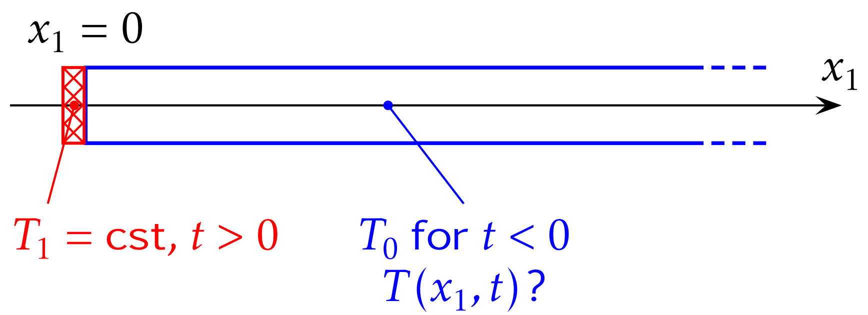
Initial condition
$T(x_1, 0) = T_0$ for $x_1 > 0$,

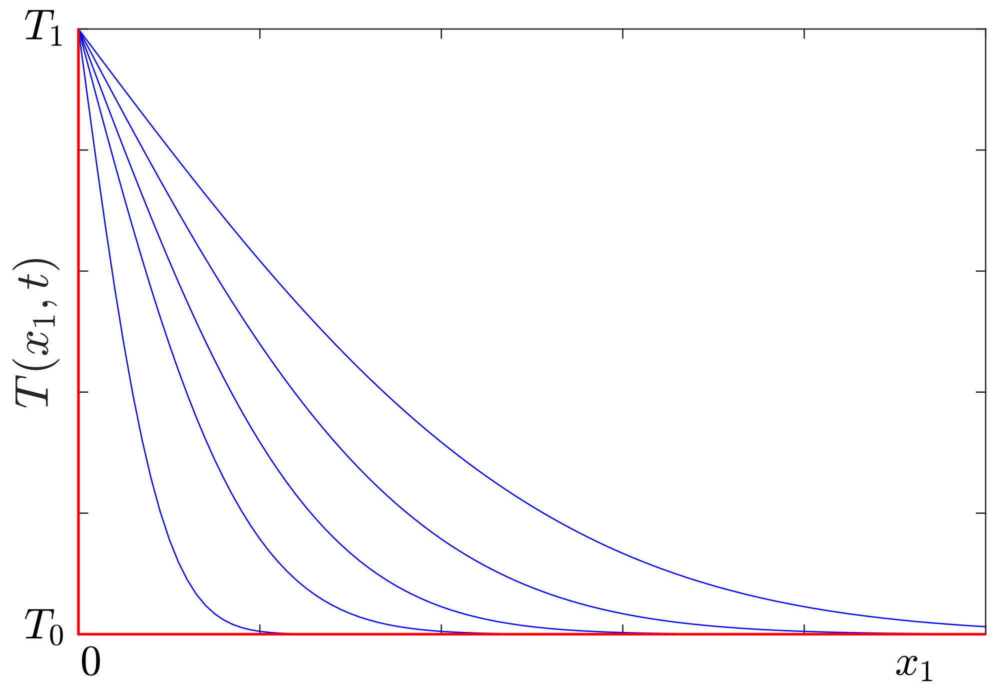
with a sudden change in temperature (thermal shock) applied at $x_1 = 0$ for $t > 0$, $T(0,t) = T_1$, associated with $T(x_1,t) \rightarrow T_0$ as $x_1 \rightarrow \infty$

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x_1^2} \qquad t > 0$$

$$T(x_1,t) = T_1 + (T_0 - T_1)\,\mathrm{erf}\left(\frac{x_1}{2\sqrt{at}}\right)$$

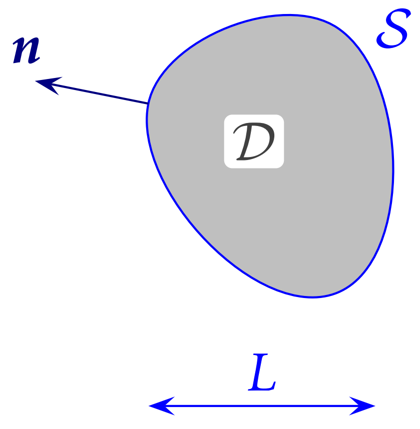
Penetration depth $\propto \sqrt{at}$





Heat diffusion in an isolated finite body

$$\frac{\partial T}{\partial t} = a\nabla^2 T \qquad \boldsymbol{x} \in \mathcal{D}$$



$T(\boldsymbol{x}, 0) = T_0(\boldsymbol{x})$ initial-value problem (non uniform)
$\boldsymbol{n} \cdot \nabla T = 0$ adiabatic boundary condition (isolated)

Estimation of the time scale for heat diffusion
$L \sim \sqrt{at} \quad \Longrightarrow \quad t \sim L^2/a$

For $t \ll L^2/a$, one has $T \sim T_0(\boldsymbol{x})$

For $t \sim L^2/a$, diffusive evolution by heat conduction

For $t \gg L^2/a$, the whole temperature field takes a constant value inside the body

For the case of a thermal shock, with $T = T_1(\boldsymbol{x}, t)$ prescribed on the surface $\mathcal{S}$, the temperature field becomes uniform for a time $t \sim L^2/a$. The time evolution of the penetrate depth is given by $\sqrt{at}$.

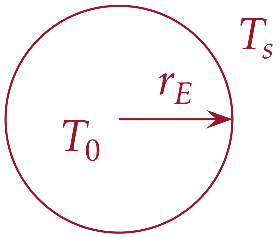
The age-of-the-Earth debate

- about 4000 years old from the Bible
  J. Keplers (1571-1630)
  I. Newton (1643-1727)
  J. Ussher (1581-1656)
  (also from Chinese imperial dynasty)

- Controversies (among others) with the geologists!
  Charles Darwin (1809-1882)
  John Perry (1850-1920)
  $t_E \geq 2$ Ga

Ma = million years ago, Ga = giga-annum

- J. Fourier (1827)



Solidification from molten rocks

$T - T_s \simeq (T_0 - T_s)\mathrm{erf}[x_1/(2\sqrt{aT})]$ (1-D)

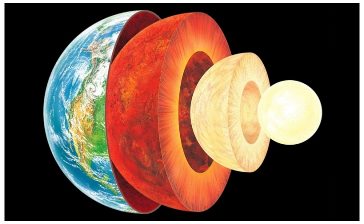
$$G_s = \left.\frac{\partial T}{\partial x_1}\right|_{x_1=0} = (T_0 - T_s)/\sqrt{\pi a t}$$

Lord Kelvin (1844, 1846) : $T_0 \simeq 2900°\mathrm{C}$, $T_S \simeq 20°\mathrm{C}$
$a = 1.2 \times 10^{-6}$ m^2.s^{-1}, $G_s \simeq 36°\mathrm{C/km}$
$\Longrightarrow$ 20 Ma $< t_E <$ 400 Ma

- E. Rutherford (1871-1937)
  radioactive heat (small)

Claire Patterson (1922-1995)
radioactive dating (1956)
$t_E \simeq 4550$ Ma ± 70 Ma

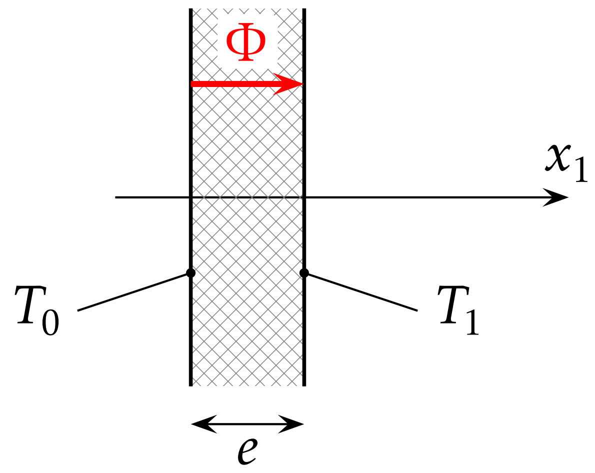
A. Wegener (1880-1930)
Earth not a solid rock!



Steady conduction

The steady solution verifies Laplace's equation $\nabla^2 T = 0$, associated with appropriate boundary conditions :

$$\frac{\partial^2 T}{\partial x_1^2} = 0 \qquad T = T_0 + \frac{T_1 - T_0}{e} x_1$$



Fourier's law $\boldsymbol{q} = \dfrac{k(T_0 - T_1)}{e}\boldsymbol{e}_1$

Heat flux (in W) accross a surface $\mathcal{S}$,

$$\Phi = \frac{k\mathcal{S}}{e}(T_0 - T_1)$$

Concept of thermal resistance $R \equiv (T_0 - T_1)/\Phi$, has units K W^{-1},
$R = e/(k\mathcal{S})$ for a layer of thickness e.

There is an electrical analogy, (temperature, heat flux) $\leftrightarrow$ (tension, intensity), sometimes used in engineering.

Advection-diffusion in a flow : Péclet number

The Péclet number is the analogue of the Reynolds number for heat transfer problems

$$\frac{\partial T}{\partial t} + \underbrace{\boldsymbol{U}\cdot\nabla T}_{\text{advection}} = \underbrace{a\nabla^2 T}_{\text{thermal diffusion}} \qquad \mathrm{Pe} \equiv \frac{UL}{a} \sim \frac{\text{advection term}}{\text{diffusion term}} \sim \frac{L^2/a}{L/U}$$

The value of the Péclet number indicates the relative importance (efficiency) of advection with respect to thermal diffusion. Two asymptotic regimes,

$\mathrm{Pe} \ll 1$ corresponding to a pure heat conduction problem.

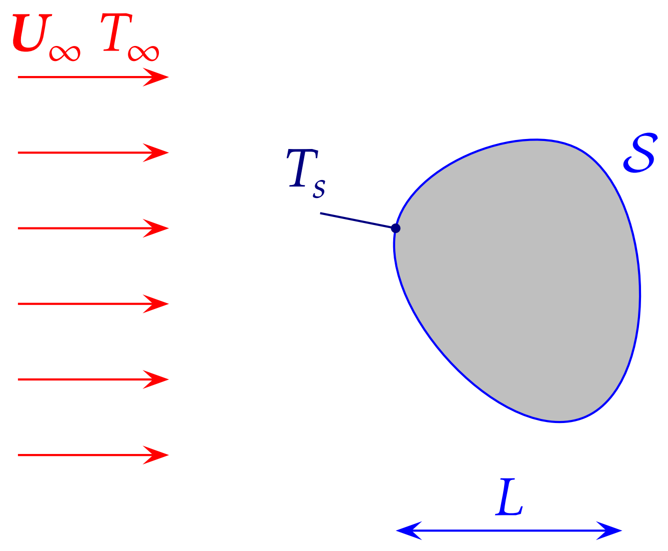
$\mathrm{Pe} \gg 1$, where advection dominates and heat conduction is negligible, except near walls with the presence of thermal boundary layers.

Advection-diffusion around an isothermal body

Transport Eq. for temperature,

$$\frac{\partial T}{\partial t} + \boldsymbol{U} \cdot \nabla T = a \nabla^2 T$$

$$\begin{cases} T = T_s = \text{cst on } \mathcal{S} \\ T \rightarrow T_\infty \text{ upstream far from the body} \end{cases}$$



By introducing the following dimensionless variables

$$\boldsymbol{x}^\star = \frac{\boldsymbol{x}}{L} \quad t^\star = \frac{U_\infty t}{L} \quad \theta = \frac{T - T_\infty}{T_s - T_\infty} \quad \boldsymbol{U}^\star = \frac{\boldsymbol{U}}{U_\infty}$$

The transport Eq. for temperature reads

$$\frac{\partial \theta}{\partial t^\star} + \boldsymbol{U}^\star \cdot \nabla^\star \theta = \frac{1}{\text{Pe}} \nabla^{\star 2} \theta \qquad \begin{cases} \theta = 1 \text{ on } \mathcal{S} \\ \theta \rightarrow 0 \text{ upstream at infinity} \end{cases}$$

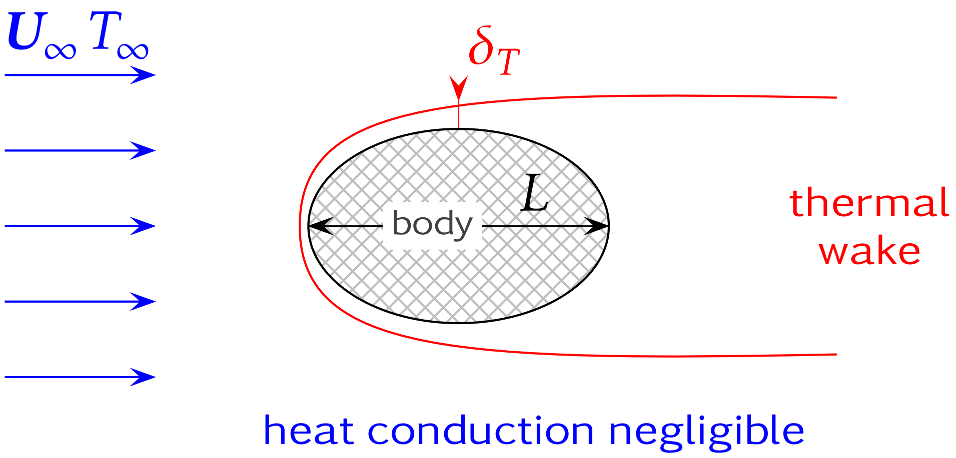
High Péclet number flow

Heat conduction is not important, thus $a = 0$, and temperature is simply advected by the flow,

$$\frac{DT}{Dt} = 0$$

In other words, temperature of fluid particles remains constant. Inlet boundary conditions (but not only) must be prescribed, *e.g.* $T = T_\infty$ here.

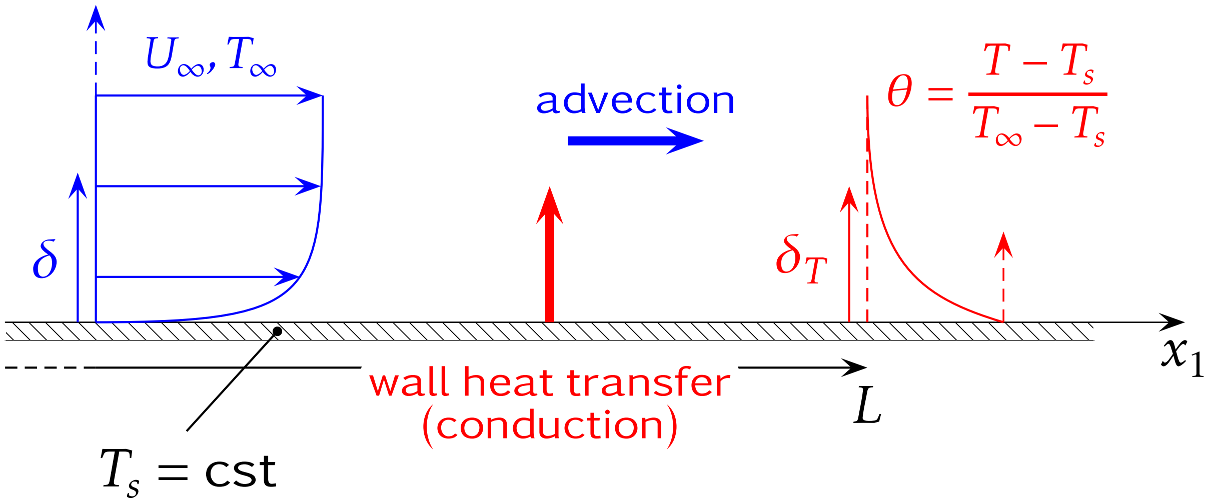
There is, however, a discontinuity of temperature on the body surface $T_\infty \neq T_s$. This singularity is an artefact of the model ($a = 0$). Physically, a thin thermal boundary layer of thickness δ_T develops around the body



The thermal wake is formed by fluid particles from the thermal boundary layer

High Péclet number flow (cont.)

The thermal boundary layer and wake are induced by conduction heat transfer between the solid body and the fluid. This thermal boundary layer is thin, a consequence of the high Péclet number flow assumption.



There is an analogy between the velocity and the thermal boundary layers, but the momentum and heat diffusivity coefficients are not the same : ν for δ, and a for δ_T

Prandtl number $\quad \mathrm{Pr} \equiv \dfrac{\nu}{a} = \dfrac{\mu c_p}{k}$

Thermal boundary layer equation

For a steady laminar flow with a high Péclet number value, and thus $\delta_T/L \ll 1$ the transport Eq. for temperature reads

$$U_1 \frac{\partial T}{\partial x_1} + U_2 \frac{\partial T}{\partial x_2} = a\left(\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2}\right)$$

Approximation of the boundary layer, with $\nabla^2 \simeq \partial^2/\partial x_2^2$ (parabolization of the differential equation)

$$\boxed{U_1 \frac{\partial T}{\partial x_1} + U_2 \frac{\partial T}{\partial x_2} = a \frac{\partial^2 T}{\partial x_2^2}}$$

Advection and diffusion terms are of the same order of magnitude, that is ($\mathcal{T}$ temperature scale and $\mathcal{U}$ velocity scale)

$$\mathcal{U}\frac{\mathcal{T}}{L} \sim a\frac{\mathcal{T}}{\delta_T^2} \quad \Longrightarrow \quad \delta_T \sim \left(\frac{aL}{\mathcal{U}}\right)^{1/2}$$

During the advection time $t = L/U_1$ (in general $U_1 \neq U_\infty$), the penetration depth of the conduction heat flux inside the flow is $\delta_T \sim (at)^{1/2} \sim (aL/U_1)^{1/2}$

Thermal boundary layer equation (cont.)

Estimation of the thickness δ_T of the thermal boundary layer

$\mathrm{Pr} \gg 1$

The viscous diffusion is faster than the thermal diffusion, $\delta_T \ll \delta$

At the edge δ_T of the thermal boundary layer, $U_1 \sim U_\infty \delta_T/\delta$, and from the previous slide, $\delta_T \sim [aL\,\delta/(U_\infty \delta_T)]^{1/2}$

$\delta_T/L \sim \mathrm{Pe}_L^{-1/3}\,\mathrm{Re}_L^{-1/6}$

$\mathrm{Pr} \ll 1$

The thermal diffusion is faster than the viscous diffusion, $\delta_T \gg \delta$

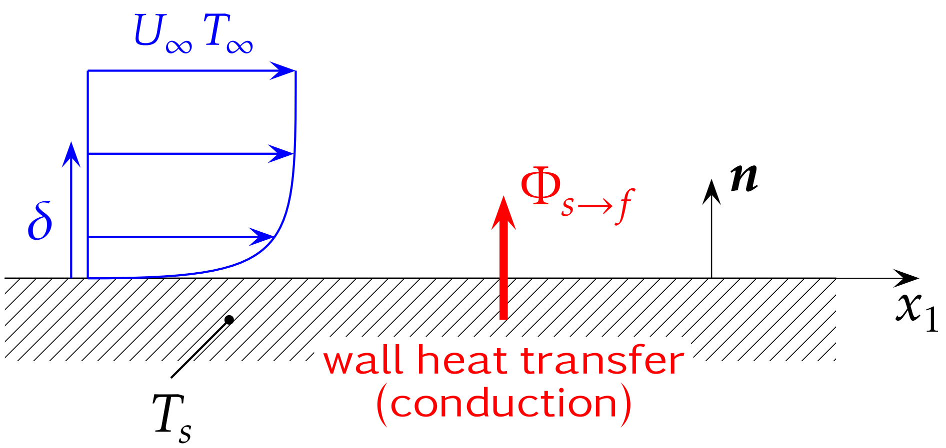
At the edge δ_T of the thermal boundary layer, $U_1 \sim U_\infty$, and $\delta_T \sim (aL/U_\infty)^{1/2}$

$\delta_T/L \sim \mathrm{Pe}_L^{-1/2}$

Both expressions can be applied for gas flow. One indeed has, $\mathrm{Pr} = \mathcal{O}(1)$, $\mathrm{Re}_L \sim \mathrm{Pe}_L$ (by observing that $\mathrm{Pr} = \mathrm{Pe}_L/\mathrm{Re}_L\,!$), and thus $\delta \sim \delta_T$

Solid-fluid heat transfer by convection

Conduction and convection are involved in flow heat transfer



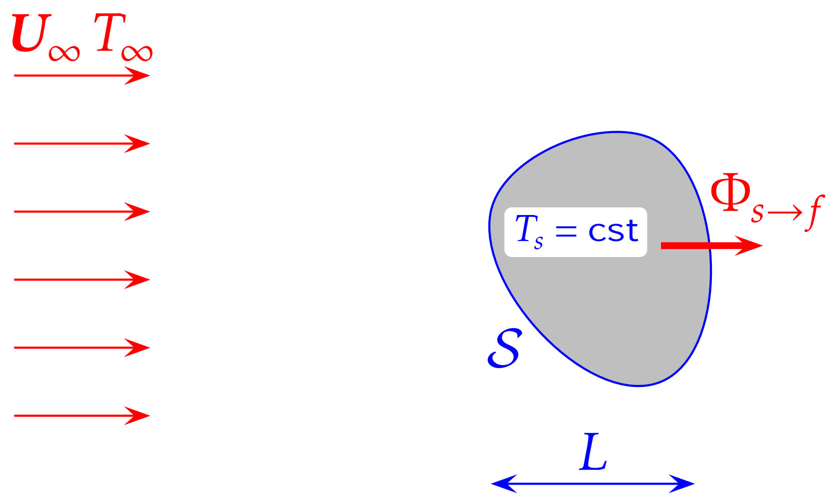
Wall heat flux

$$\Phi_{s\to f} = \int_{\mathcal{S}} \boldsymbol{q}\cdot\boldsymbol{n}\, ds = -k_f \int_{\mathcal{S}} \boldsymbol{n}\cdot\nabla T\, ds$$

(subscripts s and f stand for solid and fluid)

For the convective cooling of an isothermal heated body at T_s = cst, the total heat flux $\Phi_{s\to f}$ is given by

$$\begin{aligned}\Phi &= -k_f \int_{\mathcal{S}} \boldsymbol{n}\cdot\nabla T\, ds \\ &= -k_f(T_s - T_\infty)\int_{\mathcal{S}} \boldsymbol{n}\cdot\nabla\theta\, ds \propto T_s - T_\infty\end{aligned}$$



Convective heat transfer coefficient

The heat flux is usually modeled by introducing a (global) heat transfer coefficient $\bar{h}$ as

$$\Phi_{s\to f} = \bar{h}\,\mathcal{S}\,(T_s - T_f) \tag{26}$$

where $\bar{h}$ is an average value over the surface of the body, and has units W.m^{-2}.K^{-1} (see tables where $\bar{h}$ is provided for a sphere and rods for instance)

The dimensionless heat transfer coefficient is written as a Nusselt number Nu, defined by

$$\mathrm{Nu} = \frac{\bar{h}L}{k_f}$$

where k_f is the thermal conductivity of the fluid

Ernst Kraft Wilhelm Nusselt (1882-1957), an avid mountain climber!

