$$\frac{dP}{dz} + \frac{1}{2\mu_0}\frac{d}{dz}(B_x^2 + B_y^2) = 0 \qquad (21c)$$

Examination of Eqs. (21) shows that the pressure P must be expressed in the form

$$P = ax + by + p(z) \qquad (22)$$

where a and b are constants, and $p(z)$ is some function of z. Substituting in Eq. (21c) and integrating gives

$$p = -(1/2\mu_0)(B_x^2 + B_y^2) + C \qquad (23)$$

where C is a constant. Eqs. (21a) and (21b) now become

$$\frac{B_0}{\mu_0}\frac{dB_x}{dz} + \mu\frac{d^2u}{dz^2} = a \qquad (24a)$$

$$\frac{B_0}{\mu_0}\frac{dB_y}{dz} + \mu\frac{d^2v}{dz^2} = b \qquad (24b)$$

In addition, two of Eqs. (13) survive:

$$\frac{1}{\mu_0\sigma}\frac{d^2B_x}{dz} + B_0\frac{du}{dz} = 0 \qquad (25a)$$

$$\frac{1}{\mu_0\sigma}\frac{d^2B_y}{dz^2} + B_0\frac{dv}{dz} = 0 \qquad (25b)$$

Eqs. (24a) and (25a), which involve u and B_x only, may be combined to give

$$\frac{d^3u}{dz^3} - \frac{\sigma B_0^2}{\mu}\frac{du}{dz} = 0 \qquad (26)$$

and

$$\frac{d^3B_x}{dz^3} - \frac{\sigma B_0^2}{\mu}\frac{dB_x}{dz} = -\frac{\mu_0\sigma B_0}{\mu}a \qquad (27)$$

These two equations can be solved separately for u and B_x, respectively. Since Eqs. (24b) and (25b), which involve v and B_y, have a structure similar to that of Eqs. (24a) and (25a), it is evident that v and B_y have the same functional form, except for a constant, as u and B_x, respectively. Once B_x and B_y are determined, the pressure P can be found from Eqs. (22) and (23).

The solutions to a few problems of the two cases discussed in the foregoing are reported in the literature. Shercliff[1] obtained a complete solution for a problem of the case 1 type, namely, the flow through a straight pipe with rectangular cross section. Sherman and Sutton[2] solved a problem of the case 2 type except for the inclusion of the Hall effects in Eq. (8). The flow between parallel plates was studied by Hartmann[3] for the case of stationary plates and by Lehnert[4] for the case of relative motion between the plates. Both of these problems come under case 2 but with the additional simplifying condition that $v = 0$.

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Existence of Periodic Solutions Passing Near Both Masses of the Restricted Three-Body Problem

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The following new result is announced and an outline of its proof indicated. There exist in the restricted three-body problem with small mass ratio one-parametric families of synodically closed solution curves, which are near rotating Keplerian ellipses with arbitrary rational sidereal frequencies and appropriate positive eccentricities. By suitable selection of the parameter values, these periodic solutions can be made to come close to both attracting bodies. Thus, besides its meaning for astronomy or atomic physics possibly, the practical significance of this result for astronautics becomes apparent if one considers operating spacecraft along such paths in the Earth-sun or moon-Earth systems. (The detailed mathematical existence proof is to appear elsewhere.)

In the restricted three-body problem, one considers the motion of a particle P of negligible mass moving subject to the Newtonian attraction from two other bodies E and M, which rotate in circles about their center of gravity S. One can choose the units of time, mass, and distance such that E and M have masses $1 - \mu$ and μ, distance 1, and angular velocity 1 ($0 \leqq \mu \leqq 1$). Considering the case when P moves in the plane of E and M only and using complex position vectors drawn from S as origin, the equations of motion for P with position vector $x = x_1 + ix_2 = x(t)$ in a co-system rotating with E and M are ($\dot{} = d/dt$)

$$\ddot{x} + 2i\dot{x} - x = -(1-\mu)(x+\mu)|x+\mu|^{-3} - \mu(x+\mu-1)|x+\mu-1|^{-3} \qquad [1]$$

since $-\mu e^{it}$, $(1-\mu)e^{it}$, and $z = xe^{it} = z(t)$ are the inertial position vectors of E, M, and P at time t.

