Beyond Brownian Motion

Fractal generalizations of Brownian motion have proven to be a rich field in probability theory, statistical physics and chaotic dynamics.

Joseph Klafter, Michael F. Shlesinger, Gert Zumofen

Newtonian physics began with an attempt to make precise predictions about natural phenomena, predictions that could be accurately checked by observation and experiment. The goal was to understand nature as a deterministic, "clockwork" universe. The application of probability distributions to physics developed much more slowly. Early uses of probability arguments focused on distributions with well-defined means and variances. The prime example was the Gaussian law of errors, in which the mean traditionally represented the most probable value from a series of repeated measurements of a fixed quantity, and the variance was related to the uncertainty of those measurements.

But when we come to the Maxwell–Boltzmann distribution or the Planck distribution, the whole distribution has physical meaning. Being away from the mean is no longer an error, and a large variance is no longer an indicator of poor measurement accuracy. In fact the whole distribution is the prediction. That was a major conceptual advance.

In this article we take that idea to its extreme limit and investigate probability distributions, called Lévy distributions, with infinite variances, and sometimes even infinite means. These distributions are intimately connected with fractal random-walk trajectories, called Lévy flights, that are composed of self-similar jumps.[1] Lévy flights are as widely applied in nonlinear, fractal, chaotic and turbulent systems as Brownian motion is in simpler systems.

Brownian motion

The observation of Brownian motion was first reported in 1785, by the Dutch physician Jan Ingenhausz. He was looking at powdered charcoal on an alcohol surface. But the phenomenon was later named for Robert Brown, who published in 1828 his investigation of the movements of fine particles, including pollen, dust and soot, on a water surface. Albert Einstein eventually explained Brownian motion in 1905, his *annus mirabilis*, in terms of random thermal motions of fluid molecules striking the microscopic particle and causing it to undergo a random walk.

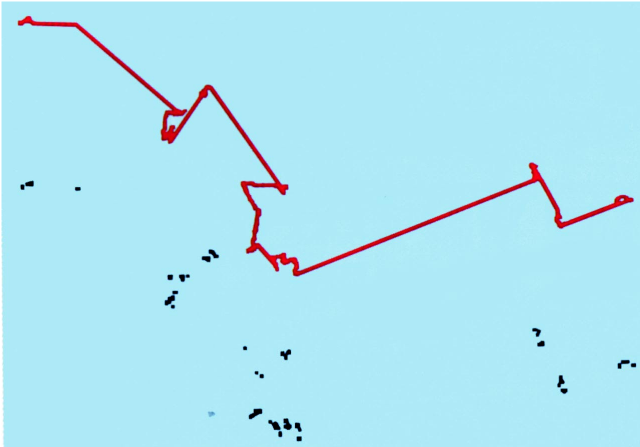
Einstein's famous paper was entitled "Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen," that is to say, "On the motion, required by the molecular-kinetic theory of heat, of particles suspended in fluids at rest." Einstein was primarily exploiting molecular motion to derive an equation with which one could measure Avogadro's number. Apparently he had never actually seen Brown's original papers, which were published in the *Philosophical Magazine*. "It is possible," wrote Einstein, "that the motions discussed here are identical with the so-called Brownian molecular motion. But the references accessible to me on the latter subject are so imprecise that I could not form an opinion about that."[2] Einstein's prediction for the mean squared displacement of the random walk of the Brownian particle was a linear growth with time multiplied by a factor that involved Avogadro's number. This result was promptly used by Jean Perrin to measure Avogadro's number and thus bolster the case for the existence of atoms. That work won Perrin the 1926 Nobel Prize in Physics.

Less well known is the fact that Louis Bachelier, a student of Henri Poincaré, developed a theory of Brownian motion in his 1900 thesis. Because Bachelier's work was in the context of stock market fluctuations, it did not attract the attention of physicists. He introduced what is today known as the Chapman–Kolmogorov chain equation. Having derived a diffusion equation for random processes, he pointed out that probability could diffuse in the same manner as heat. (See PHYSICS TODAY, May 1995, page 55.)

Bachelier's work did not lead to any direct advances in the physics of Brownian motion. In the economic context of his work there was no friction, no place for Stokes' law nor any appearance of Avogadro's number. But Einstein did employ all of these ingredients in his theory. Perhaps that illustrates the difference between a mathematical approach and one laden with physical insight.

The mathematics of Brownian motion is actually deep and subtle. Bachelier erred in defining a constant velocity v for a Brownian trajectory by taking the limit x/t for small displacement x and time interval t. The proper limit involves forming the diffusion constant $D = x^2/t$ as both x and t go to zero. In other words, because the random-walk displacement in Brownian motion grows only as the square root of time, velocity scales like $t^{-1/2}$ and there-

JOSEPH KLAFTER is a professor in the School of Chemistry at Tel Aviv University in Israel. MICHAEL SHLESINGER is chief scientist for nonlinear science at the Office of Naval Research, in Arlington, Virginia. GERT ZUMOFEN is a lecturer and senior research scientist at the Laboratory for Physical Chemistry of the Eidgenössische Technische Hochschule in Zurich, Switzerland.