(taken from Lienhard IV & Lienhard, 2017, *A heat transfer textbook*)

Interpretation of the Nusselt number

If the flow motion could be stopped (fictitiously !), $U_\infty = 0$ to remove convection, the heat flux would then be given by

$$\Phi_{s\to f} = -k_f \int_{\mathcal{S}} \boldsymbol{n}\cdot\nabla T|_f \; ds \sim k_f \frac{T_s - T_f}{L} \mathcal{S}$$

Nussel number can thus be recast as,

$$\mathrm{Nu} = \frac{\bar{h}(T_s - T_f)\mathcal{S}}{k_f(T_s - T_f)\mathcal{S}/L} \sim \frac{\Phi_{\text{convection}}}{\Phi_{\text{(fictitious) conduction}}}$$

Nusselt number indicates the ability of the flow to increase heat transfer through convection. For a small Péclet number flow, convection is negligible and Nu $= \mathcal{O}(1)$. On the contrary for Pe $\gg 1$, the wall temperature gradient takes higher values than for the case of a pure conduction heat transfer. The convective heat transfer is more efficient, Nu $\gg 1$; hence the interest of using flow motion (pump, fan, suction device) to increase heat transfers : forced convection

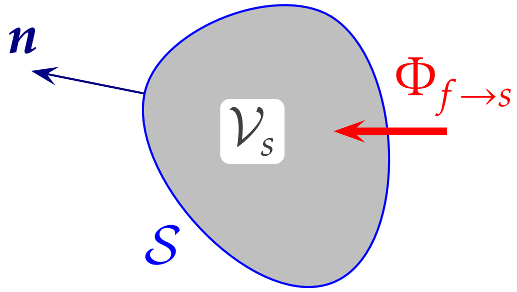
Conduction in the solid body

The temperature $T(\boldsymbol{x},t)$ is determined through the resolution of an advection-diffusion equation in the fluid, and a conduction equation in the solid. The coupling conditions at the interface are obtained by imposing the continuity of temperature and heat flux, as usual

Inside the solid,

$$\frac{\partial T}{\partial t} = a_s \nabla^2 T \qquad a_s = \frac{k_s}{\rho_s c_{ps}}$$

$$\frac{d}{dt}\int_{\mathcal{V}_s} \rho_s c_{ps} T \, d\mathcal{V} = -\int_{\mathcal{S}} \boldsymbol{q}\cdot\boldsymbol{n}\, d\mathcal{S} = \Phi_{f\rightarrow s}$$



Biot number

Biot number measures the relative importance - in the solid - of convection heat transfer with the fluid and conduction heat inside the solid,

$$\mathrm{Bi} = \frac{\bar{h}L}{k_s} \qquad \neq \qquad \mathrm{Nu} = \frac{\bar{h}L}{k_f}$$

$\mathrm{Bi} \ll 1$

Heat conduction dominates. During a first transitional step of duration $t \sim \mathcal{O}(L^2/a_s)$, the solid appears to be isolated, its temperature T_s becomes uniform by thermal conduction, and thus $T_s(\boldsymbol{x},t) \simeq T_s(t)$. A slow evolution is then observed thanks to heat exchange by convection with the flow

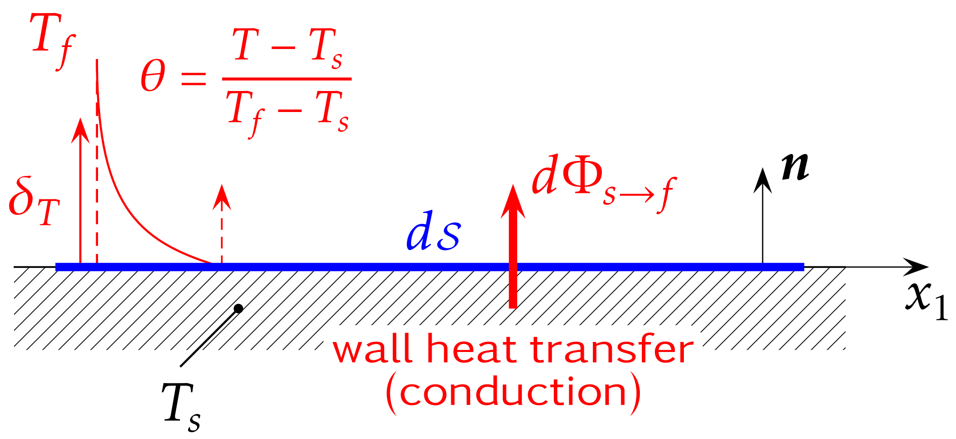
$$\underbrace{\rho_s c_{ps} \mathcal{V}_s}_{= C_s \text{ solid heat capacity}} \frac{dT_s}{dt} = \bar{h}\mathcal{S}\,(T_f - T_s) \qquad \xrightarrow{T_f = \text{cst}} \qquad T_s = T_f + T_0 \exp\left(-\frac{\bar{h}\mathcal{S}}{C_s}t\right)$$

The solid temperature exponentially tends to the (constant) fluid temperature T_f

Local solid-fluid heat transfer model

In practice, a simplified solid-fluid heat transfer model is often used, by introducing a local heat transfer coefficient h

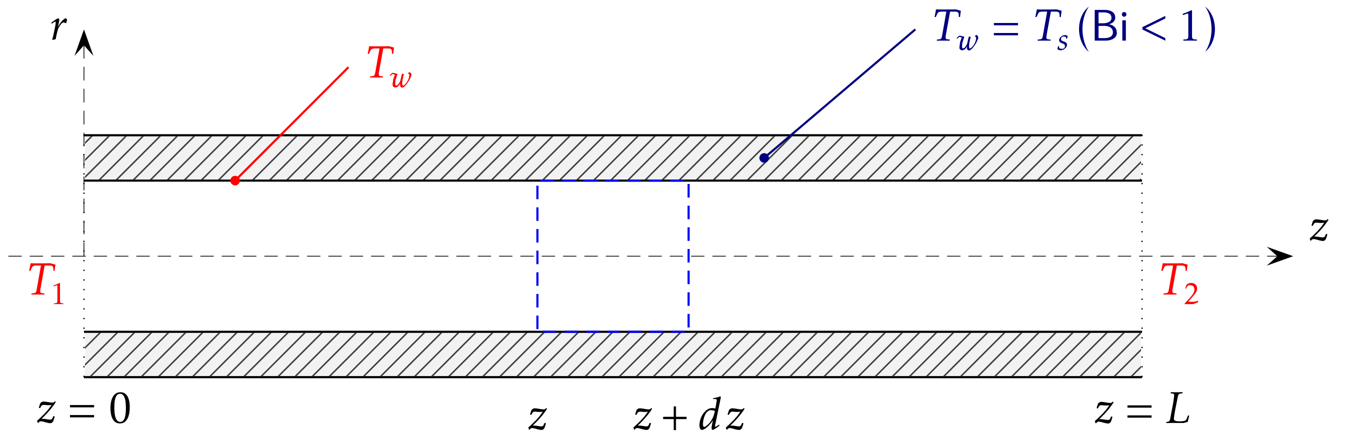
$$d\Phi_{s\to f} = h(T_s - T_f)\, ds$$
$$-k_s\, \boldsymbol{n}\cdot\nabla T|_s\, ds = h(T_s - T_f)\, ds$$



where T_s and T_f are the local temperature of the solid and of the fluid outside of the boundary layer, meaning that $\mathrm{Pe} \gg 1$.

In this local formulation, h, T_s and T_f can change along the interface solid-fluid. That should not be confused with Eq. (26) involving $\bar{h}$ applied to a whole body immersed in a flow motion

- As an illustration, heat transfer to flow in pipe



T_2 ?

heat flux ?

$T_w = 150$°C

water at $T_1 = 25$°C, $Q_v = 100$ l/mn, $L = 4.5$ m, $D = 2.5$ cm
$\mu = 10^{-3}$ Pa.s, $c_p = 4.18$ kJ/(kg.K), $\rho = 10^3$ kg.m^{-3}, $k = 0.64$ W/(m.K)

Forced convection in *turbulent* pipe flow,
$\text{Nu} = \text{Nu}(\text{Re}, \text{Pr}) \simeq 0.027\, \text{Re}^{4/5}\, \text{Pr}^{1/3}$ (Sieder & Tate)

- **Heat transfer to flow in pipe (cont.)**

  $S = \pi D^2/4 \quad U_d = Q_v/S \simeq 3.4\ \text{m.s}^{-1} \quad \text{Re}_D \simeq 8.5 \times 10^4$ (turbulent flow)

  $$a = \frac{k}{\rho c_p} \quad \text{Pe} = \frac{U_d D}{a} \simeq 5.5 \times 10^5 \quad \text{Pr} = \frac{\mu}{k/c_p} \simeq 6.5 \quad Q_m \equiv \rho U_d S \simeq 1.7\ \text{kg.s}^{-1}$$

  $$\text{Nu} = \frac{Dh}{k} \simeq 443 \gg 1 \text{ (forced convection)} \quad h \simeq 1.13 \times 10^4\ \text{W.m}^{-2}.\text{K}^{-1}$$

  Energy equation written for the temperature (steady incompressible flow)

  $\rho c_p \boldsymbol{U} \cdot \nabla T = -\nabla \cdot \boldsymbol{q}$

  and integrated over the elementary volume control $\mathcal{S}$ between z and $z + dz$, with the aim of formulating a 1-D model

  $$\underbrace{\int_{\mathcal{S}} \rho c_p T \boldsymbol{U} \cdot \boldsymbol{n}\, d\mathcal{S}}_{\text{(a)}} = \underbrace{\int_{\mathcal{S}} -\boldsymbol{q} \cdot \boldsymbol{n}\, d\mathcal{S}}_{\text{(b)}}$$

  $$\text{(a)} = \rho c_p U_d S T(z + dz) - \rho c_p U_d S T(z) = Q_m c_p dT$$

Heat transfer to flow in pipe (cont.)

For the term (b), by neglecting the heat flux along the z direction (Pe $\gg 1$)

$$\text{(b)} = \underbrace{\pi D dz\, k \left.\frac{\partial T}{\partial r}\right|_w}_{\substack{\text{conduction}\\\text{into the fluid}\\\text{(no-slip BC at the wall)}}} = \underbrace{\pi D dz\, h(T_w - T)}_{\substack{\text{local heat transfert}\\\text{coefficient } h\\\text{(forced convection)}}}$$

The energy budget takes the form $Q_m c_p dT = \pi D dz\, h(T_w - T)$, which leads to

$$\int_{T_1}^{T_2} \frac{dT}{T_w - T} = \int_0^L \frac{\pi D h}{Q_m c_p} dz \qquad \Longrightarrow \qquad \log\left(\frac{T_w - T_1}{T_w - T_2}\right) = \frac{\pi D L h}{Q_m c_p}$$

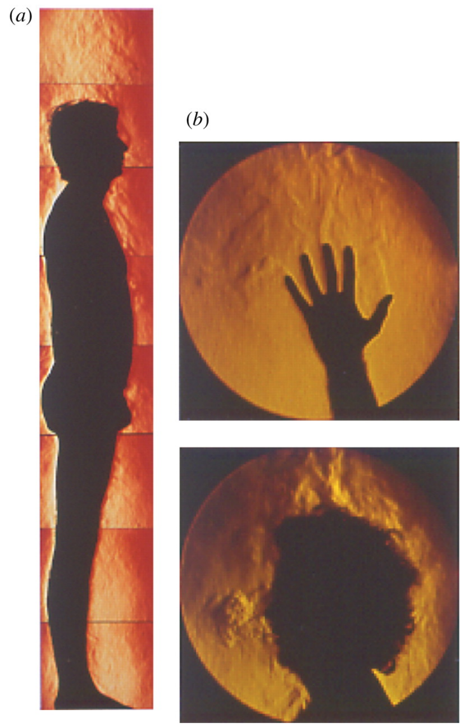
and finally $T_2 = T_w - (T_w - T_1)e^{-\frac{\pi D L h}{Q_m c_p}}$

$T_2 \simeq 80$°C, and the removed heat by the flow is given by
$\Phi = Q_m c_p (T_2 - T_1) \simeq 381$ kW

Forced and natural convection

In natural or free convection, the fluid motion is only produced by density differences in the fluid occurring due to temperature gradients, and not by any external source (pump, fan, ...) as in forced convection.

Fluid receives heat and by thermal expansion becomes less dense and rises. The driving force for natural convection is buoyancy, a result of differences in fluid density.



Schlieren visualization of a Kettle (*Spectabit Optics LLC*) and a human body to study airborne transmission of infection (Clark & de Calcina-Goff, *J. R. Soc. Interface*, 2009)

Boussinesq's approximation

In the framework of Boussinesq's approximation, the hydrostatic balance $\nabla p_0 = \rho_0 \boldsymbol{g}$ is then subtracted from Navier-Stokes Eq. (25), leading to the governing equations for natural convection,

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho_0} \nabla (p - p_0) + \nu \nabla^2 \boldsymbol{U} - \beta (T - T_0) \boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

$$\frac{\partial T}{\partial t} + \boldsymbol{U} \cdot \nabla T = a \nabla^2 T$$

A strong coupling now occurs between the velocity and temperature fields, even if the flow is incompressible.

The balance between $\boldsymbol{U} \cdot \nabla \boldsymbol{U}$ on the left hand side and the buoyancy force on the right hand side, leads to the following estimate of the velocity scale U_n of natural convection $U_n = (\beta g L \Delta T)^{1/2}$ where $\Delta T = T_s - T_0$ is the scale of temperature variations.

Grashof number

The Reynolds number based on U_n, that is $\mathrm{Re} = U_n L/\nu$, is usually replaced by the Grashof number in natural convection

$$\mathrm{Gr} = \mathrm{Re}^2 = \frac{\beta g L^3 \Delta T}{\nu^2}$$

Like the Reynolds number, the Grashof number indicates a laminar or a turbulent regime for natural convection, that is

$$\mathrm{Gr} = \frac{(L^2/\nu)^2}{(L/U_n)^2} \sim \frac{\tau^2_{\text{viscous}}}{\tau^2_{\text{convection}}}$$

- Temperature Eq. for low Mach number flows, thermal diffusivity
- Fundamental solutions for pure conduction problems
- Thermal boundary layer, Péclet and Prandtl numbers
- Forced convection, Nusselt number and local heat transfer coefficient
- Natural convection, velocity scale and Grashof number

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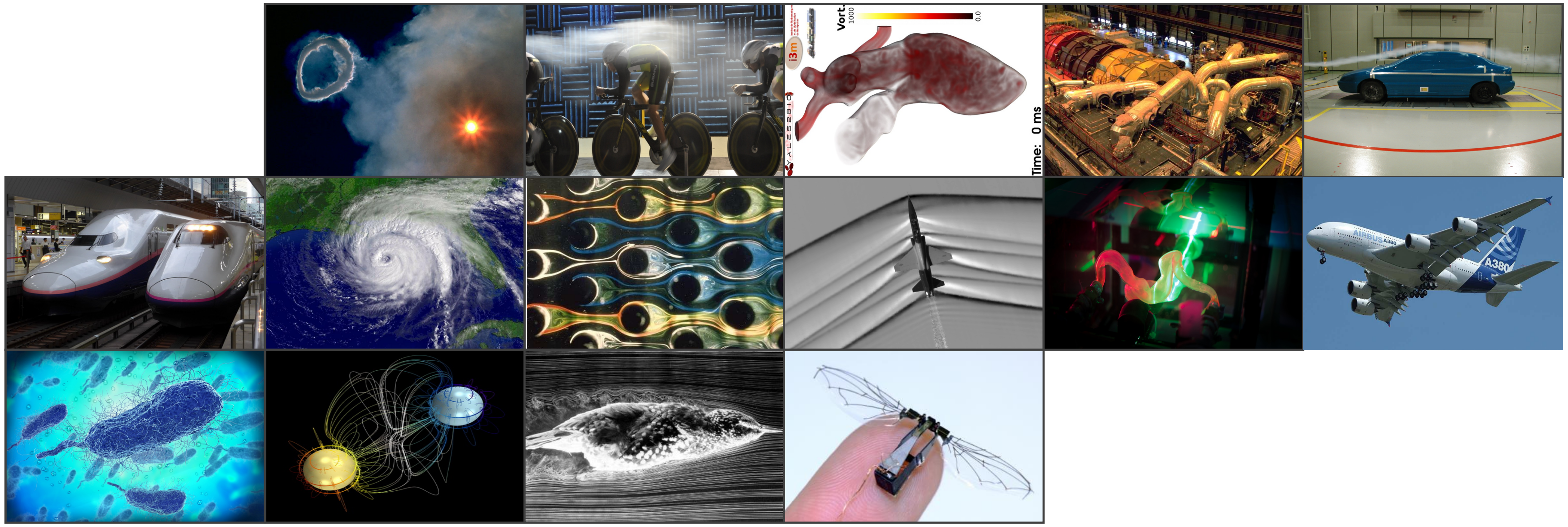
9 - Mixing of fluids



9 - Mixing of fluids

Properties of mixture

- Mass fraction
- Conservation of mass
- Equation for species
- Gouverning equations

Diffusive flux

- Binary and diluted mixtures
- Heat transfer
- Diffusive flux
- Advection-diffusion equation

Turbulent mixing

Key results

Mixture of multiple species

Mixture of N species, characterized by the mass fractions Y_α

$$Y_\alpha \equiv \frac{m_\alpha}{m} = \frac{\rho_\alpha}{\rho} \qquad \begin{cases} m_\alpha \text{ mass of species } \alpha \\ m \text{ mass of the total mixture} \\ \rho_\alpha \text{ partial density of species } \alpha \end{cases}$$

There is no implicit summation for Greek indices. The density and the velocity are respectively defined by

$$\rho = \sum_{\alpha=1}^{N} \rho_\alpha \qquad \boldsymbol{U} = \frac{1}{\rho} \sum_{\alpha=1}^{N} \rho_\alpha \boldsymbol{U}_\alpha = \sum_{\alpha=1}^{N} Y_\alpha \boldsymbol{U}_\alpha$$

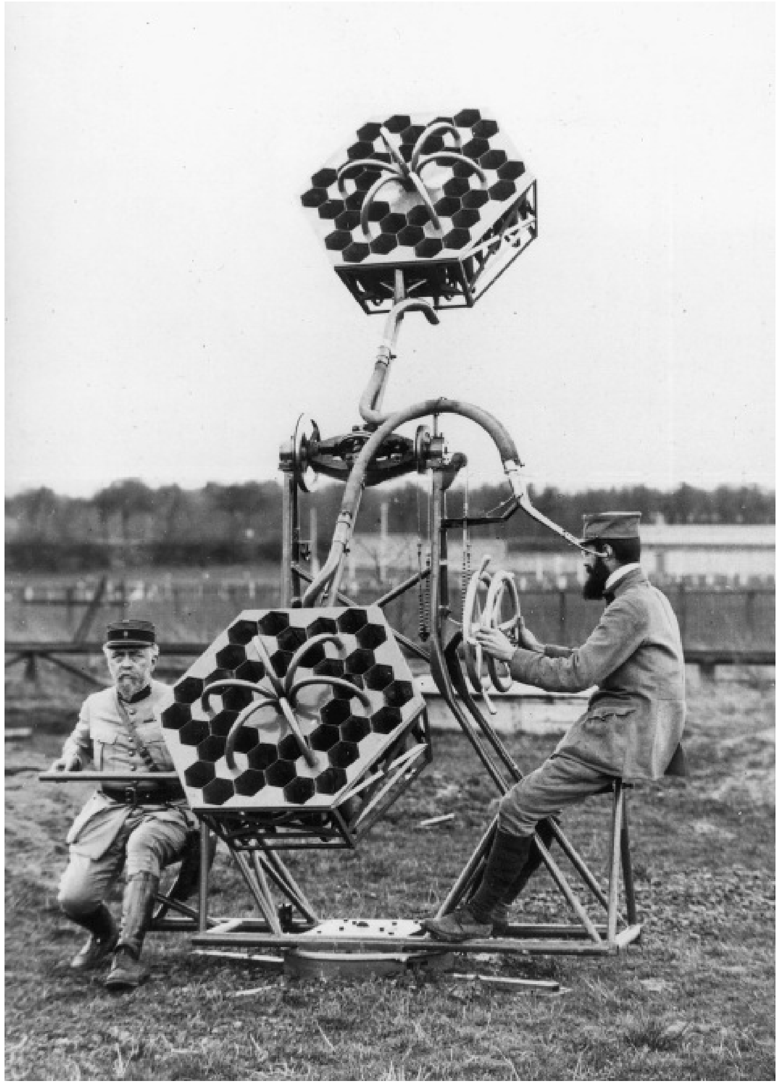
In addition,

$$\boxed{\sum_{\alpha=1}^{N} Y_\alpha = 1} \quad \Longrightarrow \quad Y_N = 1 - \sum_{\alpha=1}^{N-1} Y_\alpha$$

Link between the mass fraction Y_α and the molar concentration C_α, that is the number of moles of species α per unit volume : $\rho Y_\alpha = \rho_\alpha = M_\alpha C_\alpha$
where M_α is the molar mass of species α (kg/mol)

Avogadro's number N_A

$N_A = 6.022 \times 10^{23}$ molecules in 1 mole

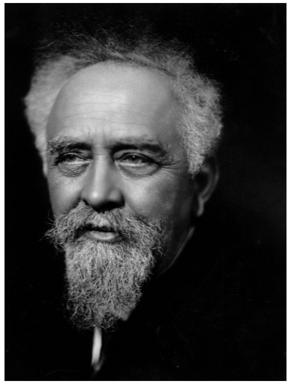


Johnson & Dudgeon, *Array Signal Processing* (1993)

Acoustic array used by the French in World War I to detect enemy aircraft ... and developed by the Sergeant Jean Perrin (right), who received the Nobel Prize in Physics (1926).