For $\mu = 0$, the solutions of [1] are well known. They correspond to Keplerian motions $z(t)$, i.e., solutions of $\ddot{z} = -z|z|^{-3}$. If $z(t)$ describes an elliptic motion, its period $T_0 = 2\pi|a^{3/2}|$ is determined by the major half-axis $a > 0$ alone. In order that the corresponding $x(t)$ be periodic, it is necessary and sufficient that T_0 be commensurable with the period of M; i.e., $a^{3/2} = m/k$ with natural m and integer k, which is chosen positive respectively negative, if $z(t)$ is direct respectively retrograde. The initial conditions

$$x(0) = a(1+\epsilon) = \xi^*$$
$$\dot{x}(0) = i(c^* - \xi^{*2})/\xi^* = i\eta^* \qquad c^{*2} = a(1-\epsilon^2) \qquad [2]$$

yield such a solution of [1] with $\mu = 0$ describing motion along a rotating ellipse of eccentricity $\epsilon (0 < \epsilon < 1)$ and period T_0. The synodical period of this solution is $T^* = 2\pi m = |k|T_0$, and it closes after $k - m$ revolutions around the origin, which is a focus of $z(t)$. This solution is denoted by $x^*(t)$ from here on.

Now the following result can be stated: There exist periodic solutions $x(t)$ of [1] for small $\mu > 0$ which are near the generating solutions $x^*(t)$ belonging to arbitrary k, m, and properly restricted ϵ. These solutions and their synodical periods T are continuous in μ and transfer into $x^*(t)$ for $\mu = 0$.

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The restrictions upon ϵ are as follows: For given $a = (m/k)^{2/3}$, there are at most finitely many ϵ in $0 < \epsilon < 1$ with $\epsilon = (1 - a^{-3})^{1/2}$ or $x^*(t) = 1$ at least once in $0 \leqq t \leqq T^*$. In the latter case, P collides with M. Such ϵ have to be omitted from (0,1).

The following steps lead to a proof of these statements:

1) The special solution curve $x = x^*(t)$, $(0 \leqq t \leqq T^*)$ intersects the real axis perpendicularly twice, namely at $t = 0$ and $t = \frac{1}{2}T^*$. But every solution curve of [1] with this property is symmetric over the real axis and thus closed. Therefore, try to determine the initially real position and pure imaginary velocity

$$x(0) = \bar{x}(0) = \xi \qquad \dot{x}(0) = -\dot{\bar{x}}(0) = i\eta \qquad [3]$$

such that the resulting solution of [1] satisfies

$$x(\tfrac{1}{2}T) = \bar{x}(\tfrac{1}{2}T) \qquad \dot{x}(\tfrac{1}{2}T) = -\dot{\bar{x}}(\tfrac{1}{2}T) \qquad [4]$$

for some $T > 0$. Then $x(t)$ will be periodic with T. (A bar denotes the conjugate complex number.) Clearly, the functions in [4] depend also upon ξ, η, and μ. Since $x^*(t)$ satisfies the periodicity conditions [3] and [4] with $T = T^*$ and $\mu = 0$, there is hope that these can be satisfied also for small $\mu > 0$ and appropriate ξ, η near ξ^*, η^* as given in [2]. This now holds indeed, but it is not obvious at all, as the detailed mathematical and historical exposition to be published elsewhere shows.