LÉVY FLIGHT RANDOM WALK of 1000 steps in two dimensions. For clarity, a dot is shown directly below each turning point. Limited resolution of this plot makes it difficult to discern indivdual turning points. They tend to cluster in self-similar patterns characteristic of fractals. Occasional long flight segments initiate new clusters. The longer the step, the less likely is its occurence. But Lévy flights have no characterisic length. (See the box on page 35.) FIGURE 1

fore is not defined in the small-t limit.

A Brownian trajectory does not possess a well-defined derivative at any point. Norbert Wiener developed a mathematical measure theory to handle this complication. He proved that the Brownian trajectory is continuous, but of infinite length between any two points. The Brownian trajectory wiggles so much that it is actually two-dimensional. Therefore an area measure is more appropriate than a length measure. Lévy flights have a dimension somewhere between zero and two.

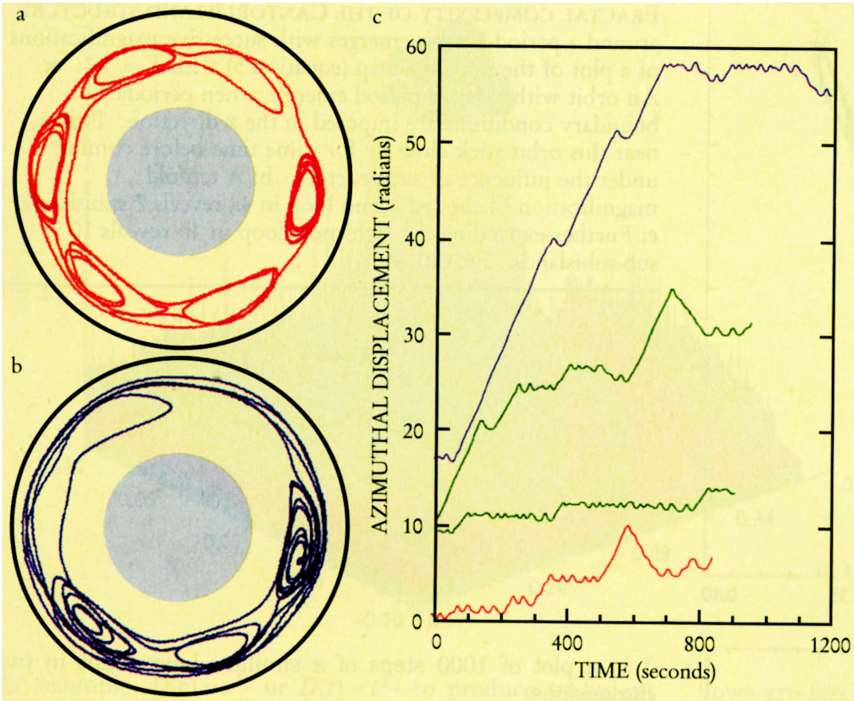
Among the methods that have been explored to go beyond Einstein's Brownian motion is fractal Brownian motion,[1] which incorporates self-similarity and produces a trajectory with a mean squared displacement that grows with time raised to a power between zero and two. It is a continuous process without identifiable jumps, and it has been used to model phenomena as diverse as price fluctuations and the water level of the Nile. Two other ventures beyond traditional Brownian motion are fractal distributions of waiting times between random-walk steps and random walks on fractal structures such as percolation lattices. These two examples lead to slower-than-linear growth of the mean squared displacement with time. But in this article we focus on random walk processes that produce faster-than-linear growth of the mean squared displacement.

Another approach to generalizing Brownian motion is to view it as a special member of the class of Lévy-flight random walks. Here we explore applications of Lévy flights in physics.[3] The interest in this area has grown with the advent of personal computers and the realization that Lévy flights can be created and analyzed experimentally. The concept of Lévy flight can usefully be applied to a wide range of physics issues, including chaotic phase diffusion in Josephson junctions,[4] turbulent diffusion,[5,6] micelle dynamics,[7] vortex dynamics,[8] anomalous diffusion in rotating flows,[9] molecular spectral fluctuations,[10] trajectories in nonlinear Hamiltonian systems,[11–17] molecular diffusion at liquid–solid interfaces,[18], transport in turbulent plasma,[19] sharpening of blurred images[20] and negative Hall resistance in anti-dot lattices.[21] In these complex systems, Lévy flights seem to be as prevalent as diffusion is in simpler systems.