He provided an estimate of Avogadro's number based on his work on Brownian motion.

Amedeo Avogadro
(1776-1856)



Jean Perrin
(1870-1942)

Avogadro's number N_A (cont.)

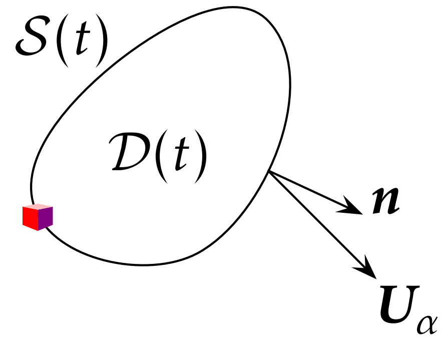
Dry air $21\% O_2$, $78.1\% N_2$, $0.9\% Ar$, that is approximately $21\% O_2$ and $79\% N_2$

$M_{\text{air}} \simeq 0.21 \times 2 \times 16 + 0.79 \times 2 \times 14 = 28.8$ g.mol^{-1}
Exact value $M_{\text{air}} = 28.97$ g.mol^{-1}, $r = R/M_{\text{air}} = 287.06$ J.kg.K^{-1}
(r ideal gas constant, $p = \rho r T$)

$$1 \text{ mole } O_2 \rightarrow 2 \times 16 + \underbrace{0.79/0.21}_{\simeq 3.76} \times 2 \times 14 = 137.3 \text{ g of air}$$

Conservation of mass

Conservation of mass for species α by considering a **material domain** $\mathcal{D}$ associated with species α



$$\frac{d}{dt}\int_{\mathcal{D}} \rho_\alpha \, dv = \int_{\mathcal{D}} R_\alpha \, dv$$

where R_α is the rate of production of species α by chemical reaction (kg.m^{-3}.s^{-1})

By applying the Reynolds theorem, the first term can be rearranged as

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{D}} \rho_\alpha \, dv &= \int_{\mathcal{D}} \frac{\partial \rho_\alpha}{\partial t} \, dv + \int_{\mathcal{S}} \rho_\alpha \boldsymbol{U}_\alpha \cdot \boldsymbol{n} \, ds \\ &= \int_{\mathcal{D}} \left(\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{U}_\alpha) \right) dv\end{aligned}$$

By substituting this development in the mass conservation Eq., one gets

$$\int_{\mathcal{D}} \left(\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{U}_\alpha) - R_\alpha \right) dv = 0$$

Conservation of mass (cont.)

By taking the limit $\mathcal{D} \to 0$

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{U}_\alpha) = R_\alpha \tag{27}$$

A fundamental identity is also obtained by summing the previous equation for all species. Indeed,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = \sum_{\alpha=1}^{N} R_\alpha \qquad \Longrightarrow \qquad \sum_{\alpha=1}^{N} R_\alpha = 0$$

in order to recover the conservation of mass for the whole mixture

Antoine Laurent de Lavoisier (1743-1794)

"Nothing is lost, nothing is created, everything is transformed"

Conservation of species

We define the mass flux vector by diffusion $\boldsymbol{J}_\alpha \equiv \rho_\alpha(\boldsymbol{U}_\alpha - \boldsymbol{U})$, that is the mass flux of species α with respect to the mixture (has units kg.m^2.s^{-1}), in order to introduce the material derivative in Eq. (27). The diffusion process is driven by the relative velocity $\boldsymbol{U}_\alpha - \boldsymbol{U}$. By construction, the following identity is satisfied

$$\sum_{\alpha=1}^{N} \boldsymbol{J}_\alpha = 0$$

Transport equation for the mass fraction Y_α

$$\rho_\alpha \boldsymbol{U}_\alpha = \rho_\alpha \boldsymbol{U} + \boldsymbol{J}_\alpha = \underbrace{\rho Y_\alpha \boldsymbol{U}}_{\text{advection}} + \underbrace{\boldsymbol{J}_\alpha}_{\text{diffusion}} \qquad (\rho_\alpha = \rho Y_\alpha)$$

From Eq. (27), one gets

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{U}) = R_\alpha - \nabla \cdot \boldsymbol{J}_\alpha$$

Conservation of species (cont.)

Using the conservation of mass, one has (as usual)

$$\frac{\partial \rho_\alpha}{\partial t} + \nabla \cdot (\rho_\alpha \boldsymbol{U}) = \frac{\partial(\rho Y_\alpha)}{\partial t} + \nabla \cdot (\rho Y_\alpha \boldsymbol{U}) = \rho\left(\frac{\partial Y_\alpha}{\partial t} + \boldsymbol{U} \cdot \nabla Y_\alpha\right)$$
$$= \rho \frac{D Y_\alpha}{D t}$$

Finally, the transport equation for species α reads

$$\boxed{\rho \frac{D Y_\alpha}{D t} = R_\alpha - \nabla \cdot \boldsymbol{J}_\alpha}$$

$$\rho\left(\frac{\partial Y_\alpha}{\partial t} + \underbrace{\boldsymbol{U} \cdot \nabla Y_\alpha\Big)}_{\text{advection}}\right. = \underbrace{R_\alpha}_{\substack{\text{chemical}\\\text{reactions}}} \quad \underbrace{-\nabla \cdot \boldsymbol{J}_\alpha}_{\text{diffusion}}$$

Gouverning equations

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{U} = 0$$	mass
$$\rho\frac{D\boldsymbol{U}}{Dt} = -\nabla p + \nabla\cdot\overline{\overline{\boldsymbol{\tau}}} + \rho\boldsymbol{g}$$	momentum
$$\rho\frac{De}{Dt} = -p\nabla\cdot\boldsymbol{U} + \overline{\overline{\boldsymbol{\tau}}}:\overline{\overline{\boldsymbol{D}}} - \nabla\cdot\boldsymbol{q} + \rho q_\star$$	internal energy
$$\rho\frac{DY_\alpha}{Dt} = R_\alpha - \nabla\cdot\boldsymbol{J}_\alpha$$	species $(\alpha = 1,\ldots,N)$

The closure of the system requires expressions for $\overline{\overline{\boldsymbol{\tau}}}$, $\boldsymbol{q}$, $\boldsymbol{J}_\alpha$, models for $q_\star$ and R_α, and an equation of state (perfect gas law for instance).

In this course, $R_\alpha \equiv 0$: no chemical reaction, only mixing of fluids

Wall boundary conditions

$\boldsymbol{U} = 0$ (no-slip condition) $\quad$ $\boldsymbol{J}_\alpha\cdot\boldsymbol{n} = 0$ (inert solid wall) $\quad$ $\boldsymbol{q}\cdot\boldsymbol{n} = 0$ (adiabatic)

Gouverning equations (cont.)

The system is over determined with N equations for species, and the additional condition

$$\sum_{\alpha=1}^{N} Y_N = 1$$

to ensure mass conservation. A (too) simplest method consists to solve the transport equations for $N-1$ species, and to compute the last mass fraction as

$$Y_N = 1 - \sum_{\alpha=1}^{N-1} Y_\alpha$$

All the numerical or modeling inconsistencies are *de facto* absorbed in the calculation of Y_N, usually a diluted species (such as N_2 in air where Y_{N_2} is large)

In the general case, complex expressions must be considered (refer to textbooks in combustion). Two simpler cases are however of interest : mixture containing only two species and transport of diluted species.

Diffusive flux

Diffusion in a binary fluid, that is composed of two species : $N = 2$, $Y_2 = 1 - Y_1$

The mass flux vector $\boldsymbol{J}_\alpha$ is calculated by Fick's law (introduced by Fick in 1855), only valid for binary diffusion,

$$\boxed{\boldsymbol{J}_\alpha = -\rho D_m \nabla Y_\alpha}$$

where D_m is the diffusion coefficient of species in m^2.s^{-1}

Adolf Fick (1829-1901)
a German physiologist

Diffusion of diluted species. Each chemical species interacts only with the bulk mixture, and not with other species.

Case for which species N is dominant (e.g. N_2 for air)

$\boldsymbol{J}_\alpha = -\rho D_{m\alpha} \nabla Y_\alpha$ Fick's law for $\alpha = 1, \ldots, N-1$

where $D_{m\alpha}$ is the diffusion coefficient of species α in species N (the bulk mixture)

Species diffusion and heat transfer

The diffusion of mass contributes to the diffusive heat flux, that is in principle calculated as $\boldsymbol{q} = -k\nabla T + \sum_\alpha h_\alpha \boldsymbol{J}_\alpha$

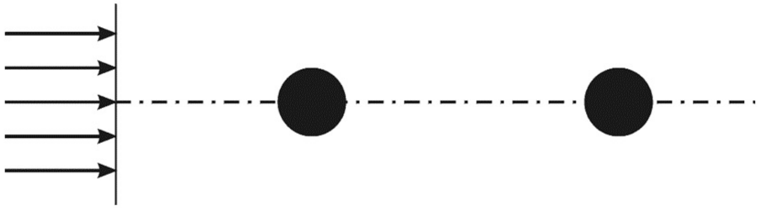
where h_α is the partial specific enthalpy,

$$h_\alpha = \left.\frac{\partial h}{\partial Y_\alpha}\right|_{p,Y_\beta} \qquad \text{(with the convention, } \beta \neq \alpha\text{)}$$

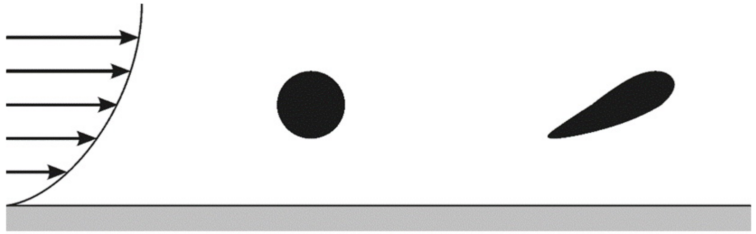
Considering a mixture of diluted species, the role of species diffusion is negligible and the heat flux is simply expressed by Fourier's law $\boldsymbol{q} = -k\nabla T$,
i.e. with no contribution of the minority species

Phenomenology of advection

Advection in a uniform flow

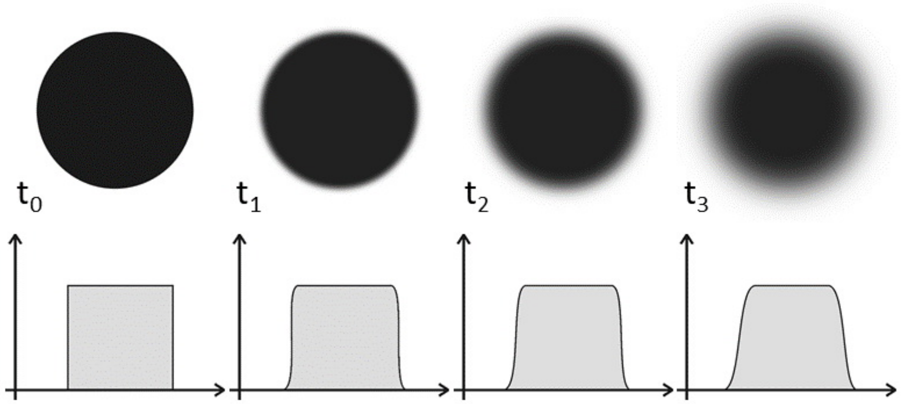


Advection in a shear flow



The volume of the marked particle and its concentration are unaltered during the transport.

Phenomenology of molecular diffusion



Diffusion in a still fluid,
diffusion makes the volume of the marked particle to increase.

The characteristic size of a cloud induced by a point injection of mass is $\sigma_d = \sqrt{2D_m t}$. By noting that $D_m \simeq 10^{-5}\text{m}^2.\text{s}^{-1}$ for various species in air and in water, the size of the cloud is $\sigma_d \simeq 0.26$ m after 1 hour of diffusion.

Diffusion is a very slow process, and is related to the chaotic motion of molecules at the microscopic scale

Advection-diffusion equation for species

Binary mixture or given diluted species in a mixture
$N = 2$, one mass fraction Y_α to describe the problem

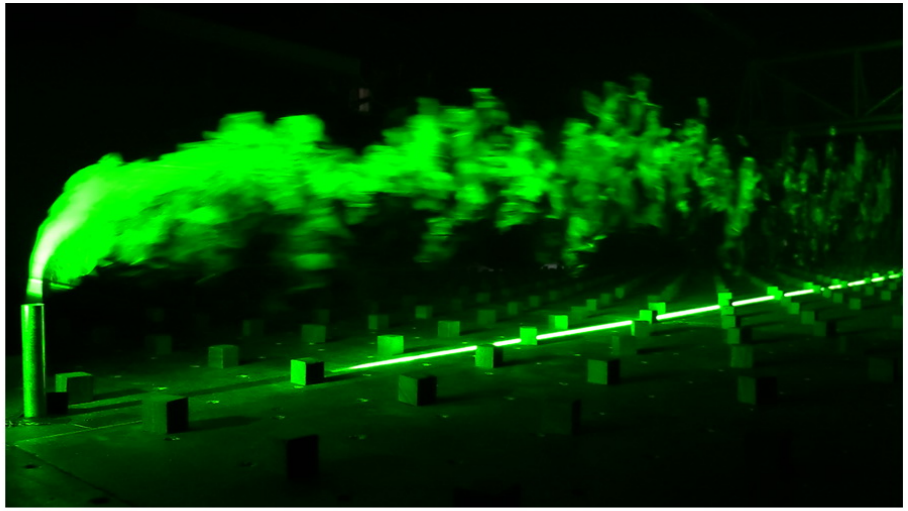
$$\rho \frac{DY_\alpha}{Dt} = -\nabla \cdot (-\rho D_m \nabla Y_\alpha) \qquad \Longrightarrow \qquad \boxed{\frac{\partial Y_\alpha}{\partial t} + \boldsymbol{U} \cdot \nabla Y_\alpha = D_m \nabla^2 Y_\alpha} \qquad (28)$$

The Péclet number Pe $= UL/D_m$ indicates the importance of advection with respect to diffusion (right-hand side) : refer to Chapter 8 for a similar discussion with transport equation for temperature.

The approximation of the boundary layer also applies for the advection-diffusion near the walls.

The Schmidt number Sc $= \nu/D_m$ measures the efficiency of the momentum diffusivity with respect to species diffusion. It plays the role of the Prandtl number for temperature, refer to slide 252 for the advection - diffusion equation of temperature

Mixing of species by turbulent flow



Turbulent plume (air and small fraction of ethane) inside a boundary layer

height of 1 m, $u_\tau \simeq 0.1$ m.s^{-1}, Re $\simeq 10^4$

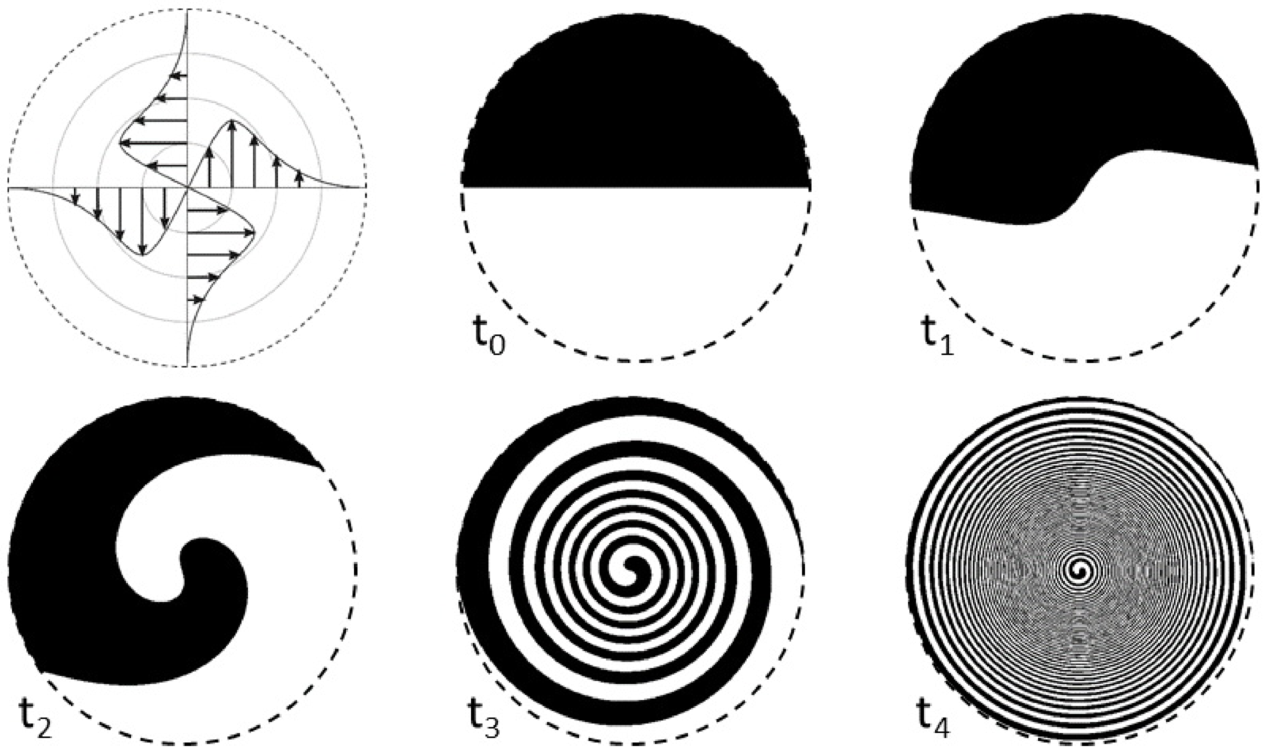
The species concentration ρ_α has to be considered as a random variable. The first order moment, *i.e.* the statistical mean $\bar{\rho}_\alpha(\boldsymbol{x},t)$ over a large number N of realizations $\rho_\alpha^{(i)}$, is defined as (see Chapter 6)

$$\bar{\rho}_\alpha(\boldsymbol{x},t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \rho_\alpha^{(i)}(\boldsymbol{x},t)$$

For practical reasons this is substituted by estimates of a temporal average or a spatial average

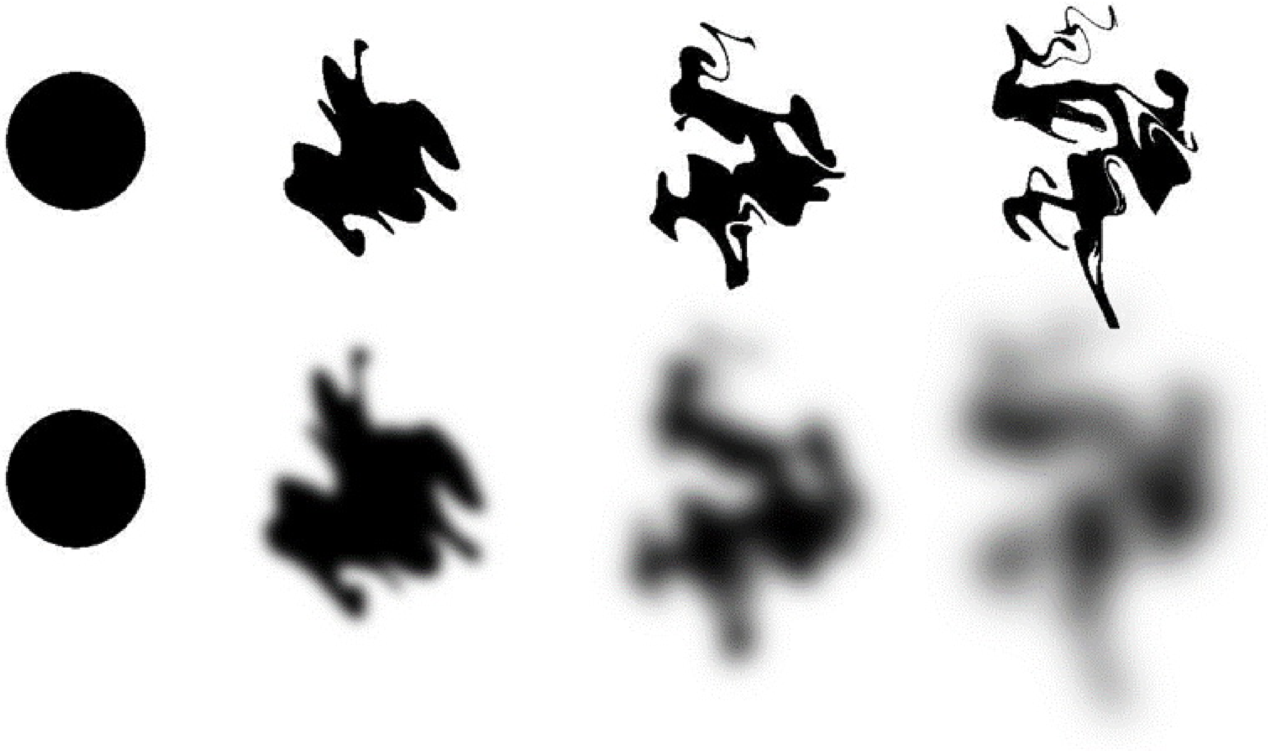
Phenomenology of turbulent mixing

Advection by a turbulent eddy



Phenomenology of turbulent mixing

Turbulent mixing of a marked volume of fluid



Turbulence enhances concentration gradients and therefore accelerates diffusive processes, that act in homogenising the concentration field. The combined role of advection and diffusion produces turbulent mixing

Advection-diffusion in a turbulent flow

Adopting the Reynolds decomposition for the concentration $\rho_\alpha = \bar{\rho}_\alpha + \rho'_\alpha$ and the velocity $\boldsymbol{U} = \bar{\boldsymbol{U}} + \boldsymbol{u}'$, we rewrite Eq. (28) as

$$\frac{\partial(\bar{\rho}_\alpha + \rho'_\alpha)}{\partial t} + \nabla \cdot [(\bar{\rho}_\alpha + \rho'_\alpha)(\bar{\boldsymbol{U}} + \boldsymbol{u}')] = D_m \nabla^2 (\bar{\rho}_\alpha + \rho'_\alpha)$$

Applying the Reynolds average operator to the r.h.s. and l.h.s. of this equation, we obtain

$$\frac{\partial \bar{\rho}_\alpha}{\partial t} + \nabla \cdot [\bar{\rho}_\alpha \bar{\boldsymbol{U}} + \overline{\rho'_\alpha \boldsymbol{u}'}] \quad = D_m \nabla^2 \bar{\rho}_\alpha \tag{29}$$

where the correlation $\overline{\rho'_\alpha \boldsymbol{u}'}$ expresses the turbulent mass flux, *i.e.* the concentration fluctuation transported by the fluctuating velocity.