2) Since [4] represents an implicit system of equations for the unknown functions $T(\mu)$, $\xi(\mu)$, $\eta(\mu)$, its solvability is essentially determined by the rank of certain associated functional matrices, besides the usually satisfied differentiability requirements, which demand particular care here, however. Neglecting this care, only the particular functional determinant

$$[\partial(x_2, \dot{x}_1)/\partial(t, \eta)] = D(t, \xi, \eta, \mu) = D \qquad [5]$$

is considered here. Its value D^* at $t = \frac{1}{2}T^* = m\pi$, $\xi = \xi^*$, $\eta = \eta^*$, $\mu = 0$ is

$$D^* = 3m\pi\eta^*\epsilon[(-1)^k - \epsilon]/(c^* - a^{-1})c^{*2} \qquad [6]$$

which is finite and not zero, if $0 < \epsilon < 1$, $\eta^* \neq 0$, and $ac^* \neq 1$. Here η^* is given by [2], and the last condition is equivalent with $\epsilon \neq (1 - a^{-3})^{1/2}$ for positive c^*. Under these conditions, then, [4] can be solved for T and η in terms of ξ, μ, and especially with $\xi = \xi^*$ and small $\mu > 0$, since [4] is equivalent with $x_2 = 0$, $\dot{x}_1 = 0$ at $t = \frac{1}{2}T$.

3) The calculation of D^* requires the explicit knowledge of the dependence of the solutions of [1] for $\mu = 0$ upon time and their initial values. This dependence, though principally known for any rotating Keplerian motion, is rather complicated and not suited at all to derive [6]. The following new variables are appropriate for this task:

$$F = \text{arctg}\, x_2/x_1 \qquad H = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 - r^2) - r^{-1}$$

$$U = x_1/r - c(\dot{x}_2 + x_1) \qquad V = x_2/r + c(\dot{x}_1 - x_2) \qquad [7]$$

where $r = (x_1^2 + x_2^2)^{1/2}$, and $c = x_1\dot{x}_2 - x_2\dot{x}_1 + r^2$ is the constant of area of the rotating Keplerian motion. The dynamical meaning of these variables is clear from the usual Keplerian motion; H is the Jacobi integral, for instance. Let h denote the value of H on any solution of [1] with $\mu = 0$. Then D in [5] can be expressed as

$$D = \frac{\partial(x_2, \dot{x}_1)}{\partial(F, V)} \cdot \frac{\partial(F, V)}{\partial(t, h)} \cdot \frac{\partial h}{\partial \eta} \qquad [8]$$

and this leads to a rather convenient calculation of D^* after transferring [1] for $\mu = 0$ into the new variables and making use of

$$U^2 + V^2 = 1 + 2Hc^2 + 2c^3$$

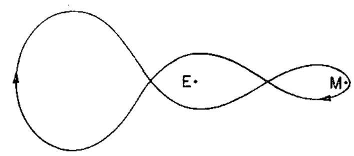

Fig. 1 Closed path of P in rotating co-system with $m = 1$, $k = 2$

as implied by [7], which will not be demonstrated here.

This completes the brief outline of the existence proof. A finer consideration shows a.o. that the only requirement upon ϵ besides $\epsilon \neq (1 - a^{-3})^{1/2}$ is expressed by $x^*(t) \neq 1$ in $0 \leqq t \leqq T^*$. It is conceivable that for fixed a there will be infinitely many ϵ for which $x^*(t)$ collides with M; these ϵ could accumulate at 0 or 1, for instance. But the commensurability of the periods of P and M for $\mu = 0$ actually allows at most finitely many such ϵ, and the proof shows that for every closed ϵ interval I not containing such exceptional values and for fixed sufficiently small $\mu > 0$, there exists a family of periodic solutions of [1] depending continuously upon the parameter ϵ in I, which transfers into the corresponding family of solutions $x^*(t)$ for $\mu \to 0$, $a = (m/k)^{2/3}$ being fixed arbitrarily.

The initial values for these solutions follow by solving [4] for T, η in terms of μ and $\xi = \xi^*$, which according to [5] and [6] is possible under the foregoing restrictions upon ϵ, if also $\eta^* \neq 0$.