Lévy flights

The basic idea of Brownian motion is that of a random walk, and the basic result is a Gaussian probability distribution for the position of the random walker after a time t, with the variance (square of the standard deviation) proportional to t. Consider an N-step random walk in one dimension, with each step of random length x governed by the same probability distribution $p(x)$, with zero mean.

ROTATING-ANNULUS APPARATUS with which Harry Swinney's group at the University of Texas investigate Lévy flights and anomalous transport in liquids. The flow is established by pumping liquid into the rotating annulus through holes in its bottom. Showing off the equipment are (clockwise) graduate student Eric Weeks and postdoc Jeff Urbach. FIGURE 2



Observing fluid flow with tracer particles in the rotating-annulus apparatus shown in figure 2 reveals **a:** a state of six stable vortices that frequently trap the particle as it circles around the annulus. **b:** In another observed flow state, the particle spends less time trapped in vortices. **c:** Plotting azimuthal displacement of several particle trajectories against time shows that the **b** trajectory (purple), exhibiting long flights between vortex captures, gets around the annulus much faster than the **a** trajectory (red). (Courtesy of H. Swinney.) **Figure 3**

The French mathematician Paul Lévy (1886–1971) posed the question: When does the probability $P_N(X)$ for the sum of N steps $X = X_1 + X_2 + \ldots + \backslash X_N$ have the same distribution $p(x)$ (up to a scale factor) as the individual steps? This is basically the question of fractals, of when does the whole look like its parts. The standard answer is that $p(x)$ should be a Gaussian, because a sum of N Gaussians is again a Gaussian, but with N times the variance of the original. But Lévy proved that there exist other solutions to his question. All the other solutions, however, involve random variables with infinite variances.

Augustine Cauchy, in 1853, was the first to realize that other solutions to the N-step addition of random variables existed. He found the form for the probability when it is transformed from real x space to Fourier k space:

$$\tilde{p}_N(k) = exp(-N|k|^{\beta}) \tag{1}$$

Cauchy's famous example is the case $\beta = 1$, which, when transformed back into x space, has the form

$$p_N(x) = \frac{1}{\pi N} N \frac{1}{(x/N)} + \; = \frac{1}{N} p_1(x/N) \tag{2}$$

This is now known as the Cauchy distribution. It shows explicitly the connection between a one-step and an N-step distribution. Exhibiting this scaling is more important than trying to describe the Cauchy distribution in terms of some pseudo-variance, as if it were a Gaussian.

Lévy showed that β in equation 1 must lie between 0 and 2 if $p(x)$ is to be nonnegative for all x, which is required for a probability. Nowadays the probabilities represented in equation 1 are named after Lévy. When the absolute value of x is large, $p(x)$ is approximately $|x|^{-1-\beta}$, which implies that the second moment of $p(x)$ is infinite when β is less than 2. This means that there is no characteristic size for the random walk jumps, *except* in the Gaussian case of $\beta = 2$. It is just this absence of a characteristic scale that makes Lévy random walks (flights) scale-invariant fractals. The box at right makes this more evident with an illustrative one-dimensional Lévy-flight probability function, and figure 1 is a plot of 1000 steps of a similar Lévy flight in two dimensions.

Despite the beauty of Lévy flights, the subject has

A simple Lévy-flight random walk

Consider a one-dimensional random walk designed to illustrate the self similarity of Lévy flights. Start with the discrete jump-probability distribution

$$p(x) = \frac{\lambda - 1}{2\lambda} \sum_{j=0}^{\infty} \lambda^{-j} [\delta(x, +b^j) + \delta(x, -b^j)]$$

where $b > \lambda > 1$ are parameters that characterize the distribution, and $\delta(x, y)$ is the Kronecker delta, which equals 1 when $x = y$ and otherwise vanishes.

This distribution function allows jumps of length 1, b, b^2, b^3 . . . . But note that whenever the length of the jump increases by an order of magnitude (in base b), its probability of occurring decreases by an order of magnitude (in base λ). Typically one gets a cluster of λ jumps roughly of length 1 before there is a jump of length b. About λ such clusters separated by lengths of order b are formed before one sees a jump of order b^2. And so it goes on, forming a hierarchy of clusters within clusters.

Figure 1 shows a two-dimensional example of this kind of random walk.