Advection-diffusion in a turbulent flow

Eq. (29) contains 4 unknowns, $\bar{\rho}_\alpha$ and the three components of $\overline{\rho'_\alpha \boldsymbol{u}'}$. To solve it we have to adopt a *closure* for the turbulent flux.

By analogy with Fick's law for the diffusive term, the simplest closure is a gradient diffusion closure which leads to,

$$-\overline{\rho'_\alpha \boldsymbol{u}'} = D_t \nabla \bar{\rho}_\alpha \tag{30}$$

where the coefficient D_t, referred to as a turbulent diffusion coefficient, depends on the statistical properties of the velocity field and not on the physical properties of the fluid

Turbulent diffusion

As for any diffusion coefficient, D_t can be expressed as the product of a velocity and a length scale,

$$D_t \simeq \mathcal{L} \times \mathcal{U}$$

with $\mathcal{L}$ the size of the largest turbulent eddies and $\mathcal{U}$ the intensity of the turbulent velocity fluctuations.

In the lower atmosphere typical values are $10 \leq \mathcal{L} \leq 100$ m and $\mathcal{U} \sim 1$ m.s^{-1}, so that $10 \leq D_t \leq 100$ in m^2.s^{-1}. Thus, D_t takes large values with respect to $D_m \simeq 10^{-5}$ m^2.s^{-1}

The turbulent transport is much more effective that molecular diffusion

Advection-diffusion for turbulent flow

We substitute Eq. (30) into Eq. (29) and finally obtain a closed form of the averaged advection-diffusion equation

$$\frac{\partial \bar{\rho}_\alpha}{\partial t} + \nabla \cdot (\bar{\rho}_\alpha \bar{\boldsymbol{U}}) = (D_t + D_m) \nabla^2 \bar{\rho}_\alpha$$

which highlights that the contribution of molecular diffusion in the spatial and temporal evolution of the mean concentration $\bar{\rho}_\alpha$ is negligible (except near the walls)

- mixture of multiple species Y_α
- conservation of mass
- conservation of species and diffusion
- governing equations
- diffusion for a binary fluid, Fick's law, diluated species
- transport equation for species, Péclet and Schmidt numbers
- advection-diffusion in a turbulent flow
- turbulent mixing
- averaged advection-diffusion equation

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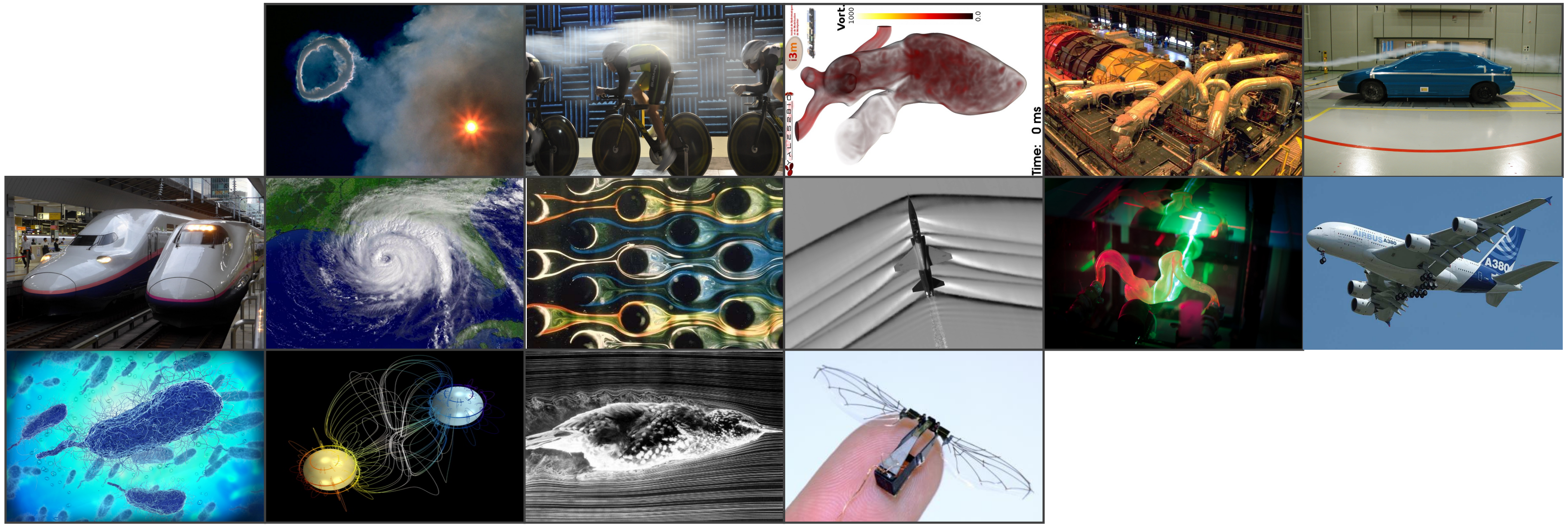
Concluding remarks



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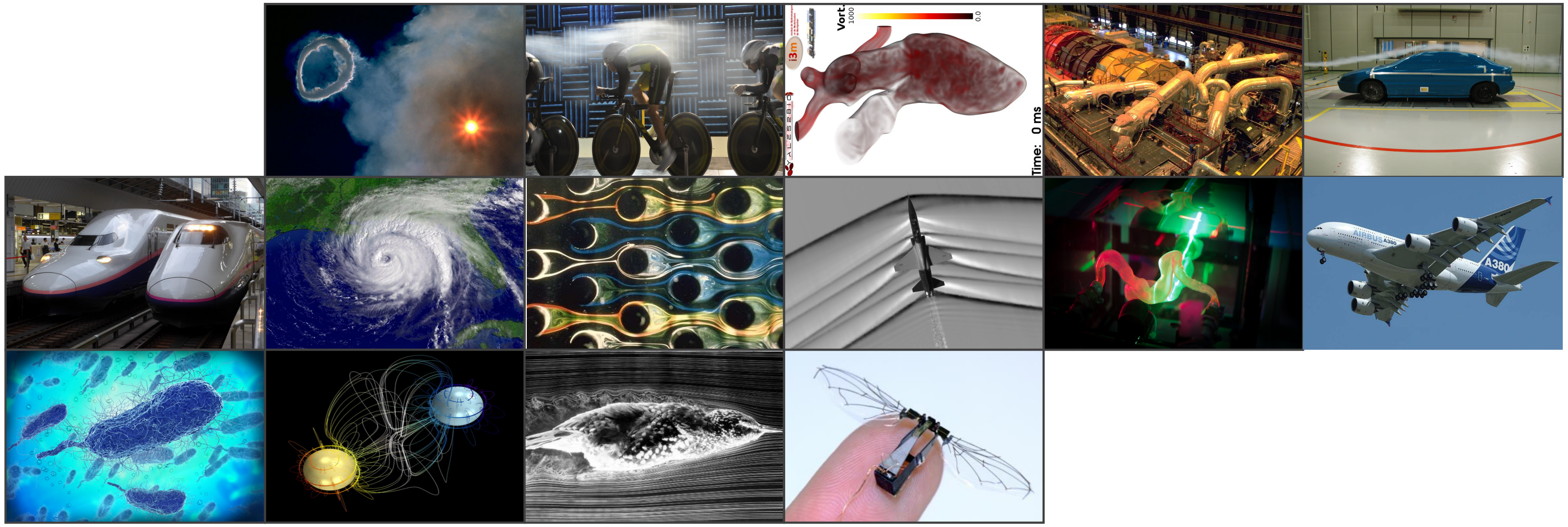
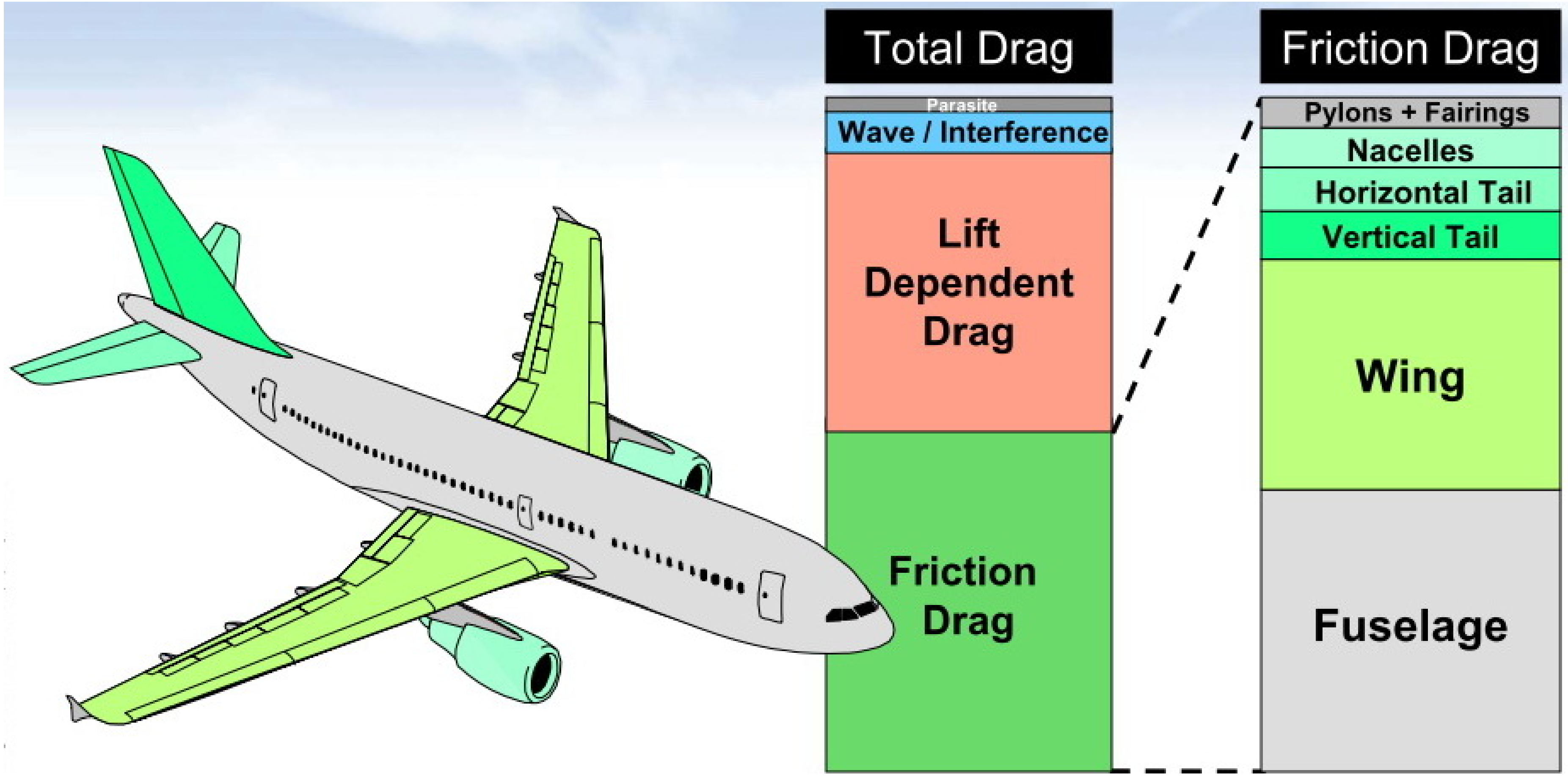
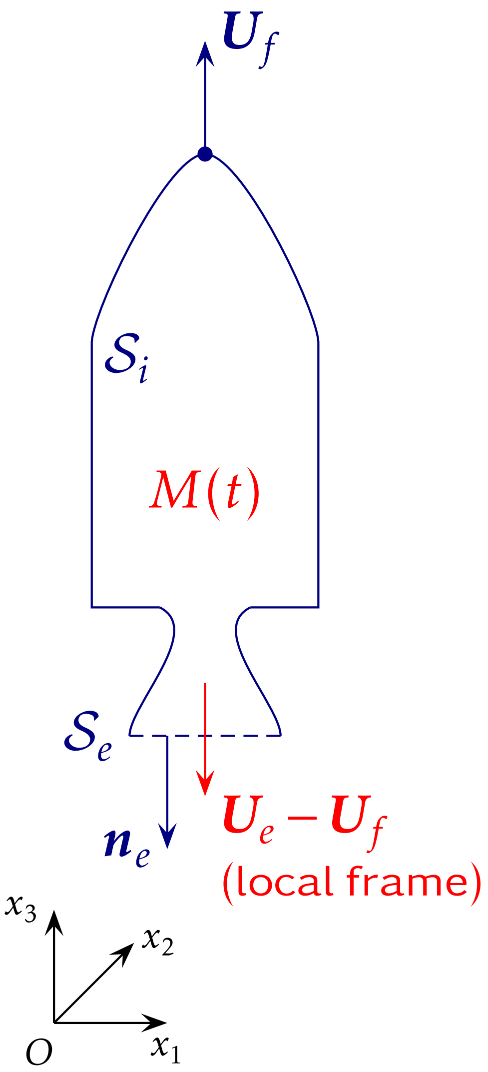


- **Typical break down of overall aircraft drag** (Hills, 2008 ; Airbus A320)
  Lift-induced drag, that is function of the angle of attack



Thrust of a rocket propelling in vacuum

Rocket velocity $\boldsymbol{U}_f(t)$ and jet exit velocity $\boldsymbol{U}_e$ in an inertial frame



Control domain $\mathcal{D}$ bounded by $\mathcal{S}_i \cup \mathcal{S}_e$, uniform flow over $\mathcal{S}_e$

Conservation of mass, Eq. (4) with $\chi = 1$ and $\boldsymbol{U}_{\mathcal{S}} = \boldsymbol{U}_f$

$$\begin{aligned} \dot{M} = \frac{d}{dt}\int_{\mathcal{D}} \rho \, dv &= \int_{\mathcal{S}_e} \rho(\boldsymbol{U}_f - \boldsymbol{U}) \cdot \boldsymbol{n} \, ds \\ &= \rho_e \mathcal{S}_e (\boldsymbol{U}_f - \boldsymbol{U}_e) \cdot \boldsymbol{n}_e < 0 \end{aligned}$$

Conservation of momentum : forces

If $\boldsymbol{F}_s$ denotes the external surfaces forces exerted on the rocket engine (no drag here to simplify the problem),

$$\boldsymbol{F}_s = -\int_{\mathcal{S}_i \cup \mathcal{S}_e} p\boldsymbol{n} \, ds = -\mathcal{S}_e p_e \boldsymbol{n}_e \quad \text{and} \quad \boldsymbol{F}_v = \int_{\mathcal{D}} \rho \boldsymbol{g} \, dv = M\boldsymbol{g}$$

Rocket propelling in vacuum (cont.)

Conservation of momentum, from Eq. (6)

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \boldsymbol{U}\, dv = \int_{\mathcal{S}_e} \rho \boldsymbol{U}(\boldsymbol{U}_f - \boldsymbol{U})\cdot \boldsymbol{n}\, ds - \mathcal{S}_e p_e \boldsymbol{n}_e + M\boldsymbol{g}$$

The two first terms can be rearranged as follows,

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \boldsymbol{U}\, dv = \frac{d}{dt}\int_{\mathcal{D}} \rho(\boldsymbol{U} - \boldsymbol{U}_f)\, dv + \underbrace{\frac{d}{dt}\left(\boldsymbol{U}_f \int_{\mathcal{D}} \rho\, dv\right)}_{= M\dot{\boldsymbol{U}}_f + \dot{M}\boldsymbol{U}_f}$$

$$\begin{aligned}\int_{\mathcal{S}_e} \rho \boldsymbol{U}(\boldsymbol{U}_f - \boldsymbol{U})\cdot \boldsymbol{n}\, ds &= \left[\rho_e \mathcal{S}_e (\boldsymbol{U}_f - \boldsymbol{U}_e)\cdot \boldsymbol{n}_e\right]\boldsymbol{U}_e \\ &= \dot{M}\boldsymbol{U}_e = \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) + \dot{M}\boldsymbol{U}_f\end{aligned}$$

The conservation of momentum can then be recast as

$$M\dot{\boldsymbol{U}}_f + \frac{d}{dt}\int_{\mathcal{D}} \rho(\boldsymbol{U} - \boldsymbol{U}_f)\, dv = \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) - \mathcal{S}_e p_e \boldsymbol{n}_e + M\boldsymbol{g}$$

Rocket propelling in vacuum (cont.)

$$M\dot{\boldsymbol{U}}_f + \frac{d}{dt}\int_{\mathcal{D}} \rho(\boldsymbol{U} - \boldsymbol{U}_f)\, dv = \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) - \mathcal{S}_e p_e \boldsymbol{n}_e + M\boldsymbol{g}$$

The (vacuum) engine thrust $\boldsymbol{T}_R$ of the rocket is defined by

$$\boldsymbol{T}_R \equiv \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) - \mathcal{S}_e p_e \boldsymbol{n}_e \simeq \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) > 0$$

By neglecting the integral term (variation of momentum inside the rocket engine) and the pressure contribution in $\boldsymbol{T}_R$

$$M\dot{\boldsymbol{U}}_f \simeq \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) + M\boldsymbol{g}$$

Ariane V – P230 at lift-off

Specific impulsion $I_s = 275.4$ s ($T_R = \dot{M}gI_s$)

$U_e \simeq 2700$ m.s^{-1}, $T_R \simeq -\dot{M}U_e \simeq 6500$ kN

Ariane V, $M = 780$ tons, 2 solid boosters P230 providing 92% of the lift-off thrust

Rocket propelling in vacuum (cont.)

