period-three window, corresponding to Figure 9.7(f). The period-three orbit then period-doubles as part of another period-doubling cascade. In the center of the figure a period doubling of a period-four attractor is followed by a period-halving bifurcation.

9.4 CHUA'S CIRCUIT

A rather simple electronic circuit became popular for the study of chaos during the 1980's (Matsumoto et al., 1985). It allows almost all of the dynamical behavior seen in computer simulations to be implemented in an electronics lab and viewed on an oscilloscope. As designed and popularized by L. Chua, an electronic engineering professor at the University of California at Berkeley, and the Japanese scientist T. Matsumoto, it is an RLC circuit with four linear elements (two capacitors, one resistor, and one inductor) and a nonlinear diode, which can be modeled by a system of three differential equations. The equations for Chua's circuit are

$$\begin{aligned} \dot{x} &= c_1(y - x - g(x)) \\ \dot{y} &= c_2(x - y + z) \\ \dot{z} &= -c_3 y, \end{aligned} \tag{9.6}$$

where $g(x) = m_1 x + \frac{m_0 - m_1}{2}(|x+1| - |x-1|)$.

Another way to write $g(x)$, which is perhaps more informative, is

$$g(x) = \begin{cases} m_1 x + m_1 - m_0 & \text{if } x \le -1 \\ m_0 x & \text{if } -1 \le x \le 1 \\ m_1 x + m_0 - m_1 & \text{if } 1 \le x \end{cases} \tag{9.7}$$

The function $g(x)$, whose three linear sections represent the three different voltage-current regimes of the diode, is sketched in Figure 9.9. This piecewise linear function is the only nonlinearity in the circuit and in the simulation equations. We will always use slope parameters satisfying $m_0 < -1 < m_1$, as drawn in the figure.

Typical orbits for the Chua circuit equations are plotted in Figure 9.10. All parameters except one are fixed and c_3 is varied. Two periodic orbits are created simultaneously in a Hopf bifurcation, which we will study in Chapter 11. They begin a period-doubling cascade, as shown in Figure 9.10(b)-(c) and reach chaos in Figure 9.10(d). The chaotic attractors fill out and approach one another as c_3 is varied, eventually merging in a crisis, one of the topics of Chapter 10.

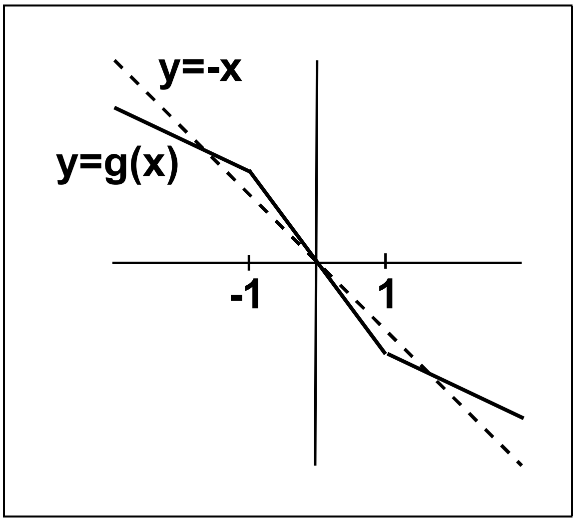

Figure 9.9 The piecewise linear $g(x)$ for the Chua circuit.
Equilibria correspond to intersections of the graph with the dotted line $y = -x$.

Color Plate 17 shows a circuit diagram of Chua's circuit. Color Plate 18 shows the computer-generated attractor for parameter settings $c_1 = 15.6, c_2 = 1$, $c_3 = 25.58$, $m_0 = -8/7$, $m_1 = -5/7$. Color Plate 19 shows a projection of the experimental circuit attractor in the voltage-current plane, and Color Plate 20 shows an oscilloscope trace of the voltage time series.

9.5 FORCED OSCILLATORS

One way to produce chaos in a system of differential equations is to apply periodic forcing to a nonlinear oscillator. We saw this first for the pendulum equation in Chapter 2. Adding damping and periodic forcing to the pendulum equation

$$\ddot{\theta} + \sin\theta = 0$$

produces

$$\ddot{x} + c\dot{\theta} + \sin\theta = \rho\cos t,$$

which has apparently chaotic behavior for many parameter settings.