The Fourier transform of $p(x)$ is

$$\tilde{p}(k) = \frac{\lambda - 1}{2\lambda} \sum_{j=0}^{\infty} \lambda^{-j} \cos(b^j k)$$

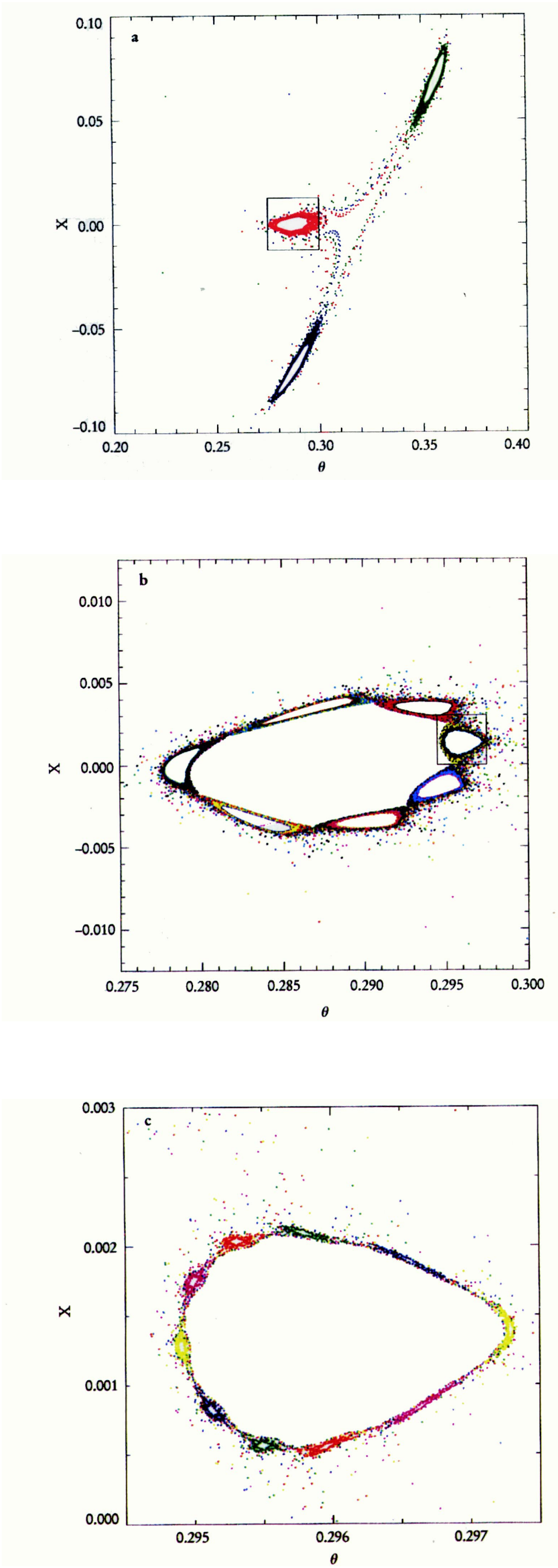
which is the famous self-similar Weierstrass function. So we could call this the Weierstrass random walk. The self-similarity of the Weierstrass curve appears explicitly through the equation

$$\tilde{p}(k) = \frac{1}{\lambda} \tilde{p}(bk) + \frac{\lambda - 1}{\lambda} \cos(k)$$

which has a small-k solution

$$\tilde{p}(k) \approx \exp(-|k|^{\beta})$$

with $\beta = \log \lambda / \log(b)$. That is the form of the Lévy probability in equation 1 of the main text.



FRACTAL COMPLEXITY OF THE CANTORI-ISLAND STRUCTURE around a period-3 orbit emerges with successive magnifications of a plot of the standard map (equation 5) with $K = 1.1$. **a:** An orbit with a 3-step period emerges when periodic boundary conditions are imposed in the x direction. Points near this orbit stick close by for some time before coming under the influence of substructure. **b:** A tenfold magnification of the red island loop in 4a reveals 7 subislands. **c:** Further expanding the rightmost loop in 4b reveals 10 sub-subislands. FIGURE 4

been largely ignored in the physics literature, mostly because the distributions have infinite moments. The first point we wish to make is that one should focus on the scaling properties of Lévy flights rather than on the infinite moments. The divergence of the moments can be tamed by associating a velocity with each flight segment. One then asks how far a Lévy walk has wandered from its starting point in time t, rather than what is the mean squared length of a completed jump. The answer to the first question will be a well-behaved time-dependent moment of the probability distribution, while the answer to the second is infinity. Specifically, a Lévy random walker moving with a velocity v, but with an infinite mean displacement per jump, can have a mean squared displacement from the origin that varies as v^2t^2. (See the box on page 37.) Even faster motion is possible if the walker accelerates, as we shall see when we come to the phenomenon of turbulent diffusion.