Rocket motion equation (Tsiolkovsky, 1897)

$$M\dot{\boldsymbol{U}}_f \simeq \dot{M}(\boldsymbol{U}_e - \boldsymbol{U}_f) + M\boldsymbol{g}$$

Assuming $\dot{M} \simeq$ cst, the conservation of mass provides $\boldsymbol{U}_f - \boldsymbol{U}_e =$ cst, and the motion is thus given by

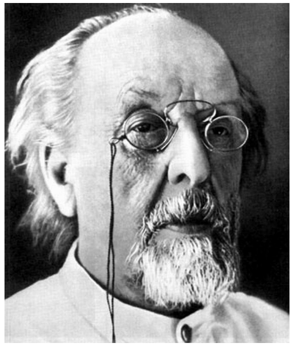
$$\dot{U}_f = (U_e - U_f)\frac{\dot{M}}{M} - g$$

$$\int_0^{U_f} dU_f = (U_e - U_f)\ln\left(\frac{M}{M_0}\right) - \int_0^t g\,dt$$

$$U_f - U_{f0} = (U_e - U_f)\ln\left(\frac{M}{M_0}\right) - gt$$

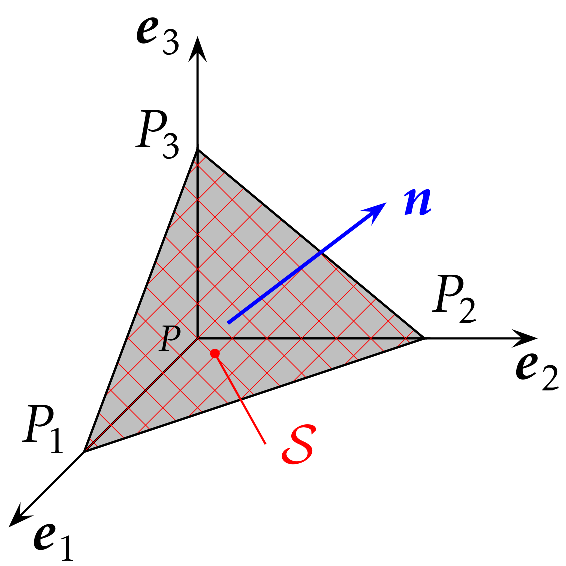
Konstantin Tsiolkovsky (1857 - 1935)

Russian and Soviet rocket scientist, one of the founding fathers of modern rocketry and astronautics



"The Earth is the cradle of humanity, but mankind cannot stay in the cradle forever"

Cauchy's tetrahedron theorem



$$\mathcal{S}_0 = \mathcal{A}(P_1, P_2, P_3)$$

$$\mathcal{S}_1 = \mathcal{A}(P, P_2, P_3) = \mathcal{S}_0 \cos(\boldsymbol{n} \cdot \boldsymbol{e}_1) = \mathcal{S}_0 n_1$$

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$$

Momentum equation (based on Newton's second law) for a tetrahedral fluid particle $\boldsymbol{x}_P$

$$\rho \mathcal{V} \frac{D\boldsymbol{U}}{Dt} = \int_{\mathcal{S}} \boldsymbol{T}\, ds + \int_{\mathcal{D}} \rho \boldsymbol{g}\, dv$$

$$\rho \mathcal{V} \frac{D\boldsymbol{U}}{Dt} = \boldsymbol{T}(\boldsymbol{n})\mathcal{S}_0 + \boldsymbol{T}(-\boldsymbol{e}_j)\mathcal{S}_j + \rho \boldsymbol{g} \mathcal{V}$$

When $\mathcal{V} \to 0$, $\mathcal{V} \sim \epsilon^3$ and $\mathcal{S}_j \sim \epsilon^2$. The dominant term in ϵ must be zero, *i.e.* $\boldsymbol{T}(\boldsymbol{n}) = \boldsymbol{T}(\boldsymbol{e}_j) n_j$

$$T_i(\boldsymbol{n}) = T_i(\boldsymbol{e}_j) n_j = \sigma_{ij} n_j$$

where σ_{ij} is the i-th component of $\boldsymbol{T}$ along the direction j. There is a linear dependence of the stress vector with $\boldsymbol{n}$, leading to

$$\boldsymbol{T} = \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}$$

Conservation of the angular momentum

For a fluid particle of volume $\mathcal{V}$, the torque exerted by the force $\boldsymbol{F} = \boldsymbol{T} + \rho \boldsymbol{g}$ about a fixed point (origin of the coordinate system here) is $\boldsymbol{x} \times \boldsymbol{F}$. The conservation of the angular momentum reads

$$\rho \mathcal{V}\, \boldsymbol{x} \times \frac{D\boldsymbol{U}}{Dt} = \int_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{T}\, d\mathcal{S} + \int_{\mathcal{D}} \boldsymbol{x} \times \rho \boldsymbol{g}\, d\mathcal{V}$$

It can be shown that the antisymmetric part of the stress tensor $\overline{\overline{\boldsymbol{\sigma}}}$ must necessarily vanish for the angular momentum balance to be satisfied, that is $\sigma_{ij} = \sigma_{ji}$.

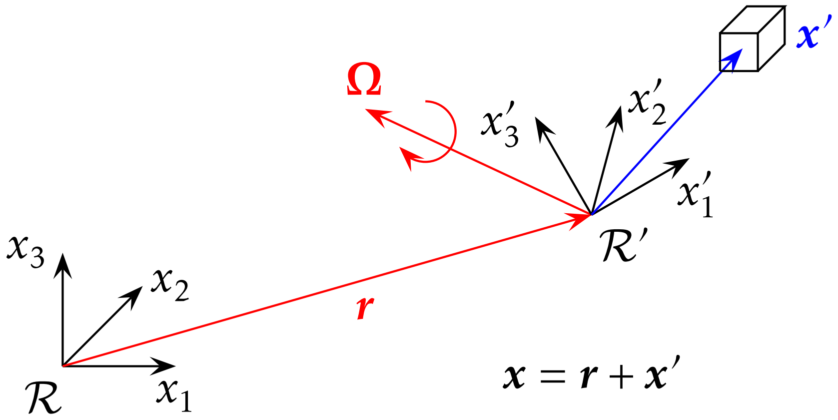
Local budget of a quantity χ

$$\frac{\partial(\rho\chi)}{\partial t} + \nabla \cdot (\rho \chi \boldsymbol{U}) = \rho \frac{D\chi}{Dt}$$

Noninertial reference frame $\mathcal{R}'$

Absolute velocity

$$\boldsymbol{U}(\boldsymbol{x}_{/\mathcal{R}}) = \boldsymbol{U}_r(\boldsymbol{x}'_{/\mathcal{R}'}) + \underbrace{\frac{d\boldsymbol{r}}{dt} + \boldsymbol{\Omega}_{\mathcal{R}'/\mathcal{R}} \times \boldsymbol{x}'}_{\boldsymbol{x}'/\mathcal{R}' \text{ fixed}}$$



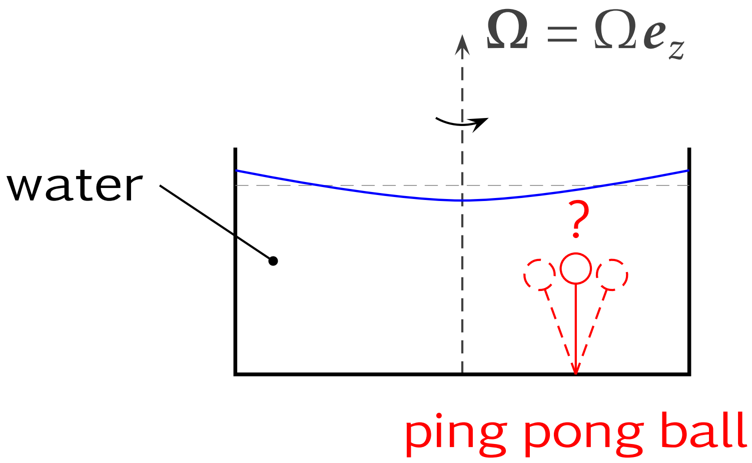
Acceleration

$$\boldsymbol{a}(\boldsymbol{x}_{/\mathcal{R}}) = \boldsymbol{a}_r(\boldsymbol{x}'_{/\mathcal{R}'}) + \boldsymbol{a}_e(\boldsymbol{x}'_{/\mathcal{R}}) + \underbrace{2\boldsymbol{\Omega}_{\mathcal{R}'/\mathcal{R}} \times \boldsymbol{U}_r(\boldsymbol{x}'_{/\mathcal{R}'})}_{\text{Coriolis}}$$

$$\boldsymbol{a}_e(\boldsymbol{x}'_{/\mathcal{R}}) = \frac{d^2\boldsymbol{r}}{dt^2} + \left.\frac{d\boldsymbol{\Omega}}{dt}\right|_{\mathcal{R}'} \times \boldsymbol{x}' + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}')}_{\text{centrifugal force}}$$

Gaspard-Gustave de Coriolis (1792-1843)

A funny illustration involving buoyancy force



Centrifugal force (see slide 315)

$\boldsymbol{\Omega} = \Omega \boldsymbol{e}_z \qquad \boldsymbol{x} = r\boldsymbol{e}_r + h\boldsymbol{e}_z$

$\boldsymbol{\Omega} \times \boldsymbol{x} = r\Omega \boldsymbol{e}_\theta$

$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) = -r\Omega^2 \boldsymbol{e}_r$

From Dauxois & Raynal (2005)

Cylindrical coordinates $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$

Note that the free surface has a parabolic shape

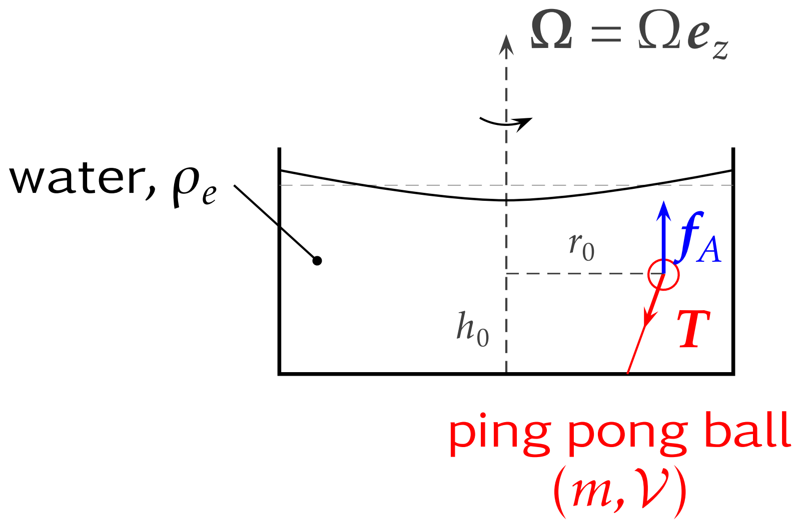
Euler's equation in the local frame

$-\rho\Omega^2 r\boldsymbol{e}_r = -\nabla p + \rho\boldsymbol{g}$

$$\begin{cases} -\rho\Omega^2 r = -\dfrac{\partial p}{\partial r} \\ 0 = -\dfrac{1}{r}\dfrac{\partial p}{\partial \theta} \\ 0 = -\dfrac{\partial p}{\partial z} - \rho g \end{cases}$$

$$p(r,z) = \frac{\rho\Omega^2}{2} r^2 - \rho g z + \text{cst}$$

Archimedes' principle (version 1)



Ball position, $\boldsymbol{x}_0 = r_0 \boldsymbol{e}_r + h_0 \boldsymbol{e}_z$

Acceleration, $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}_0) = -\Omega^2 r_0 \boldsymbol{e}_r$

Balance of the ball,

$-m\Omega^2 r_0 \boldsymbol{e}_r = m\boldsymbol{g} + \boldsymbol{T} + \boldsymbol{f}_A$

Buoyant force $\boldsymbol{f}_A$,

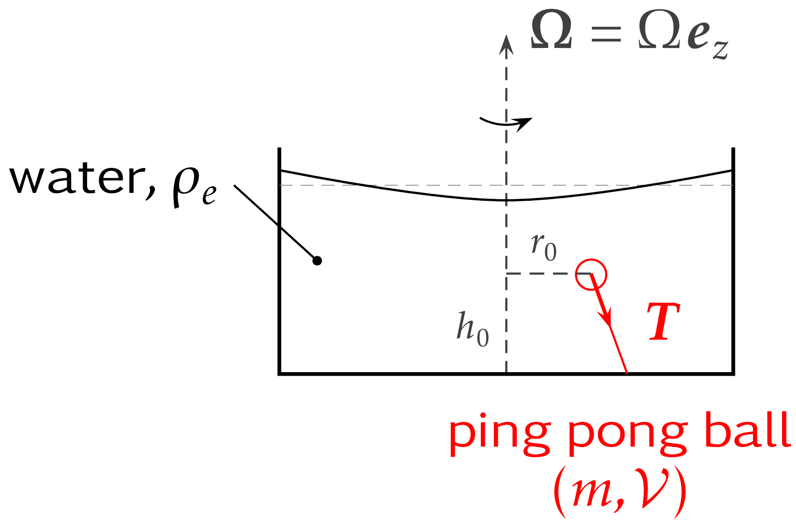
$0 = \boldsymbol{f}_A + \rho_e \mathcal{V} \boldsymbol{g}$

At the end, $\boldsymbol{T} = (\rho_e \mathcal{V} - m)\boldsymbol{g} - m\Omega^2 r_0 \boldsymbol{e}_r$

It can be observed that $\boldsymbol{T} \cdot \boldsymbol{e}_r = -m\Omega^2 r_0 < 0$, the ball has thus deviated from the z-axis ... but this result is not in agreement with experiment!

Why?

Archimedes' principle (version 2)



Ball position, $\boldsymbol{x}_0 = r_0 \boldsymbol{e}_r + h_0 \boldsymbol{e}_z$
Acceleration, $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}_0) = -\Omega^2 r_0 \boldsymbol{e}_r$

Balance of the ball,
$-m\Omega^2 r_0 \boldsymbol{e}_r = m\boldsymbol{g} + \boldsymbol{T} + \boldsymbol{f}_A$

Buoyant force $\boldsymbol{f}_A$,
$-\rho_e \mathcal{V} \Omega^2 r_0 \boldsymbol{e}_r = \boldsymbol{f}_A + \rho_e \mathcal{V} \boldsymbol{g}$

At the end, $\boldsymbol{T} = (\rho_e \mathcal{V} - m)(\boldsymbol{g} + \Omega^2 r_0 \boldsymbol{e}_r)$

For $m < \rho_e \mathcal{V}$ (mass of the ball lighter than the corresponding displaced water volume, a reasonable assumption for a ping pong ball), one has
$\boldsymbol{T} \cdot \boldsymbol{e}_r = (\rho_e V - m)\Omega^2 r_0 \boldsymbol{e}_r > 0$

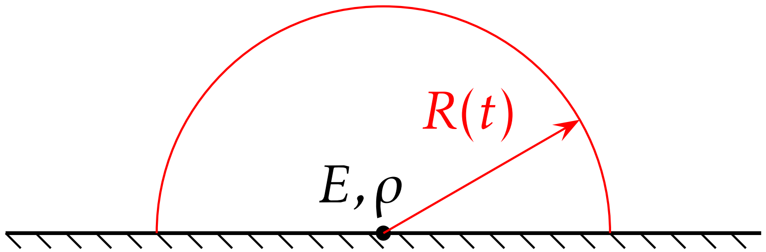
So the ball has now moved closer to the z-axis ... in agreement with experiment

A famous application of the Π-theorem

(Cavendish Lab.,
March 1955)

(Stanford, Aug. 1968)

Estimation of the energy released by the first atomic bomb from pictures published on the cover of *Life Magazine* in 1945 (Trinity nuclear test, New Mexico; Manhattan project)



According to Geoffrey Taylor (1886-1975), the time evolution of the blast wave $R(t)$ should primarily be determined by released energy E and the density of air ρ

Blast wave produced by a strong explosion

The formation of a blast wave by a very intense explosion
I. Theoretical discussion

By Sir Geoffrey Taylor, F.R.S.

(*Received* 6 *October* 1949)

Summary and introduction

This paper was written early in 1941 and circulated to the Civil Defence Research Committee of the Ministry of Home Security in June of that year. The present writer had been told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission—the name atomic bomb had not then been used—and the work here described represents his first attempt to form an idea of what mechanical effects might be expected if such an explosion could occur. In the then common explosive bomb mechanical effects were produced by the sudden generation of a large amount of gas at a high temperature in a confined space. The practical question which required an answer was: Would similar effects be produced if energy could be released in a highly concentrated form unaccompanied by the generation of gas? This paper has now been declassified, and though it has been superseded by more complete calculations, it seems appropriate to publish it as it was first written, without alteration, except for the omission of a few lines, the addition of this summary, and a comparison with some more recent experimental work, so that the writings of later workers in this field may be appreciated.

An ideal problem is here discussed. A finite amount of energy is suddenly released in an infinitely concentrated form. The motion and pressure of the surrounding air is calculated. It is found that a spherical shock wave is propagated outwards whose radius R is related to the time t since the explosion started by the equation

$$R = S(\gamma)\, t^{\frac{2}{5}} E^{\frac{1}{5}} \rho_0^{-\frac{1}{5}},$$

- **Blast wave produced by a strong explosion (cont.)**

$\mathcal{F}(R, t, \rho, E) = 0 \qquad \Longrightarrow n = 4$

	R	t	ρ	E
L	1	0	−3	2
T	0	1	0	−2
M	0	0	1	1
Θ	0	0	0	0

$\Longrightarrow r = 3$

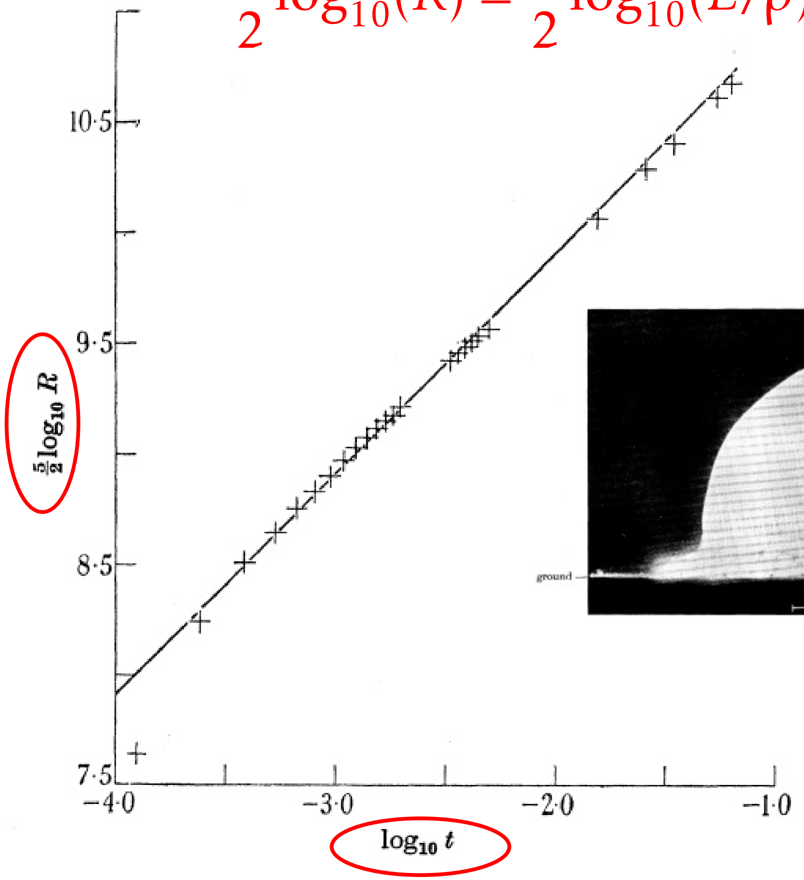
$n - r = 1$ parameter Π_1

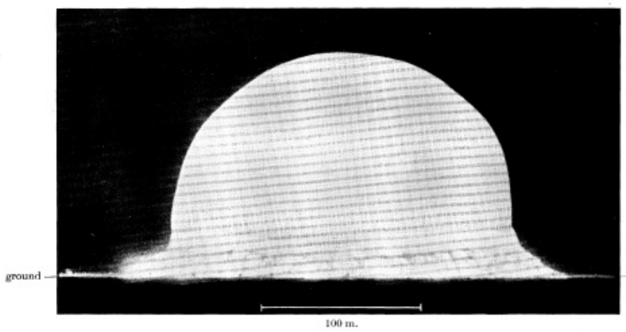
$\Pi_1 = \text{cst}$

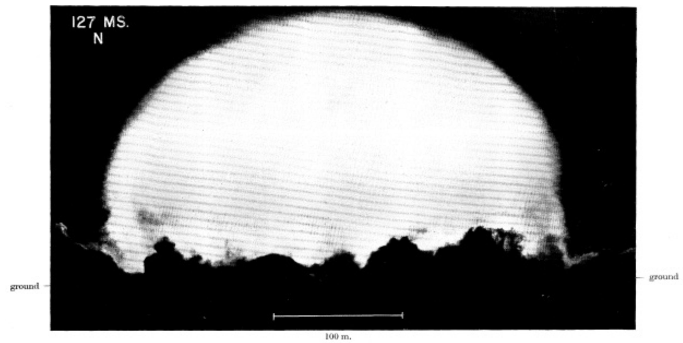
$$\Pi_1 = \frac{R}{t^{2/5}\rho^{-1/5}E^{1/5}} = \text{cst} \ (\simeq 1.03)$$

Blast wave produced by a strong explosion (cont.)

$$\frac{5}{2}\log_{10}(R) = \frac{1}{2}\log_{10}(E/\rho) + \log_{10}(t)$$







Estimation of the energy released by the nuclear explosion $E \simeq 10^{14}$ J (using a 1-D shock-wave problem ; 1 kt TNT $= 4.184 \times 10^{12}$ J)

Boundary layer approximation

An alternative (equivalent) view based on dimensionless variables of order of magnitude $\mathcal{O}(1)$. The quantities x_2 and U_2 must be stretched by the factor $\sqrt{\mathrm{Re}_L}$, where $\mathrm{Re}_L \equiv \mathcal{U}_1 L/\nu$ (Reynolds number)

$$\tilde{x}_1 = \frac{x_1}{L} \qquad \tilde{x}_2 = \frac{x_2}{L}\sqrt{\mathrm{Re}_L} \qquad \tilde{U}_1 = \frac{U_1}{\mathcal{U}_1} \qquad \tilde{U}_2 = \frac{U_2}{\mathcal{U}_1}\sqrt{\mathrm{Re}_L}$$

