We are starting to see a bewildering variety of models. An excellent survey of possible forms for laws of predation and population growth is given in "Stability and Complexity in Model Ecosystems" by R. M. May [37].* May neatly describes the possibilities as components of a "build-a-model" toy. (This text, which is extremely rewarding to read, also has illustrations showing the introduction of random processes into the models, and systems with time delay.) One particular model has been chosen here from those constructed by May. It can be expressed as

$$\left.\begin{aligned}\frac{dx}{dt} &= ax - bx^2 - \frac{cxy}{x + d},\\ \frac{dy}{dt} &= ey\left(1 - f\frac{y}{x}\right).\end{aligned}\right] \tag{4.13.1}$$

If we let

$$\xi = x/d, \quad \eta = fy/d, \quad \tau = at, \tag{4.13.2}$$

then the system can be written as

$$\left.\begin{aligned}\frac{d\xi}{d\tau} &= \xi - (bd/a)\xi^2 - (c/fa)\frac{\xi n}{1 + \xi},\\ \frac{d\eta}{d\tau} &= (e/a)\eta\left(1 - \frac{\eta}{\xi}\right).\end{aligned}\right] \tag{4.13.3}$$

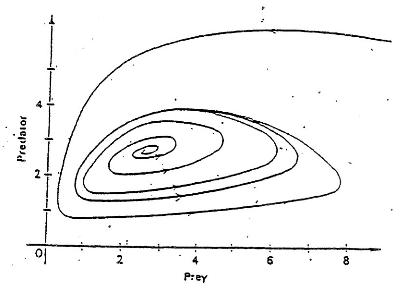

Figure 4.7 Two orbits for the system (4.13.3) using the numerical parameters of (4.13.4). Each approaches the limit cycle, which is represented by the solid curve.

A special case considered by May and illustrated in Figure 4.7 has the numerical values

$$(bd/a) = 0.1, \quad (c/fa) = 1, \quad (e/a) = 1/6. \tag{4.13.4}$$

This case has an equilibrium at $\xi = \eta = \frac{1}{2}(-1 + \sqrt{41})$ which is unstable. But the orbits do not depart from it indefinitely, and they all approach what is called a *limit cycle*. In fact, orbits starting from outside the limit cycle also approach it, so, for obvious reasons, it is called a *stable* limit cycle. An orbit on the limit cycle remains there forever.

This is the first time in these projects that we have met a limit cycle, but it will not be the last: limit cycles are an important feature of many nonlinear systems. There is a mathematical background for limit cycles based on the "Poincaré-Bendixson theorem" which you can look up in many references. Its application to predator–prey problems is due to Kolmogorov, and this is described by May. The theorem gives sufficient conditions for a system to have a stable equilibrium or a stable limit cycle.

For the project, start with the numerical values given in (4.13.4) and confirm the details of the figure and the properties of the limit cycle. According to May, the limit cycle is more likely if (1) a/e is large, i.e., the prey reproduce more vigorously than the predators, or (2) a/b is large, i.e., the influence of the environment on the maximum prey population is weak. So try reducing these quantities, and see if you can follow the system to one with a stable equilibrium. Alternatively, see what happens if the quantities are increased.