Lévy walks in turbulence

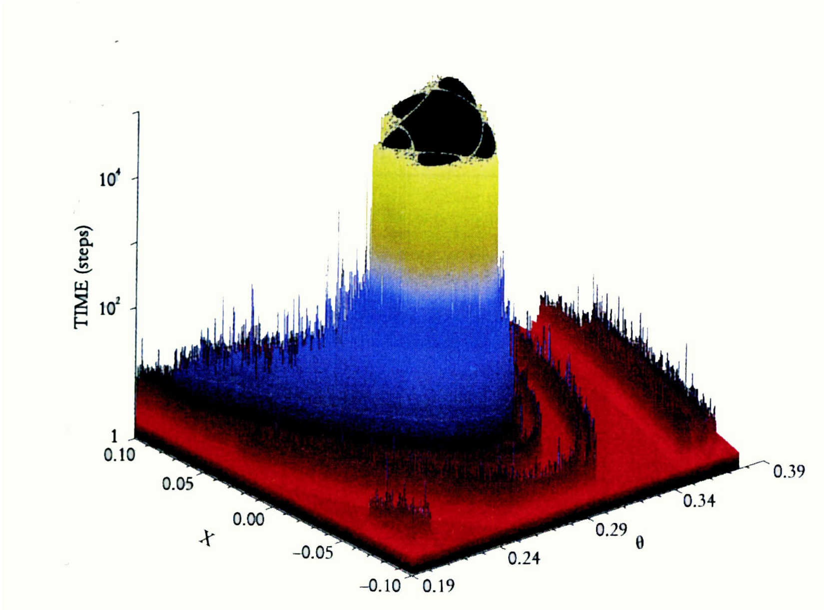
To employ Lévy flight for trajectories, one introduces velocity through a coupled spatial–temporal probability density $\psi(r, t)$ for a random walker to undergo a displacement r in a time t. We write

$$\psi(r, t) = \psi(t \mid r)\, p(r) \qquad (3)$$

The factor $p(r)$ is just the probability function, discussed above, for a single jump. The $\psi(t \mid r)$ factor is the probability density that the jump takes a time t, given that its length is r. Let us, for simplicity, make $\psi(r, t)$ the Dirac delta function $\delta(t - |r|/v(r))$, which ensures that $r = vt$. Such random walks, with explicit velocities, visit all points of the jump on the path between 0 and r. They are called Lévy walks as distinguished from Lévy flights, which visit only the two endpoints of a jump.

The velocity v need not be a constant; it can depend on the size of the jump. A most interesting case is turbulent diffusion. In 1926 Lewis Fry Richardson published his discovery that the mean square of the separation r between two particles in a turbulent flow grows like t^3. Dimensional analysis of Brownian motion tells us that $\langle r^2(t) \rangle = Dt$. This means that the diffusion constant D can be endowed with a specific space or time dependence—for example, $D(r) \approx r^{4/3}$ or $D(t) \approx t^2$—to produce turbulent diffusion.

Richardson chose the $D(r) \approx r^{4/3}$ route. Note that a



DISTRUBUTION OF TIME it takes a standard-map trajectory (with K = 1.03) to leave the Cantori structure around a period-5 orbit, plotted as a function of the trajectory's initial $(x, \theta\Pi$ position. The time distribution exhibits a complex fractal hierarchy of sticking regions. The black areas mark islands of stability inside the period-5 orbit, from which a trajectory will never leave. **FIGURE 5**

diffusion constant has dimensions of $[rv]$. Therefore Richardson's 4/3 power law dimensionally implies that $v^2(r)$ scales like $r^{2/3}$. Then the Fourier transform, $v^2(k)$, must scale like $k^{-5/3}$.

This last result, first stated in 1941, is Kolmogorov's well-known inertial-range turbulence spectrum. Although it looks as if Richardson could have predicted the Kolmogorov spectrum, the two scaling laws do not necessarily imply each other. Only if the Kolmogorov scaling $(v(r) \approx r^{1/3})$ is combined with Lévy flights $(p(r) \approx r^{1-\beta})$, so that the mean absolute value of r is infinite, does one recover Richardson's result: $< r^2(t) > = t^3$. If $p(r)$ decays fast enough so that all of its moments are finite, then one recovers the standard Brownian law, even with Kolmogorov scaling. The main point here is that the Lévy walk with Kolmogorov scaling describes aspects of turbulent diffusion. The box on this page shows an example of the possible scaling laws for the constant-velocity Lévy random walks found in dynamical systems.

An important feature of the Lévy-walk approach to turbulent diffusion is that it provides a method for simulating trajectories of turbulent particles. Fernand Hayot6 has used this procedure to describe turbulent flow in pipes by means of Lévy-walk trajectories with Kolmogorov velocity scaling in a lattice gas simulation.

Two-dimensional fluid flow

In two-dimensional computer simulations of fluid flow, James Viecelli[8] has found Lévy walks and enhanced diffusion with mean squared displacement scaling like t^γ with $\gamma = 1.67$ for point vortices all spinning in the same sense at high temperatures. At low temperatures the vortices form a rotating triangular lattice. As the temperature is raised, the motion of the vortices eventually becomes turbulent. The vortices cluster and rotate around a common center, but now with Lévy-walk paths. If both clockwise and counterclockwise vortices are present, the scaling exponent γ becomes 2.6. Oppositely spinning vortices pair and the pairs move in straight-line Lévy walks, changing direction when they collide with other pairs.