By choosing $\tilde{p} = p/(\rho\mathcal{U}_1^2)$, the Navier-Stokes equation along x_2 reads

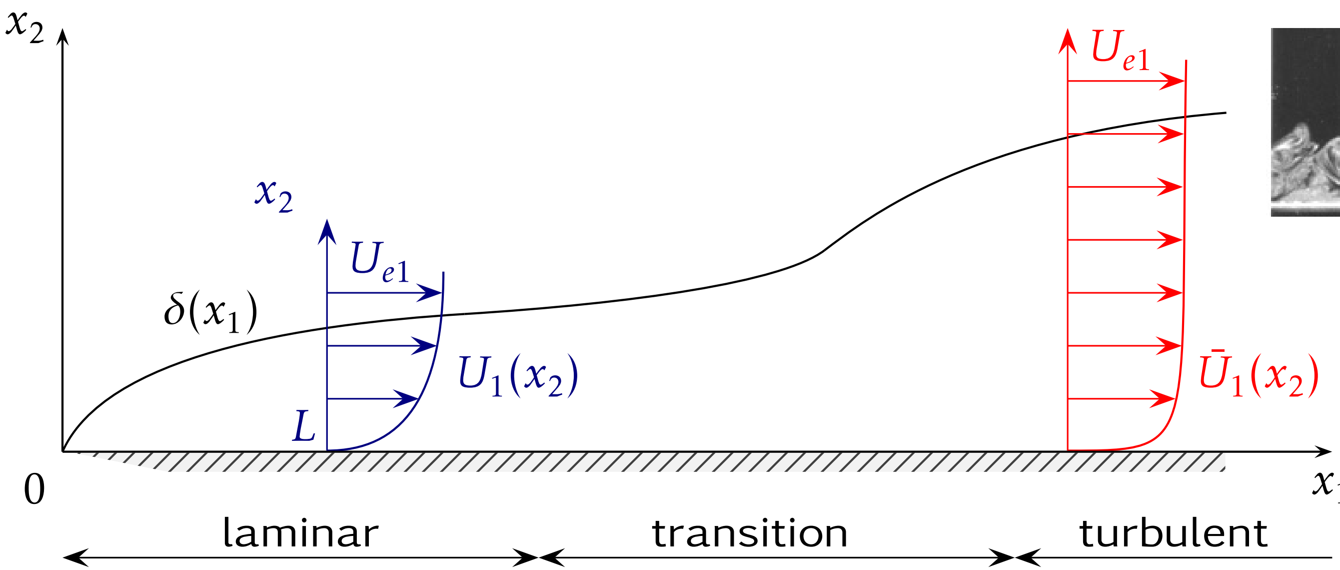
$$\frac{1}{\mathrm{Re}_L}\left(\tilde{U}_1\frac{\partial\tilde{U}_2}{\partial\tilde{x}_1} + \tilde{U}_2\frac{\partial\tilde{U}_2}{\partial\tilde{x}_2}\right) = -\frac{\partial\tilde{p}}{\partial\tilde{x}_2} + \frac{1}{\mathrm{Re}_L}\left(\frac{1}{\mathrm{Re}_L}\frac{\partial^2\tilde{U}_2}{\partial\tilde{x}_1^2} + \frac{\partial^2\tilde{U}_2}{\partial\tilde{x}_2^2}\right)$$

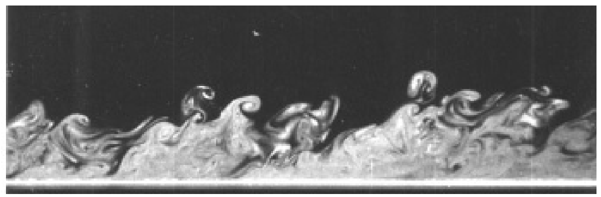
When $\mathrm{Re}_L \gg 1$, the remaining term is

$$\frac{\partial\tilde{p}}{\partial\tilde{x}_2} \simeq 0$$

Zero-pressure-gradient boundary layer on a flat plate

Transition for $\mathrm{Re}_{x_1} \simeq 3.2 \times 10^5$ or equivalently for $\mathrm{Re}_\delta = U_{e1}\delta/\nu \simeq 2800$





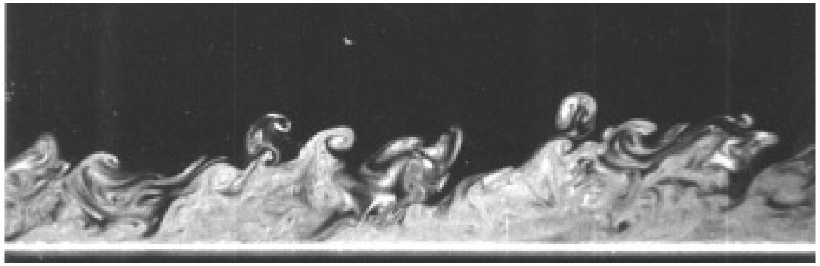
(F. Laadhari, LMFA,
$\mathrm{Re}_{\delta_\theta} \simeq 1000$
$U_{e1} = 2.1$ m.s^{-1} $\delta \simeq 7$ cm)

In laminar regime, molecular diffusion $\tau \sim \delta^2/\nu$ in the transverse direction, compared with the turbulent regime, turbulent diffusion δ/u' with $u' \simeq 0.1\,U_{e1}$

Lee, Kwon, Hutchins & Monty (Melbourne University)

Turbulent boundary layer : short overview

Two different scales must be introduced to correctly describe a turbulent boundary layer



Outer scales – u' with $u'/U_{e1} \simeq 0.1$ and δ
The convection along the mean flow is balanced by the turbulent diffusion in the transverse direction

Inner scales – u_τ and $l_\nu = \nu/u_\tau$
u_τ is the friction velocity defined from the mean wall shear stress $\bar{\tau}_w = \rho u_\tau^2$
$(\bar{\tau}_w = \mu d\bar{U}_1/dx_2$ at the wall)

Short overview (cont.)

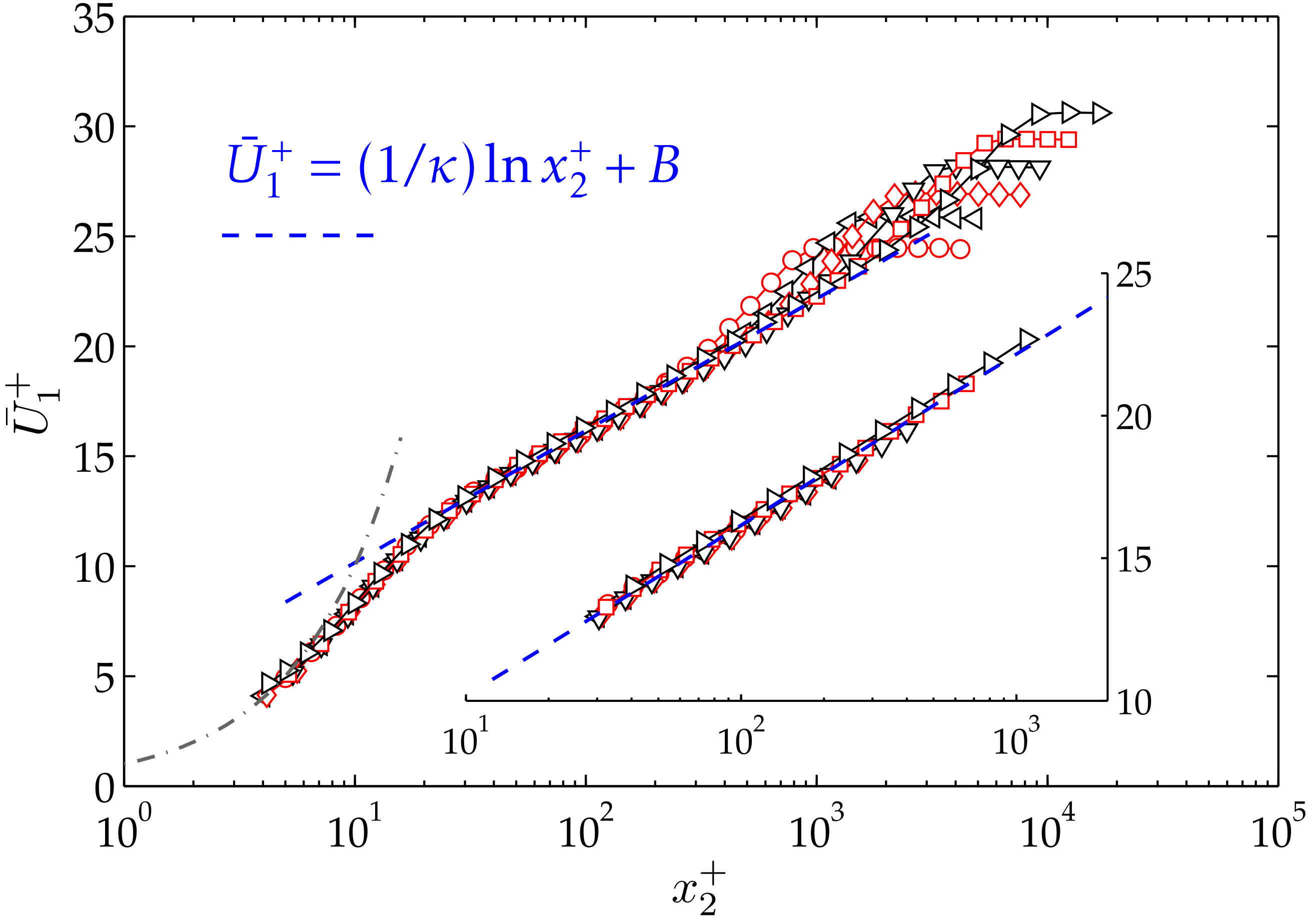
Ratio between the two length scales,

$$\mathrm{Re}^{+} = \frac{\delta}{l_\nu} = \frac{u_\tau \delta}{\nu} \qquad \text{Kármán number}$$

Notations in wall unit

$$x_2^{+} = \frac{x_2}{l_\nu} = \frac{x_2 u_\tau}{\nu} \qquad \bar{U}_1^{+} = \frac{\bar{U}_1}{u_\tau}$$

Mean velocity profile the logarithmic law (inner scales)



log-law
$x_2^+ \geq 30$ & $x_2/\delta \leq 0.20$

For a zero-pressure-gradient boundary layer,

$\kappa \simeq 0.384 \quad B \simeq 4.17$

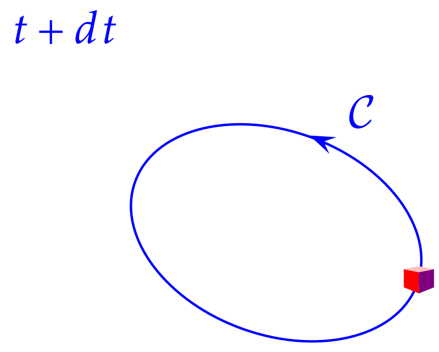
(Bailly & Comte-Bellot, Turbulence, *Springer*, 2015, data from Osterlünd, 1999)

$\mathrm{Re}_{\delta_{0.95}}$	1.7×10^4	2.8×10^4	4.3×10^4	6.9×10^4	1.1×10^5	1.9×10^5
$\mathrm{Re}^+_{\delta_{0.95}}$	684	1092	1594	2462	3944	6147
	○	◁	◊	▽	□	▷

Kelvin's circulation theorem (1869)

What is the time evolution of the circulation $\Gamma_{\mathcal{C}}$ around a material closed curve $\mathcal{C}$, that is a curve moving with the fluid ?

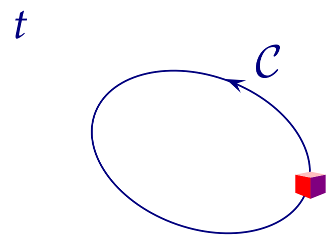
By considering the material derivative of Γ_c,



$$\frac{D\Gamma_{\mathcal{C}}}{Dt} = \frac{D}{Dt}\oint_{\mathcal{C}} \boldsymbol{U}\cdot d\boldsymbol{l} = \underbrace{\oint_{\mathcal{C}} \frac{D\boldsymbol{U}}{Dt}\cdot d\boldsymbol{l}}_{\text{(a)}} + \underbrace{\oint_{\mathcal{C}} \boldsymbol{U}\cdot\frac{Dd\boldsymbol{l}}{Dt}}_{\text{(b)}}$$

For an inviscid flow, using Euler Eq.

$$\text{(a)} = \oint_{\mathcal{C}} \frac{D\boldsymbol{U}}{Dt}\cdot d\boldsymbol{l} = \oint_{\mathcal{C}}\left(-\frac{1}{\rho}\nabla p + \boldsymbol{g}\right)\cdot d\boldsymbol{l} = \int_{\mathcal{S}} \nabla\times\left(-\frac{1}{\rho}\nabla p + \boldsymbol{g}\right)\cdot \boldsymbol{n}\, ds$$

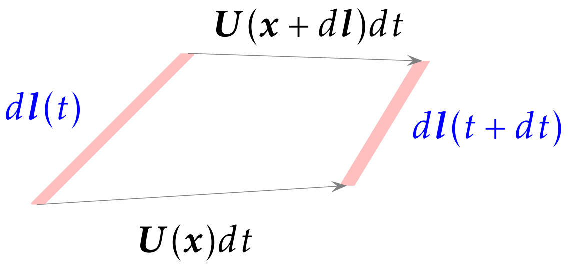


The gravity force is conservative, $\boldsymbol{g} = -\nabla\Psi$, and thus does not contribute : the curl of any gradient is zero. In addition, the first term in (a) can be recast as

$$\nabla\times\left(-\frac{1}{\rho}\nabla p\right) = \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Kelvin's circulation theorem (cont.)

By definition, the second term (b) is zero



$$d\boldsymbol{l}(t+dt) - d\boldsymbol{l}(t) = \boldsymbol{U}(\boldsymbol{x}+d\boldsymbol{l})dt - \boldsymbol{U}(\boldsymbol{x})dt$$

$$dl_i(t+dt) - dl_i(t) = \frac{\partial U_i}{\partial x_j} dl_j\, dt \quad \Longrightarrow \quad \frac{D d\boldsymbol{l}}{Dt} = d\boldsymbol{l} \cdot \nabla \boldsymbol{U}$$

And consequently,

$$\text{(b)} = \oint_{\mathcal{C}} \boldsymbol{U} \cdot \frac{D d\boldsymbol{l}}{Dt} = \oint_{\mathcal{C}} \boldsymbol{U} \cdot (d\boldsymbol{l} \cdot \nabla \boldsymbol{U}) = \oint_{\mathcal{C}} \nabla(U^2/2) d\boldsymbol{l} = 0$$

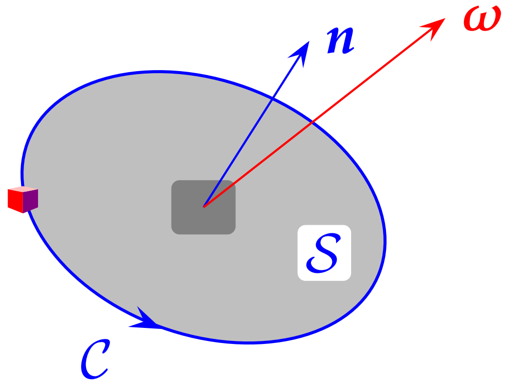
by observing once again that $\nabla \times (\nabla -) = 0$

Kelvin's circulation theorem (cont.)

For an inviscid flow submitted to conservative body forces, the circulation around a material closed curve $\mathcal{C}$ is thus governed by

$$\frac{D\Gamma_\mathcal{C}}{Dt} = \frac{D}{Dt}\oint \boldsymbol{U}\cdot d\boldsymbol{l} = \int_\mathcal{S} \frac{1}{\rho^2}\nabla\rho\times\nabla p\cdot\boldsymbol{n}\, d\mathcal{S} = 0 \qquad \text{for barotropic flows, } \rho = \rho(p)$$

Note that constant density, isothermal, and isentropic flows are barotropic. As a result, the material circulation $\Gamma_\mathcal{C}$ is preserved,



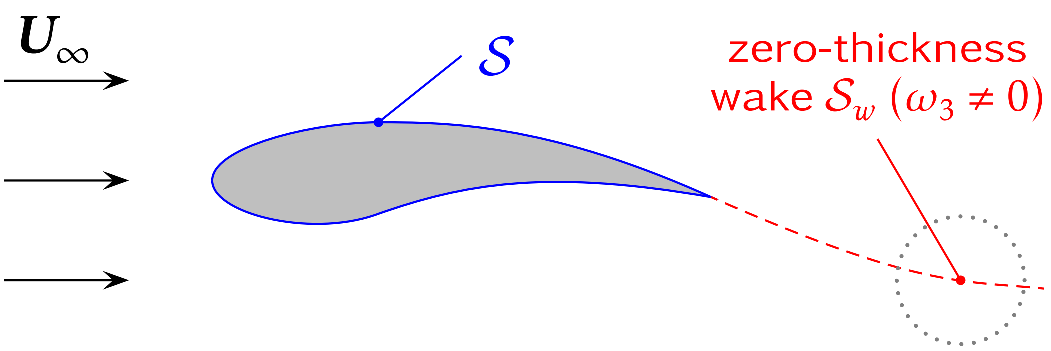
$$\frac{D\Gamma_\mathcal{C}}{Dt} = 0$$

$$\Gamma_c = \oint_\mathcal{C} \boldsymbol{U}\cdot d\boldsymbol{l} = \int_\mathcal{S} \boldsymbol{\omega}\cdot\boldsymbol{n}\, d\mathcal{S} = \text{cst}$$

Japanese smoke ring experiment : giant vortex box of 30 m! (starting vortex)

Model flow around a streamlined body

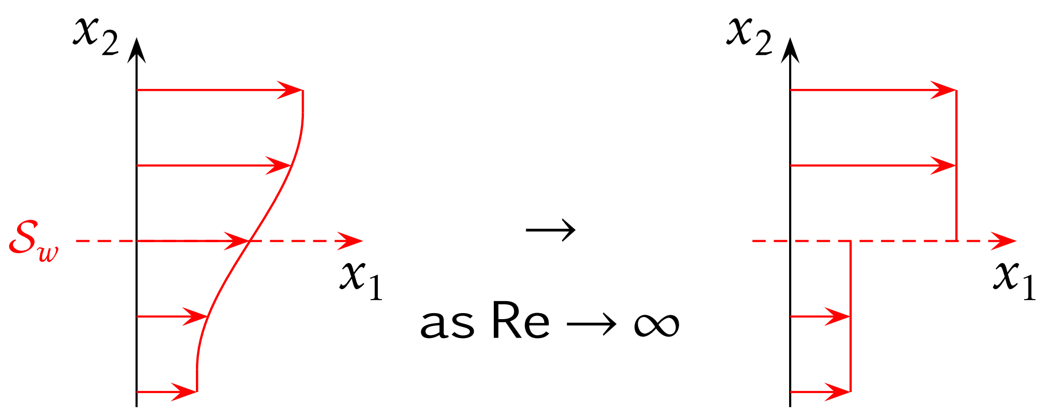
Until the end of this chapter, the case of a streamlined body when Re $\to \infty$ is considered. The flow is never detached from the profile, with a zero-thickness boundary layer and wake.



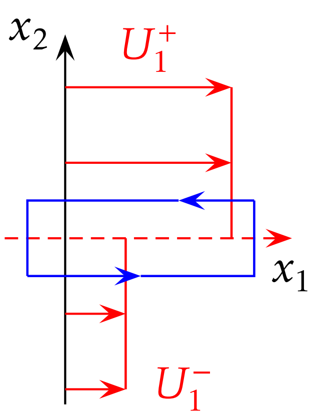
$\mathcal{S}_w$ is a vortex sheet, where all the vorticity is concentrated. The flow is irrotational and inviscid everywhere else.

Mixing layer between two streams
$\boldsymbol{U} = (U(x_2), 0, 0)$

$$\omega_3 = -\frac{dU}{dx_2}$$



Vortex sheet model



A vortex sheet only makes sense for an inviscid flow (viscosity would destroy discontinuities and impose a non-zero thickness)

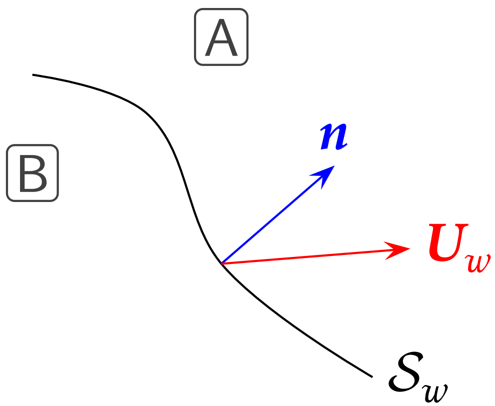
The jump in tangential velocity discontinuity across the vortex sheet ($x_2 = 0$) is $\gamma_s = U_1^- - U_1^+$,

and the vorticity can be expressed as $\omega = \gamma_s \delta(x_2)$ (s^{-1})

$\Gamma = [U_1^- - U_1^+] l_1 = \int_{-l_2}^{+l_2} \gamma_s \delta(x_2) l_1 dx_2$ over $[0, l_1] \times [-l_2, +l_2]$

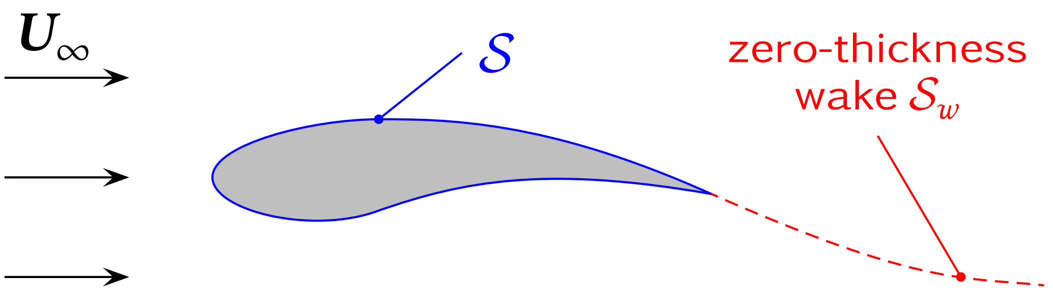
More generally, the velocity discontinuity reads $\boldsymbol{\gamma}_s = \boldsymbol{U}_B - \boldsymbol{U}_A$, and the vortex sheet has a local velocity $\boldsymbol{U}_w$.