The actual calculation of these values requires the knowledge of the general solution of [1] for small μ and initial values [3] near [2] at $t = \frac{1}{2}T$ with T near T^*. This can always be achieved at least by high-speed computers carrying out a numerical integration of [1]. Thereby, ξ can be chosen arbitrary as long as ϵ resulting from $\xi = a(1 + \epsilon) = \xi^*$ is between 0 and 1, the corresponding η^* in [2] does not vanish, and ϵ is not one of the exceptional values defined previously. Thus the initial position ξ can be chosen independent of μ, and only the initial velocity η and the period T will depend upon μ and ξ, assuming always that $a = (m/k)^{2/3}$ is given. However, this leads to a difficulty, when η approaches zero, which happens for a value of $\xi = \xi^*$ near that value, which produces $\eta^* = 0$ in [2]. Then D in [8] will be zero, but this formula shows also that h or η^2 should be used instead of η to solve the appropriately modified periodicity conditions [4]. Then η follows uniquely for fixed $\mu > 0$ from h by observing that η is a strictly monotonic function of ξ near $\eta = 0$, as can be shown. This situation can occur only when $2a \leqq 1$.

The solution curves of [1] become interesting for astronautical applications if they pass at prescribed distances near E and M. By selecting $a = (m/k)^{2/3}$ and ϵ according to

$$\tfrac{1}{2}\delta < a(1 - \epsilon) < 2\delta \qquad 1 < a(1 + \epsilon) < 1 + \delta \qquad [9]$$

for instance, with suitable small $\delta > 0$, this can be achieved for $x^*(t)$ and thus also for the corresponding $x(t)$, which is reasonably near $x^*(t)$ still for $\mu = \frac{1}{80} \approx$ the value of μ for the case E = Earth, M = moon. For example, the choice

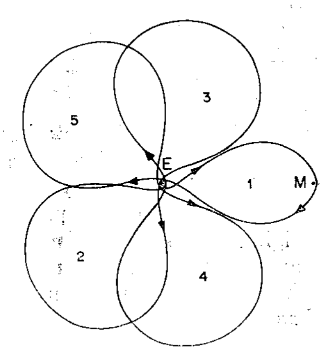

Fig. 2 Closed path of P in rotating co-system with $m = 2$, $k = 5$; numbers show loops made by P in succession

$m = 2$, $k = +5$ gives for $a^{-1} - 1 < \epsilon < 1$ a highly interesting family of solutions passing close to E and M, with no exceptional ϵ values in this range. Solution curves passing near E and M can also be generated, starting from $\xi^* = a(1 - \epsilon)$ and η^* by [2], when $k - m$ is odd. The corresponding solutions $x^*(t)$ are obtained from the previous ones using [2] by rotation through $-\pi(1 - m/k)\cdot \text{sign}\, k$, but for $\mu > 0$ this relation will be distorted. Such solutions can be made to come close to M at a later time (not at apogee). The choice $m = 1$, $k = +2$ yields promising flight paths for solar probes returning to Earth after a little less than 1 yr.

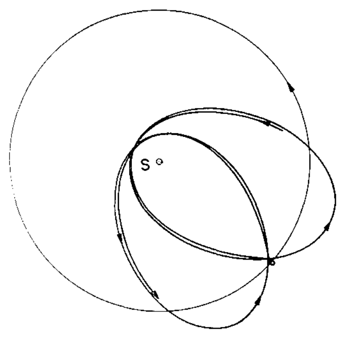

Fig. 3 Circular paths of E, M, and the path from Fig. 1 in inertial co-system nearly from $t = 0$ to $t = 2T$; capture and rejection of P by M

The perturbations encountered on actual flights, for which the restricted three-body problem is only an idealization, could be controlled essentially by employing continuous electric or other propulsions. Practical uses of certain of the periodic flight paths, whose existence has been established mathematically, can be conceived readily. This paper mentions only a radiation protected heavy Earth-moon ferry vehicle perpetually on such a path, to be supplied or boarded by passengers after rendezvous with much smaller crafts near E or M, which themselves could return to their bases, for instance. This would relieve the necessity to launch a heavy space ship from Earth or to assemble it in Earth orbit for every new flight to the moon and landing back on Earth. Thus tourist trips to the moon and back become practical and more economical.