These results suggest that two-dimensional rotating flows are fertile territory for seeking anomalous diffusion. They might serve to approximate large-scale global atmospheric and oceanic flows. We predict that Lévy walks will become increasingly important for understanding global environmental questions of transport and mixing in the atmosphere and the oceans.

Harry Swinney and coworkers at the University of Texas have been investigating quasi-two-dimensional fluid flow in a rotating laboratory vessel.[9] (See figure 2.) In their experiment, a fluid-filled annulus rotates as a rigid body. The flow is established by pumping fluid through holes in the bottom of the annulus. In this nearly

Mean squared displacement

The mean squared displacement in simple Brownian motion in one dimension has a linear dependence on time

$$< x^2(t) > \approx 2Dt$$

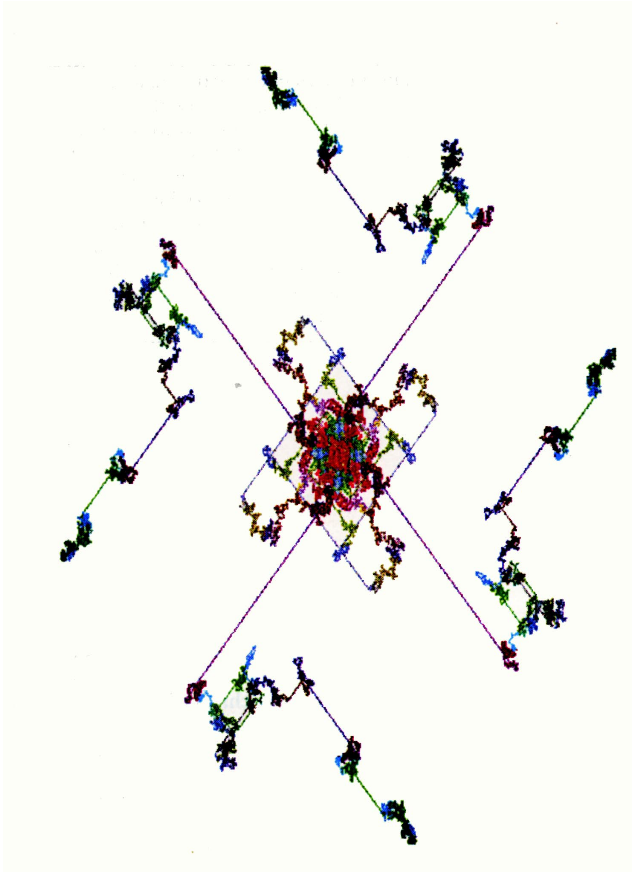
where D is the diffusion constant and $< x^2(t) >$ is the second moment of the Gaussian distribution that governs the probability of being at site x at time t. The Gaussian is the hallmark of Brownian motion. In the case of Lévy flights, by contrast, the mean squared displacement diverges. Therefore we consider the more complicated random walk described by the probability distribution of equation 3, which accounts for the velocity of the Lévy walk. For the Kolmogorov turbulence case, the velocity depended on the jump size. But here we consider the simpler constant-velocity case with

$$\psi(r,t) \approx 1(1+t)^{1+\beta}$$

being the probability density for time spent in a flight segment. For various values of β, that leads to the following different time dependences of the mean square displacement:

$$< r^2(t) > \approx \begin{cases} t^2 \; for \; 0 < \beta < 1 \\ t^2 / \ln t \; for \; \beta = 1 \\ t^{3-\beta} \; for \; 1 < \beta < 2 \\ t \ln t \; for \; \beta = 2 \\ t \; for \; for \; \beta > 2 \end{cases}$$

The enhanced diffusion observed in various dynamical systems, such as the standard map (equation 5 in the main text), corresponds to the case $1 < \beta < 2$. Similar results, with time dependences ranging from t^3 down to t, are found in turbulent diffusion.



LÉVY FLIGHT BEHAVIOR is seen in the trajectory of a computer-generated Zaslavsky map with fourfold symmetry.[13] The random walk starts near the origin and the color is changed every 1000 steps. **FIGURE 6**

two-dimensional flow, neutrally buoyant tracer particles can be carried along in Lévy walks even though the velocity field is laminar. In fact, Swinney and company were able to make direct measurements of Lévy walks and enhanced diffusion in this experimental system. (See figure 3.) An instability of the axisymmetrically pumped fluid led to a stable chain of six vortices, as shown in figure 3a. Tracer particles were followed to produce probability distributions for the distance and time a particle can travel before it gets trapped in a vortex