One remains here that the boundary conditions at the interface $\mathcal{S}_w$ for inviscid flows are imposed by the continuity of the pressure $p_A = p_B$ and of the normal component of the velocity $\boldsymbol{U}_A \cdot \boldsymbol{n} = \boldsymbol{U}_B \cdot \boldsymbol{n}$



The interface motion satisfies $\boldsymbol{U}_w \cdot \boldsymbol{n} = \boldsymbol{U} \cdot \boldsymbol{n}$

Model flow around a streamlined body (cont.)



Flow governing equations (except in $\mathcal{S}_w$)

$$\begin{cases} \dfrac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\dfrac{1}{\rho} \nabla p_\star \\ \nabla \cdot \boldsymbol{U} = 0 \end{cases}$$

These equations are associated with the boundary conditions
$\boldsymbol{U} \cdot \boldsymbol{n} = 0$ on $\mathcal{S}$ and $\boldsymbol{U} \to \boldsymbol{U}_\infty$ upstream

In addition, the continuity of p and $\boldsymbol{U}$ through the vortex sheet $\mathcal{S}_w$ must be satisfied (+ interface motion if necessary)

This problem is difficult to solve, but an instructive solution for understanding physics can be derived for a 2-D steady flow

Flow around an airfoil : context

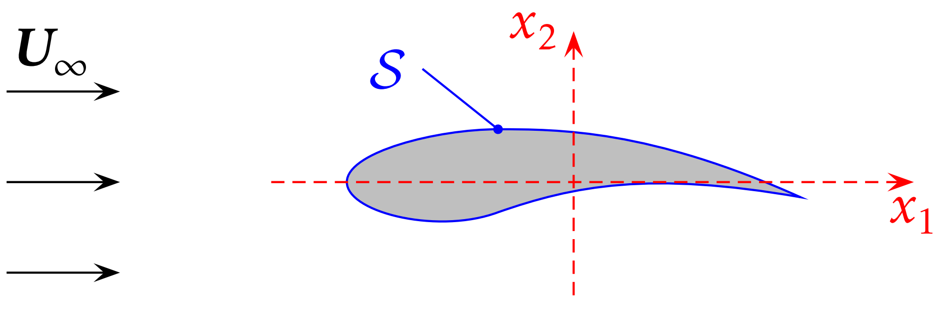
Assumptions : 2-D steady flow around an airfoil at high Reynolds number with $\boldsymbol{U} = (U_1(x_1,x_2), U_2(x_1,x_2))$

Using the strongest formulation of Bernoulli's theorem (see slide 173)

$$p + \frac{1}{2}\rho U^2 = \text{cst} \qquad (\rho\Psi = 0, \text{ no gravity force})$$

As a consequence, the continuity of p implies that $U = \text{cst}$ across the vortex sheet $\mathcal{S}_w$, and since $\boldsymbol{U}\cdot\boldsymbol{n}$ must also be continuous, $\boldsymbol{U}$ is continuous across $\mathcal{S}_w$

The vortex sheet can thus be removed from the problem, the wake $\mathcal{S}_w$ does not contribute to the solution. The flow is irrotational everywhere.



Introduction of a stream function ψ

For a 2-D incompressible flow

$$U_1 = \frac{\partial \psi}{\partial x_2} \qquad U_2 = -\frac{\partial \psi}{\partial x_1}$$

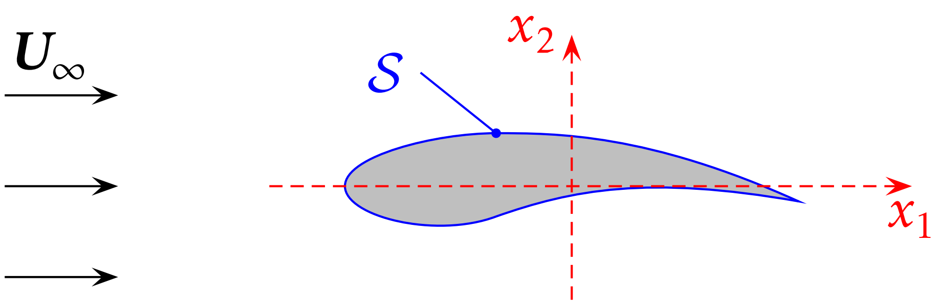
The relationship $\nabla \cdot \boldsymbol{U} = 0$ is thus satisfied by construction. Moreover, $\boxed{\nabla^2 \psi = -\omega}$ and since $\boldsymbol{U} \cdot \nabla \psi = 0$ by construction also, $\psi =$ cst along a streamline.

Here the desired flow field is irrotational, and thus $\nabla^2 \psi = 0$

On the airfoil surface $\mathcal{S}$, $\boldsymbol{U} \cdot \boldsymbol{n} = 0$. The surface is thus a streamline with $\psi =$ cst. Usually, the constant is taken to be $\psi = 0$ on $\mathcal{S}$ by convention.

Far from the airfoil, $\psi \to \psi_\infty = U_\infty x_2$, leading to $\boldsymbol{U}_\infty = U_\infty \boldsymbol{e}_1$.

Introduction of a stream function (cont.)



$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = 0$$

$\psi = 0$ on $\mathcal{S}$

$\psi \sim U_\infty x_2$ far from the airfoil

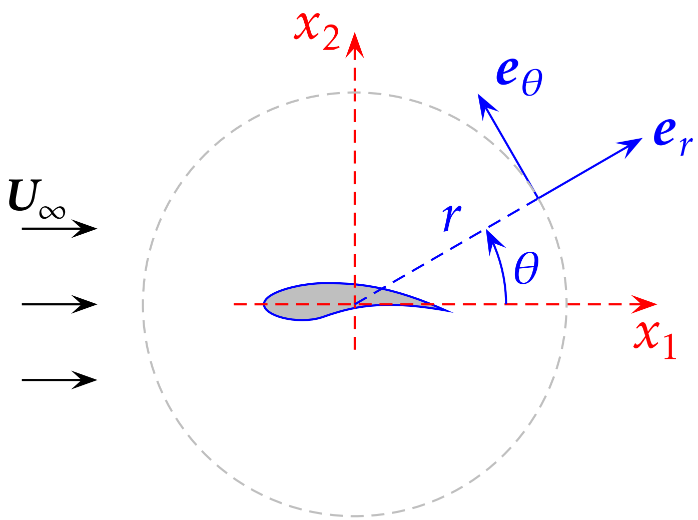
That problem is not so difficult to solve (at least numerically!) : Laplace equation associated with mixed-type boundary conditions.

When ψ is known, the velocity field is obtained from ψ, and the pressure from Bernoulli's theorem $p + \rho U^2/2 = \text{cst}$.

Determination of the far field flow

Using polar coordinates to express $\nabla^2\psi = 0$,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$



The solution is periodic in θ, and can be searched as a Fourier series

$$\psi = \sum_{n=0}^{\infty} C_n(r) e^{in\theta}$$

The $C_n(r)$ functions must satisfy the following equation

$$r\frac{d}{dr}\left(r\frac{dC_n}{dr}\right) - n^2 C_n = 0 \quad \Longrightarrow \quad \begin{cases} C_0 = A_0 \ln r + B_0 & n = 0 \\ C_n = A_n r^{|n|} + B_n r^{-|n|} & n \neq 0 \end{cases}$$

The red terms contribute to the near field only, close to the airfoil. They are thus disregarded here (linked to the detailed airfoil geometry).

Determination of the far field flow (cont.)

The stream function ψ must also match ψ_∞ for large values in r

$$\psi_\infty = U_\infty x_2 = U_\infty r \sin\theta = \frac{U_\infty r}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \quad \Longrightarrow \quad A_1 = A_{-1} = \frac{U_\infty}{2i}$$

and $A_n = A_{-n} = 0$ for $n > 1$ (Note that the term C_0 does not contribute to $\boldsymbol{U}$ as $r \to \infty$). The stream function then reads

$$\psi = U_\infty r \sin\theta + A_0 \ln(r) + B_0 + \mathcal{O}(r^{-1})$$

By reminding that the velocity vector is given by

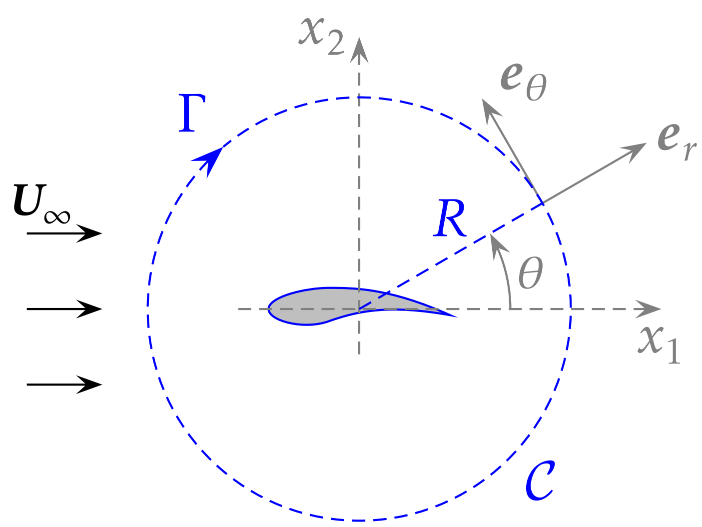
$$\boxed{U_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad U_\theta = -\frac{\partial\psi}{\partial r}}$$

the velocity field can be expressed as

$$\boldsymbol{U} = U_\infty \cos\theta\, \boldsymbol{e}_r + \left(-U_\infty \sin\theta - \frac{A_0}{r}\right)\boldsymbol{e}_\theta + \mathcal{O}(r^{-2}) = \boldsymbol{U}_\infty - \frac{A_0}{r}\boldsymbol{e}_\theta + \mathcal{O}(r^{-2})$$

Far field flow and circulation

Since the flow is irrotational, the circulation Γ is only a property of the airfoil. By considering a circle of radius R surrounding the airfoil, using clockwise sense here for $\mathcal{C}$,



$$\Gamma = \int_{\mathcal{C}} \boldsymbol{U} \cdot d\boldsymbol{x} = -\int_0^{2\pi} \left(-\frac{A_0}{R}\boldsymbol{e}_\theta\right) \cdot R d\theta \boldsymbol{e}_\theta = 2\pi A_0$$

$$\boxed{\boldsymbol{U} = \boldsymbol{U}_\infty - \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta + \mathcal{O}(r^{-2})}$$

Far from the airfoil, the flow field is equivalent to a point vortex.

The pressure field is given by

$$p = p_\infty + \frac{1}{2}\rho(U_\infty^2 - U^2) = p_\infty + \frac{\rho\Gamma}{2\pi r}\boldsymbol{U}_\infty \cdot \boldsymbol{e}_\theta + \mathcal{O}(r^{-2})$$

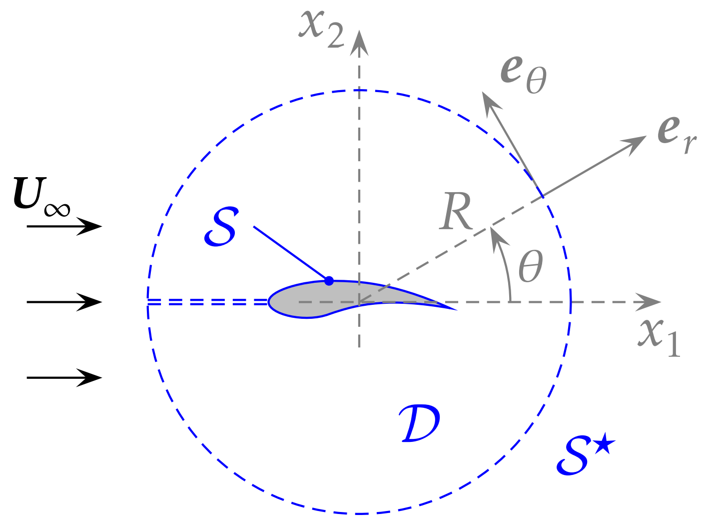
Force applied to the airfoil

Integral formulation of the momentum conservation

$$\frac{d}{dt}\int_{\mathcal{D}} \rho \boldsymbol{U}\, dv = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n}\, ds + \int_{\mathcal{S}} \rho \boldsymbol{U}(\boldsymbol{U}_{\mathcal{S}} - \boldsymbol{U}) \cdot \boldsymbol{n}\, ds$$

It is considered a fixed ($\boldsymbol{U}_{\mathcal{S}} = 0$) fluid domain $\mathcal{D}$ bounded by $\mathcal{S} \cup \mathcal{S}^\star$, for a steady flow (lhs = 0), and $\overline{\overline{\boldsymbol{\sigma}}} \cdot \boldsymbol{n} = -p\boldsymbol{n}$. Hence

$$\int_{\mathcal{S}\cup\mathcal{S}^*} \underbrace{p\boldsymbol{n} + \rho \boldsymbol{U}(\boldsymbol{U} \cdot \boldsymbol{n})}_{\bullet}\, ds = 0$$



The force $\boldsymbol{F}$, per unit length in the spanwise direction, applied to the airfoil is then given by

$$\boldsymbol{F} = -\int_{\mathcal{S}} \bullet\, ds = \int_{\mathcal{S}^\star} \bullet\, ds = -\int_0^{2\pi} \left[p\boldsymbol{e}_r + \rho \boldsymbol{U}(\boldsymbol{U} \cdot \boldsymbol{e}_r)\right]_{r=R} R d\theta$$

Force applied to the airfoil (cont.)

From the determination of the stream function, one has
$\boldsymbol{U} \cdot \boldsymbol{e}_r \sim \boldsymbol{U}_\infty \cdot \boldsymbol{e}_r \sim U_\infty \cos\theta$ for large values in R

and on the surface $\mathcal{S}^\star$ $(r = R)$, $\quad \boldsymbol{U} \sim \boldsymbol{U}_\infty - \dfrac{\Gamma}{2\pi R}\boldsymbol{e}_\theta \qquad p \sim p_\infty + \dfrac{\rho\Gamma}{2\pi R}\boldsymbol{U}_\infty \cdot \boldsymbol{e}_\theta$

The force $\boldsymbol{F}$ per unit length in the spanwise direction is then given by

$$
\begin{aligned}
\boldsymbol{F} &\sim -\int_0 2\pi\left[\frac{\rho\Gamma}{2\pi}(-U_\infty \sin\theta)\boldsymbol{e}_r + \rho U_\infty \cos\theta\left(-\frac{\Gamma}{2\pi}\right)\boldsymbol{e}_\theta\right] R d\theta \\
&\sim \frac{\rho\Gamma U_\infty}{2\pi}\int_0 2\pi\left[\sin\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \cos\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right] d\theta \\
&\sim \frac{\rho\Gamma U_\infty}{2\pi} 2\pi \boldsymbol{e}_2
\end{aligned}
$$

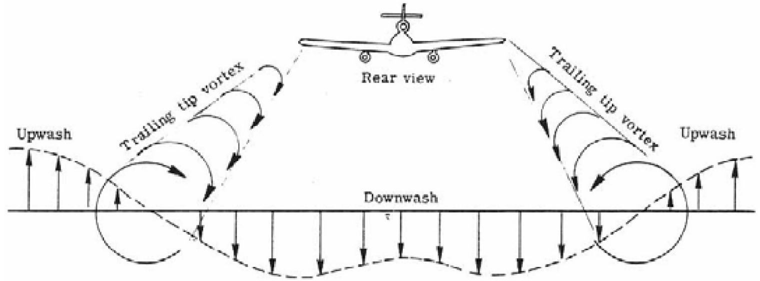
$\boldsymbol{F} \sim (0, \rho\Gamma U_\infty)$ Kutta-Joukowski theorem

There is no drag force (as expected thanks to the flow model used), and the lift force is found to be directly proportional to the circulation.

Wing-tip vortex

Tip vortex behind an agricultural airplane (aerial spraying)

Boeing 767-370/ER

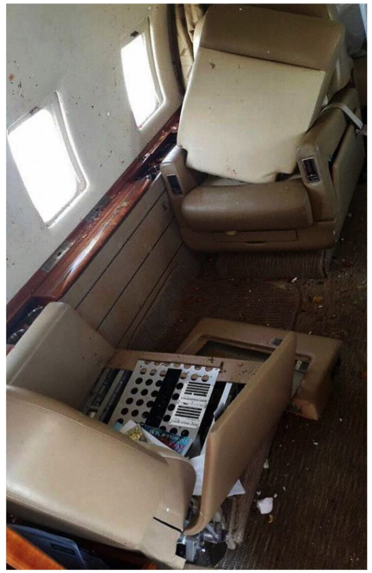


(Talay, T.A., Introduction to the aerodynamics of flight, NASA SP-367, 1975)

Emirates A380-800 over Arabian Sea on Jan 7th 2017, wake turbulence sends business jet in uncontrolled descent

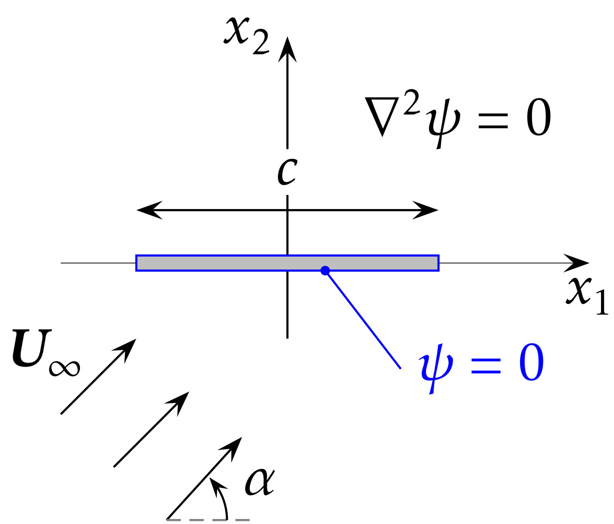
www.avherald.com

The CL-604 passed 1000 feet below an Airbus A380-800 while enroute over the Arabian Sea, when a short time later (1-2 minutes) the aircraft encountered wake turbulence sending the aircraft in uncontrolled roll turning the aircraft around at least 3 times, both engines flamed out, the Ram Air Turbine could not deploy possibly as result of G-forces and structural stress, the aircraft lost about 10,000 feet until the crew was able to recover the aircraft exercising raw muscle force, restart the engines and divert to Muscat.



wingspan of 19.6 m (Canadair Challenger 604) versus 79.7 m (A380)

Flat plate airfoil



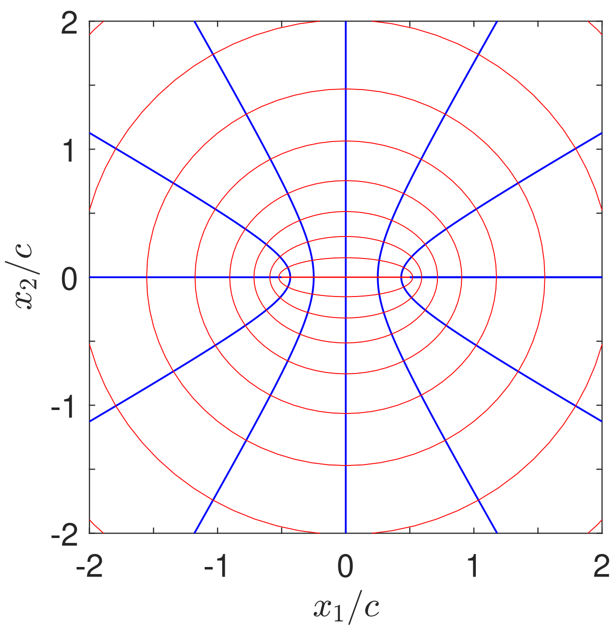
Introduction of elliptic coordinates (ξ, η)

$$\begin{cases} x_1 = \dfrac{1}{2}c\cosh(\xi)\cos(\eta) \\ x_2 = \dfrac{1}{2}c\sinh(\xi)\sin(\eta) \end{cases} \qquad \xi \geq 0 \quad 0 \leq \eta \leq 2\pi$$

The curves η = cst are hyperbolae, whereas the curves ξ = cst are ellipses. When $\xi \to \infty$,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \frac{1}{4}ce^{\xi}\begin{pmatrix} \cos\eta \\ \sin\eta \end{pmatrix}$$

polar coordinates are retrieved with $r \sim \frac{1}{4}ce^{\xi}$ and $\theta \sim \eta$. The value $\xi = 0$ corresponds to the flat-plate chord c.