A second interesting example is the double-well Duffing equation from Chapter 7:

$$\ddot{x} - x + x^3 = 0. \tag{9.8}$$

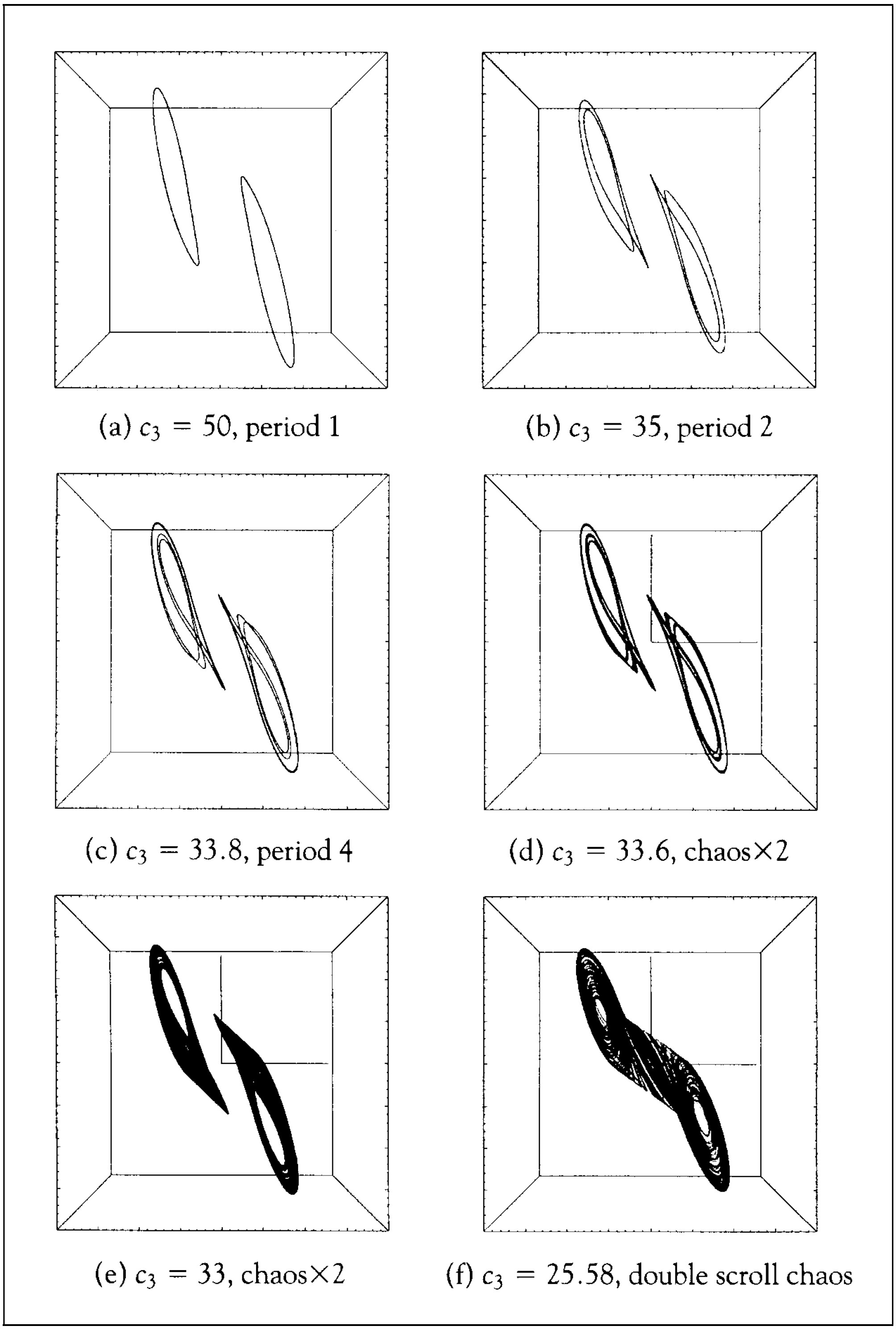

Figure 9.10 Chua circuit attracting sets.
Fixed parameters are $c_1 = 15.6$, $c_2 = 1$, $m_0 = -8/7$, $m_1 = -5/7$. The attracting set changes as parameter c_3 changes. (a) $c_3 = 50$, two periodic orbits. (b) $c_3 = 35$, the orbits have "period-doubled". (c) $c_3 = 33.8$, another doubling of the period. (d) $c_3 = 33.6$, a pair of chaotic attracting orbits. (e) $c_3 = 33$, the chaotic attractors fatten and move toward one another. (f) $c_3 = 25.58$, a "double scroll" chaotic attractor. This attractor is shown in color in Color Plate 18.