We use n, the number of vortices passed by a tracer particle, as a convenient measure of the travel distance r. When a particle leaves a vortex and then passes n vortices in a counterclockwise (or clockwise) fashion before it is trapped again, we count that as a random walker jumping n units to the right (or left) at a constant velocity. In Lévy-walknotation,taking $\psi_{\text{walk}}(t\,|\,r) = \delta(t - |\,n\,|)$, the University of Texas experimenters found

$$p_{\text{walk}}(n) \approx |\,r|^{-2.3} = |\,n|^{1-\beta} \qquad (4)$$

and they found that the mean time the particle spends trapped in a vortex is finite. Under these conditions Lévy-walk theory implies that the variance $< r^2(t) > - < r(t) >^2$ should scale like $t^{3-\beta}$. (See the box on page 37.) That turns out to be quite consistent with the experimental data, and it gives us a beautiful example of enhanced diffusion.

Lévy walks in nonlinear dynamics: Maps

The Lévy-walk case of dynamic scale-invariant circular motion with constant speed was introduced in 1985 by Theo Geisel and coworkers at the University of Regensburg, in Germany.[4] Their investigations involved the study of chaotic phase diffusion in a Josephson junction by means of a dynamical map, that is to say, a simple rule for generating the nth discrete value of a dynamical variable from its predecessor. With a constant voltage across the junction, the phase rotates at a fixed rate, but it can change direction intermittently. N complete clockwise voltage-phase rotations correspond to a random walk with a jump of N units to the right. Analysis of experimental data led Geisel and his colleagues to a consideration of the nonlinear map $x_{t+1} = (1+\varepsilon)x_t + a x_t^z - 1$, with ε small. Averaging over initial conditions, they found that

$$< x^2(t) > \approx \begin{cases} t^2 \text{ for } z > 2 \\ t^\gamma \text{ for } 3/2 < z < 2 \\ t \ln t \text{ for } z = 3/2 \\ t \text{ for } 1 < z < 3/2 \end{cases}$$

where $\gamma = 3 - (z-1)^{-1}$. This scaling behavior of the mean square displacement corresponds to constant-velocity Lévy walk with a flight-time distribution between reversals given by $\psi(t) \approx t^{-1-\beta}$, where $\beta = 1/(z-1)$.

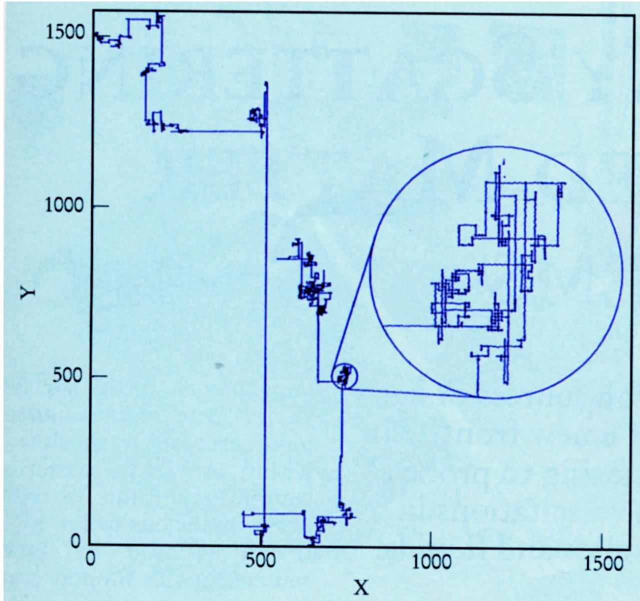
In the above example, one can go directly from z, the exponent of the map, to the temporal exponent $\beta = 1/(1-z)$ for the mean-square displacement. In most nonlinear dynamical systems, however, the connection between the equations and the kinetics is not obvious or simple. For example, the so-called standard map:

$$x_{n+1} = x_n + K\sin(2\pi\theta_n) \qquad (5)$$

$$\theta_{n+1} = \theta_n + x_{n+1}$$

has no explicit exponent in sight. But when one plots the progression of points x, θ, noninteger exponents abound in the description of the orbit kinetics, hinting at complex dynamics.

Plotting successive (x versus θ) points, at first one might see simple, nearly closed circular orbits, eventually giving way to thoroughly chaotic wandering. But before chaos sets in one can see orbits that exhibit hierarchical periodicities characteristic of the laminar segments of Lévy walks. The structures that emerge depend sensitively on the parameter K. With $K = 1.1$, for example, we see (in figure 4a) a period-3 orbit. The heirarchical, fractal nature of its three loops (called Cantori islands, because



Through an egg-crate potential in the x, y plane, a computer-generated 10^5-step random walk exhibits Lévy flight. The second moment of the step-length distribution is infinite. Between long flights, the trajectory gets trapped intermittently in individual potential wells. **Figure 7**

they form something like a Cantor set of tori) is revealed in the successively magnified plots of figures 4b and 4c. This fractal regime, describing complex kinetics with Lévy statistics, has been dubbed "strange kinetics."[16]

Figure 5 shows the distribution of the time a standard-map x-versus-θ trajectory spends around a period-5 Cantori orbit that emerges when $K = 1.03$. The time distribution exhibits a complex fractal hierarchy of sticking regions in the phase space. The probability densities for spending time in laminar states near period-3 and period-5 orbits have, respectively, the Lévy scaling behaviors $t^{-2.2}$ and $t^{-2.8}$.