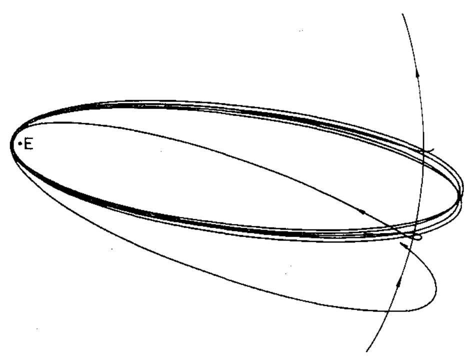

Fig. 4 Circular arc of M and the path from Fig. 2 as seen from E under space-fixed orientation from $t = 0$ to nearly $t = \frac{6}{5}T$; capture and rejection of P by M

A few illustrations are included which are calculated for the case $\mu = \frac{1}{82}$. Figs. 1 and 2 show synodically closed solution curves of [1] for $m = 1$, $k = 2$ and $m = 2$, $k = 5$, respectively, in the rotating co-system. Fig. 3 shows for the first case the circular paths of E and M and the path of P in an inertial co-system with origin at S. Fig. 4 shows for the second case the paths of P and M in a co-system with origin at E and space-fixed orientation. Note the short time capture of P by M with subsequent rejection which happens after elapse of every T units of time, i.e., every time when P has completed nearly k Keplerian elliptic orbits with focus at E.

Radiation Environment Following a Nuclear Attack

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This article describes an analytical model that can predict postblast nuclear radiation levels following an air burst of any size and proximity. It is also an attempt to assemble in one place all the salient subexpressions and considerations that must be incorporated into a computer program for complete prosecution of the model's details. The initial condition assumed is that an overhead nuclear detonation, sufficiently high to preclude significant fallout, has just occurred. An oblate, spheroid-shaped cloud is rising, growing, and emitting gamma rays at an ever-decreasing rate. The radiation dose rate at any discrete time i and altitude j is then calculated by the expression

$$\dot{D}_{ij} = \sum_{k=H_{\text{bot}\,i}}^{H_{\text{top}\,i}} \sum_{l=0}^{r_{0i}} \sum_{m=1}^{4} 1.602 \times 10^{-8} F_{im} \times \left(\frac{\mu_a}{\rho}\right)_m B_{jklm} E_m S_{vi} \frac{\exp[-(\mu/\rho)_m \bar{\rho}_{ijkl} R_{ijkl}]}{4\pi (R_{ijkl})^2}$$

A sample case of a 20-MT burst is considered, and curves of dose rates vs altitudes at several discrete instants in time are generated.

THE effect that nuclear radiation may have on electronic devices is of crucial concern to designers of weapons systems. Several informative articles dealing with susceptibilities of components and circuits to radiation have appeared recently in unclassified literature (2,3),[3] but to incorporate these considerations intelligently into his system concept, the designer also needs to know the greatest amount of radiation the system might receive.

It is, of course, impossible to predict numbers, sizes, and proximities of nuclear detonations that might occur in the event of attack. However, one can hypothesize an array of this sort and obtain radiation environment curves, which will enable the designer to make definitive statements concerning component selection, "hold" times, etc., required for successful missions immediately following a nuclear attack.

By definition, ". . . a typical air burst takes place at such a height that the fireball, even at its maximum, is well above the surface of the earth" (1).

Description of Analytical Model

The initial condition assumed is that an overhead nuclear detonation sufficiently high to preclude significant fallout has just occurred. An oblate spheroid-shaped cloud is rising, growing, and emitting gamma rays at an ever-decreasing rate. Also, the system, in this case a missile, has been protected from the burst's primary radiation pulse and peak air overpressure. Such would be the case for a hardened underground silo (4). It is desired to find radiation levels as functions of times and altitudes following this detonation.

Define the following:

i = time after air burst, sec
j = distances of a fictitious aboveground radiation "detector," cm. At each discrete time i chosen following the burst, this detector will be moved from one

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