Flat plate airfoil (cont.)

Laplace's equation $\nabla^2\psi = 0$ for the stream function $\psi(\xi,\eta)$ in elliptic coordinate

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\eta^2} = 0$$

The stream function is periodic in η, $\psi(\xi,\eta+2\pi) = \psi(\xi,\eta)$, and is also associated with boundary conditions $\psi = 0$ at $\xi = 0$, and

$$\psi \sim U_\infty(x_2\cos\alpha - x_1\sin\alpha) \sim \frac{1}{4}cU_\infty e^{\xi}\sin(\eta-\alpha) \qquad \text{as } \xi \to \infty$$

that is far from the plate, where α is the angle of attack of the incoming flow.

It can shown that the solution to this problem is

$$\psi = A\xi + \frac{1}{2}cU_\infty\sinh(\xi)\sin(\eta-\alpha)$$

where A is an unknown constant, linked to the circulation $A = \Gamma/(2\pi)$ and thus to the lift force.

The Kutta condition

For a given configuration (airfoil shape and angle of incidence), the value of Γ is such that the flow leaves smoothly the trailing edge (an assumption of the flow model). That is often an intuitive condition, but the general statement is that the strength of the vortex sheet must be zero at the trailing edge, $\gamma_s(\boldsymbol{x}_{te}) = 0$ (see slide 332)

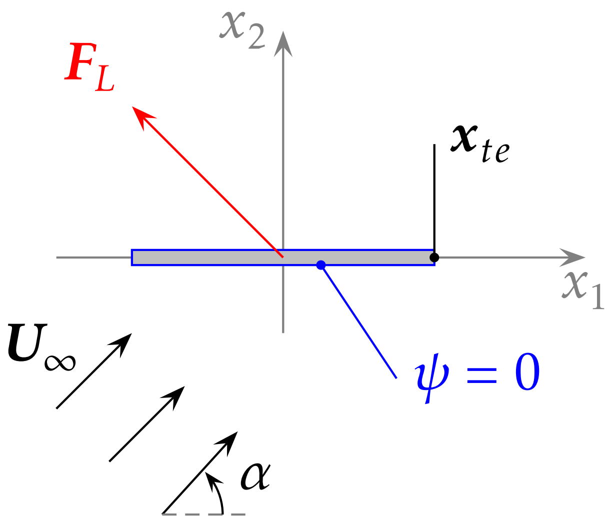
For the flat plate airfoil, the velocity must be finite at $\boldsymbol{x}_{te}$. This condition allows to determine the value of the circulation and thus provides a unique solution. The slip velocity along the surface is given by (see Appendix 350)

$$U_1(\xi = 0, \eta) = \frac{1}{\sin(\eta)} \left[\frac{\Gamma}{\pi c} + U_\infty \sin(\eta - \alpha) \right]$$

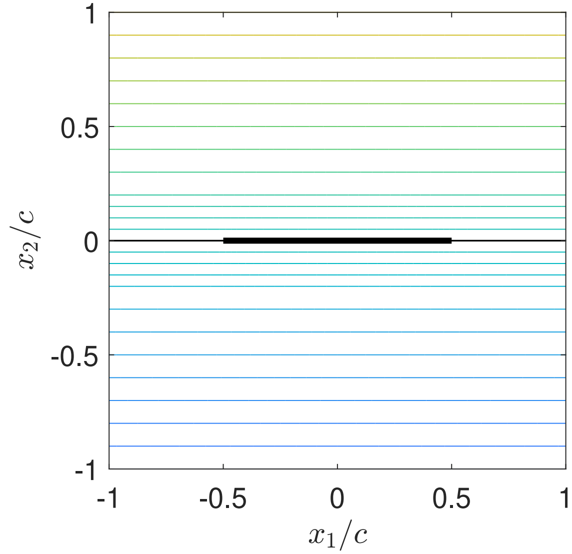
This velocity is infinite at the trailing edge in $\eta = 0$, except if $\Gamma = \pi c U_\infty \sin\alpha$. The lift force (per unit length in the spanwise direction) is thus $F_L = \pi \rho c U_\infty^2 \sin\alpha$, and lift coefficient C_L in 2-D is provided by

$$C_L = \frac{F_L}{c \frac{1}{2} \rho U_\infty^2} = 2\pi \sin\alpha \qquad (C_D = 0)$$

The Kutta condition (cont.)



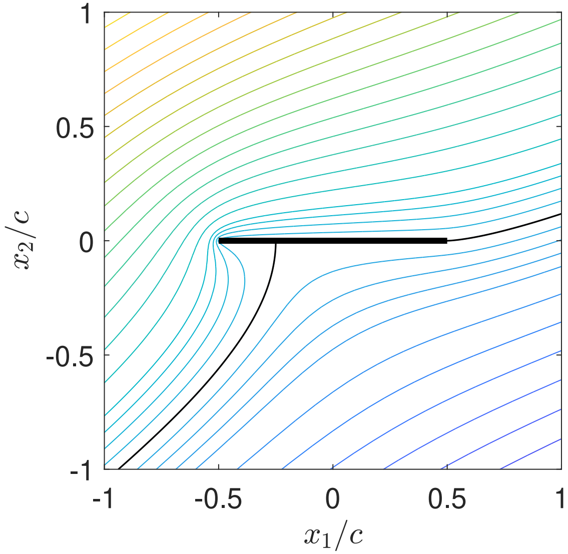
$\alpha = 0° \, (\Gamma = 0)$



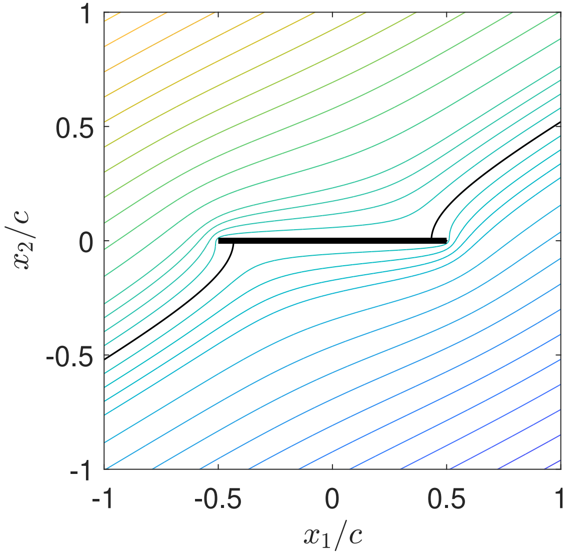
Kutta condition : the unphysical divergence is cancelled by letting one of the two stagnation points meeting the trailing edge.

Note also that $U_1(\xi = 0, \eta = 0) \neq 0$ at the TE

$\alpha = 30° \, (\Gamma = \pi c U_\infty \sin\alpha)$



$\alpha = 30° \, (\Gamma = 0!)$



Stalled airfoil and reduction in the lift coefficient

From Deutsches Zentrum für Luft- und Raumfahrt (DLR, 1915)

See also slides 140 - 141

Further reading

(about the generation of lift)

Babinsky, H., 2003, How do wings work ?, *Physics Education*, **38**(6), 497-503

M. Wilhelm Kutta
(German mathematician, 1867-1944)

Nikolai E. Joukowski
(Russian physicist, 1847-1921)

Determination of the slip velocity for the flat plate airfoil

$$U_1 = \frac{\partial \psi}{\partial x_2} = \frac{\partial \psi}{\partial \xi}\frac{\partial \xi}{\partial x_2} + \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x_2} \quad \text{in } \xi = 0$$

$$\psi = A\xi + \frac{1}{2}cU_\infty \sinh(\xi)\sin(\eta - \alpha)$$

$$\begin{cases} \dfrac{\partial \psi}{\partial \xi} = A + \dfrac{1}{2}cU_\infty \cosh(\xi)\sin(\eta - \alpha) \\ \dfrac{\partial \psi}{\partial \eta} = \dfrac{1}{2}cU_\infty \sinh(\xi)\cos(\eta - \alpha) \end{cases}$$

$$U_1(\xi = 0) = \left[A + \frac{1}{2}cU_\infty \sin(\eta - \alpha)\right] \left.\frac{\partial \xi}{\partial x_2}\right|_{\xi=0}$$

$$x_2 = \frac{1}{2}c \sinh(\xi)\sin(\eta)$$

$$dx_2 = \frac{1}{2}c\cosh(\xi)\sin(\eta)d\xi + \frac{1}{2}c\sinh(\xi)\cos(\eta)d\eta \qquad \left.\frac{\partial \xi}{\partial x_2}\right|_{\xi=0} = \frac{2}{c\sin(\eta)}$$

$$\Longrightarrow \quad U_1(\xi = 0) = \frac{1}{\sin(\eta)}\left[\frac{2A}{c} + U_\infty \sin(\eta - \alpha)\right]$$

Viscous stress tensor & bulk viscosity

$$
\begin{aligned}
\overline{\overline{\boldsymbol{\tau}}} &= \mu\left[\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^t\right] + \lambda(\nabla\cdot\boldsymbol{u})\overline{\overline{\boldsymbol{I}}}\\
&= \mu\left[\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^t - \frac{2}{3}(\nabla\cdot\boldsymbol{U})\overline{\overline{\boldsymbol{I}}}\right] + \left(\lambda + \frac{2}{3}\mu\right)(\nabla\cdot\boldsymbol{U})\overline{\overline{\boldsymbol{I}}}\\
&= 2\mu\boldsymbol{s} + \mu_b(\nabla\cdot\boldsymbol{U})\overline{\overline{\boldsymbol{I}}}
\end{aligned}
$$

λ second viscosity

$\mu_b = \lambda + (2/3)\mu$ bulk viscosity

Stokes's hypothesis (1845) $\mu_b = 0$

The bulk viscosity can be deduced experimentally from absorption of sound (refer to the remark in slide 214 about the normal mean stress)

$\mu_b \equiv 0$ for monoatomic gases, and $\mu_b = \mu_b(T)$ otherwise ($\mu_b \simeq 0.6\mu$)

Karim & Rosenhead (1952), Landau & Lifchitz (1989), Pierce (1991)

Conservative form of governing equations

for the 1-D compressible inviscid flow model

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{U}) = 0 \\ \dfrac{\partial (\rho \boldsymbol{U})}{\partial t} + \nabla \cdot (\rho \boldsymbol{U} \boldsymbol{U}) = -\nabla p \\ \dfrac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho h_t \boldsymbol{U}) = 0 \end{cases}$$

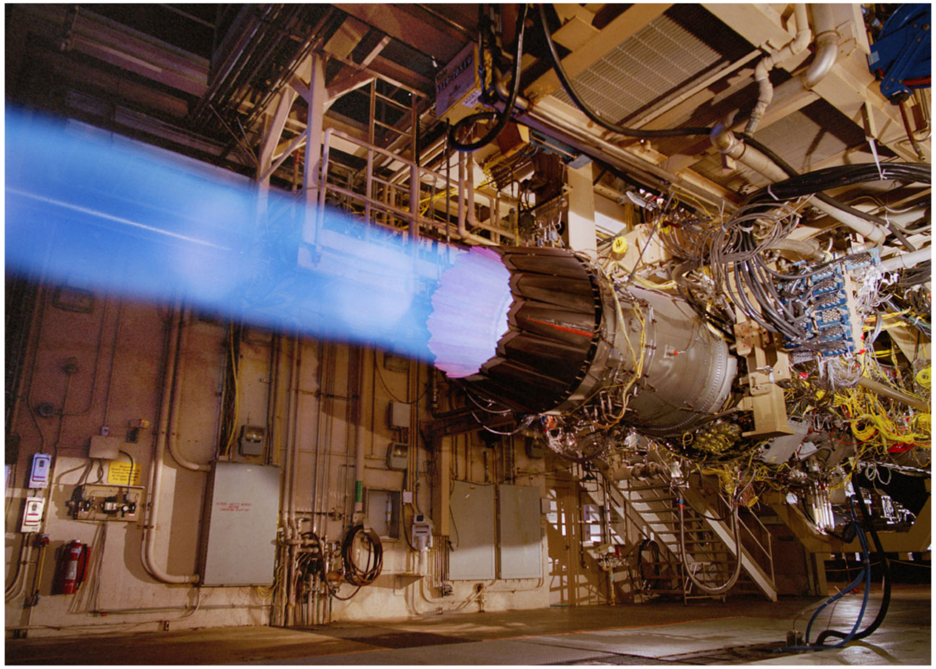
Using the conservation of mass, for any variable φ

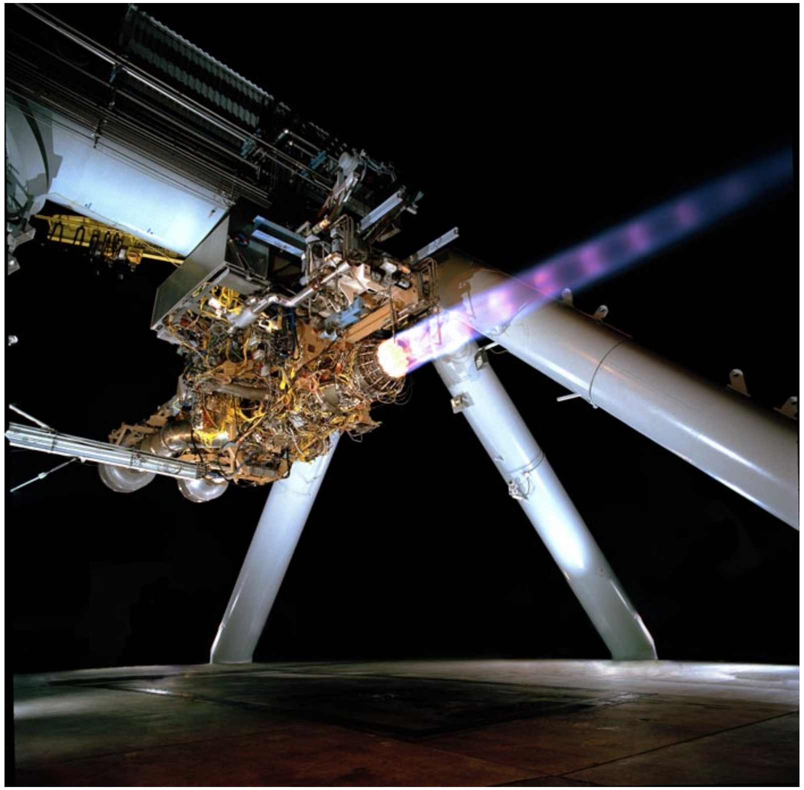
$$\frac{\partial (\rho \varphi)}{\partial t} + \nabla \cdot (\rho \varphi \boldsymbol{U}) = \rho \left(\frac{\partial \varphi}{\partial t} + \boldsymbol{U} \cdot \nabla \varphi \right) = \rho \frac{D\varphi}{Dt}$$

Military and supersonic transport aircrafts

Pratt & Whitney FX631 jet engine (F-35 Joint Strike Fighter)

Kleine & Settles, *Shock Waves* (2008)





http://www.jsf.mil

Terminology : adversion versus convection

Dictionnaire de l'Académie des Sciences (9e édition)

- ADVECTION n. f. XXe siècle. Emprunté du latin *advectio,* « transport ». Déplacement horizontal d'une masse d'air. L'advection de l'air maritime tiède au-dessus du continent

- CONVECTION n. f. XIXe siècle. Emprunté du bas latin *convectio,* « action de transporter », de *convehere,* « charrier ». PHYS. Circulation globale d'un fluide. Spécialt. Transport de chaleur lié aux mouvements d'un fluide. échauffement par convection. Convection forcée, activée par un moyen mécanique, pompe ou ventilateur. - GÉOPHYSIQUE. Courants de convection, produits par les différences de température à l'intérieur du manteau de l'écorce terrestre. Les phénomènes de convection au sein de la Terre. - MÉTÉOR. Certains vents sont des courants de convection naturelle. - ASTRON. La convection au sein du Soleil.

Mixture properties - Thermodynamics

The thermodynamic state of a mixture of N species is determined by $N+1$ independent thermodynamic variables. As an illustration, one has for the entropy $s = (\rho, e, Y_1, \ldots, Y_{N-1})$. In addition, only ρ, e and Y_α are defined independently of any thermodynamic equilibrium, refer to slide 222.

The Gibbs relation (see slide 225) can be generalized as follows

$$T ds = de + p d\left(\frac{1}{\rho}\right) - \sum_{\alpha=1}^{N} \mu_\alpha dY_\alpha$$

where μ_α is the chemical potential (J/kg) of species α.

The specific enthalpy $h = e + p/\rho$ is given by

$$h = \sum_{\alpha=1}^{N} Y_\alpha h_\alpha \qquad dh = \frac{1-\beta T}{\rho} dp + c_p dT + \sum_{\alpha=1}^{N} h_\alpha dY_\alpha$$

Mixture properties - Thermodynamics (cont.)

where β is the thermal expansion coefficient

$$\beta \equiv -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg|_{p,Y_\alpha} = \frac{1}{T} \text{ for a perfect gas}$$

The generalization of the definition of thermodynamic variables is straightforward, *e.g.* for the specific heat at constant pressure

$$c_p = \frac{\partial h}{\partial T}\bigg|_{p,Y_\alpha}$$

where $|_{Y_\alpha}$ denotes the partial derivative with Y_α constant for $\alpha = 1, \ldots, N$

Partial specific enthalpy h_α

$$h_\alpha = \frac{\partial h}{\partial Y_\alpha}\bigg|_{p,Y_\beta} \qquad \text{(with the convention, } \beta \neq \alpha\text{)}$$

Svante August Arrhenius (1859-1927)

Nobel Prize in Chemistry in 1903 (electrolytic theory of dissociation)

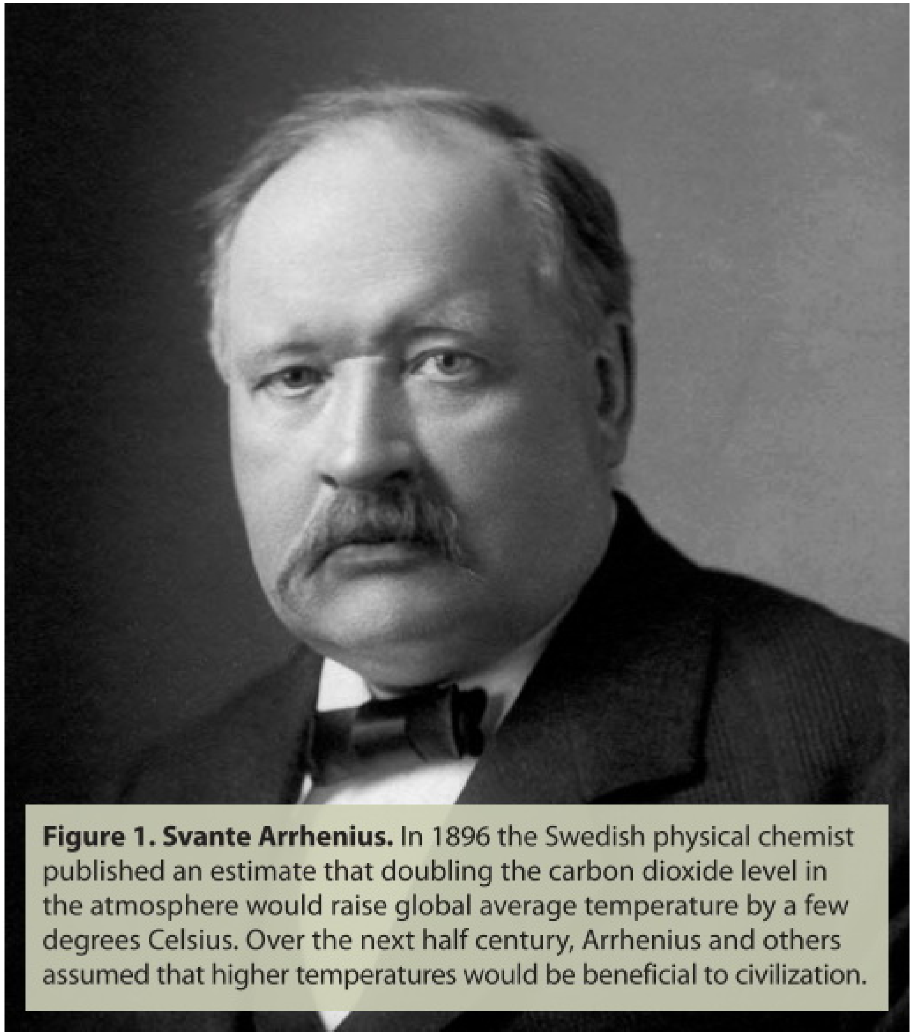


Figure 1. Svante Arrhenius. In 1896 the Swedish physical chemist published an estimate that doubling the carbon dioxide level in the atmosphere would raise global average temperature by a few degrees Celsius. Over the next half century, Arrhenius and others assumed that higher temperatures would be beneficial to civilization.

Physics Today, Sept. 2015,
Climate change impacts : The growth of understanding (S. Weart)

... Through the first half of the 20th century, when global warming from the greenhouse effect was itself only a speculation, the handful of scientists who thought about it supposed any warming would be for the good. For example, Svante Arrhenius published the first calculations in 1896 and claimed that the world *may hope to enjoy ages with more equable and better climates.* Others tended to agree that global warming, or any effect of the progress of human industry, could only lead to a beneficent future.

End of appendices