Another example of Lévy walks in dynamical systems is provided by the orbits in a map devised by George Zaslavsky of the Courant Institute.[12,13] These walks form an intricate fractal web throughout a two-dimensional phase space. The Zaslavsky map has only simple trigonometric nonlinearities but, like the standard map, it requires noninteger exponents for the characterization of its trajectories. Figure 6 shows a Lévy-like trajectory in a 4-fold symmetric Zaslavsky map.

Potentials

One might expect that motion in a simple periodic potential should look simple. But closer inspection of the phase-space structure of trajectories in a simple two-dimensional egg-crate potential reveals a behavior as rich as what we get from the standard and Zaslavsky maps[11–13,15]. It turns out that Lévy walks emerge quite naturally in such potentials, and one finds enhanced diffusion. See figure 7. Although the trajectories possess long walks with Lévy scaling in the x and y directions of the egg-crate array, the motion is complicated by intermittent trapping in the potential wells. This trapping is related to the observed vortex sticking of tracer particles in the Swinney group's experiment. In neither case is the distribution of such "localizing" events broad enough to affect the predicted enhanced-diffusion exponent.

Lévy walks also appear for time-dependent potentials. Igor Aranson and colleagues at the Institute for Applied Physics in Nizhny Novgorod (formerly Gorky) in Russia, investigated a potential that varies sinusoidally in time and found that it produces Lévy walks similar to those one gets with static egg-crate potentials. All of these examples indicate that we are exploring an exciting, wide-open field when we venture beyond the traditional confines of Brownian motion.

We thank Harry Swinney for providing his experimental results and for sharing his insights into Lévy walks in fluids. We also thank George Zaslavsky for illuminating discussions and for suggesting the Lévy walk shown in figure 6.

References

1. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco (1982).
2. A. Einstein, Annalen der Physik 17, 549 (1905).
3. *Lévy Flights and Related Topics in Physics*, M. Shlesinger, G. Zaslavsky, U. Frisch, Eds., Springer, Berlin (1995).
4. T. Geisel, J. Nierwetberg, A. Zacherl, Phys. Rev. Lett. **54**, 616 (1985).
5. M. Shlesinger, B. West, J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987). M. Shlesinger, J. Klafter, Y. M. Wong, J. Stat. Phys. **27**, 499 (1982).
6. F. Hayot, Phys. Rev. A. **43**, 806 (1991).
7. A. Ott, J. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990). J. Bouchaud, A. Georges, Phys. Reports **195**, 127 (1980).
8. J. Viecelli, Phys. Fluids A **5**, 2484 (1993).
9. T. Solomon, E. Weeks, H. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
10. G. Zumofen, J. Klafter, Chem. Phys. Lett. **219** 303 (1994).
11. T. Geisel, A. Zacherl, G. Radons, Z. Phys. B. **71**, 117 (1988).
12. A. Chernikov, B. Petrovichev, A. Rogalsky, R. Sagdeev, G. Zaslavsky, Phys. Lett. A **144**, 127 (1990).
13. D. Chaikovsky, G. Zaslavsky, Chaos **1**, 463 (1991).
14. I. Aranson, M. Rabinovich, L. Tsimring, Phys. Lett. A **151**, 523 (1990).
15. J. Klafter, G. Zumofen, Phys. Rev. E **49**, 4873 (1994).
16. M. Shlesinger, G. Zaslavsky, J. Klafter, Nature **363**, 31 (1993).
17. R. Ramashanker, D. Berlin, J. Gollub, Phys. Fluids A **2**, 1955 (1980).
18. O. Baychuk, B. O'Shaughnessy, Phys. Rev. Lett **74**, 1795 (1985). S. Stapf, R. Kimmich, R. Seitter, Phys. Rev. Lett. **75**, 2855 (1995).
19. G. Zimbardo, P. Veltrei, G. Basile, S. Principato, Phys. Plasma **2**, 2653 (1995). R. Balescu, Phys. Rev. E **51**, 4807 (1995).
20. A. Carasso, in *Mathematical Methods in Medical Imaging II*, SPIE vol. 2035 (1993), p. 255.
21. R. Fleischmann, T. Geisel, R. Ketzmerick, Europhys. Lett. **25**, 219(1994).

■