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CEAS Aeroacoustics Award Lecture

A discussion around adjoint method to compute sound propagation and jet noise

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- Adjoint method to compute sound propagation

  – Green-Lagrange's identity

  – An illustration with Lilley's equation

  – Connection with the Flow Reversal Theorem

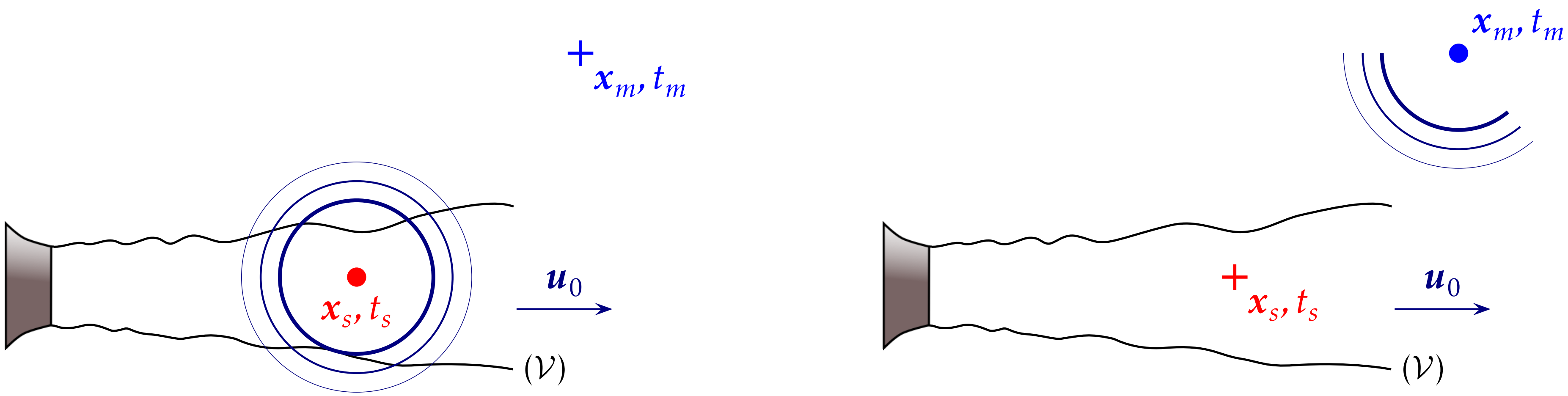
  – Self-adjoint wave equations for sound propagation

  – Our motivation for jet noise (finally!)

Spieser & Bailly, 2020, Sound propagation using an adjoint based method, to appear in *J. Fluid Mech.*

Adjoint method in aeroacoustics

The adjoint method is the proper generalization of the reciprocity principle when $(\boldsymbol{x}_m, t_m) \leftrightarrow (\boldsymbol{x}_s, t_s)$



Reciprocity principle

Hermann von Helmholtz
(1821–1894)

Den Hauptnutzen gewährt aber die Uebertragung des Theorems von *Green* *) auf die hier vorliegenden Verhältnisse, welches sich schon für die Theorie der Electricität und des Magnetismus so aufserordentlich fruchtbar gezeigt hat. Von allgemeinen Sätzen, die daraus herfliefsen, sollen nur folgende hervorgehoben werden: 1) *Die Schallbewegung in jedem allseitig begrenzten Raume, welcher keine Erregungspunkte enthält, kann stets dargestellt werden als ausgehend von Erregungspunkten, die nur längs der Oberfläche des Raumes in einer oder zwei einander unendlich nahen Schichten ausgebreitet sind.* 2) *In jedem Raume, dessen sämmtliche Dimensionen verschwindend klein gegen die Wellenlänge sind, können für das Geschwindigkeitspotential der Luftbewegung die analytischen Formen der electrischen Potentialfunctionen gesetzt werden, welche von jenem nur unendlich wenig unterschieden sind.* 3) *Wenn in einem theils von endlich ausgedehnten festen Wänden begrenzten, theils unbegrenzten Raume Schall im Punkte a erregt wird, so ist das Geschwindigkeitspotential in einem anderen Punkte b so grofs, als es in a sein würde, wenn dieselbe Schallerregung in b stattfände.*

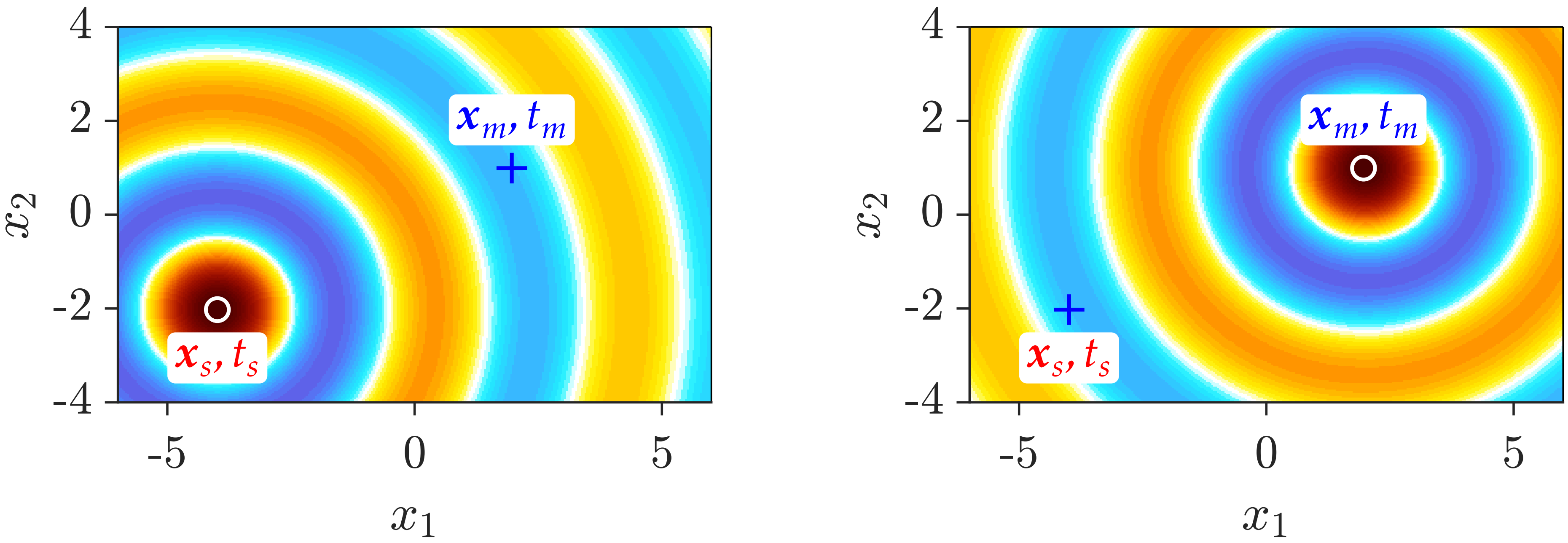
Theorie der Luftschwingungen in Röhren mit offenen Enden, in *Journal für die reine und angewandte Mathematik*, **57** (1860)

3) In a space partially bounded by rigid walls and open elsewhere, if a sound is generated at point A then the velocity potential at another point B is equal to that at point A if the same sound excitation had occurred at point B.

Reciprocity principle

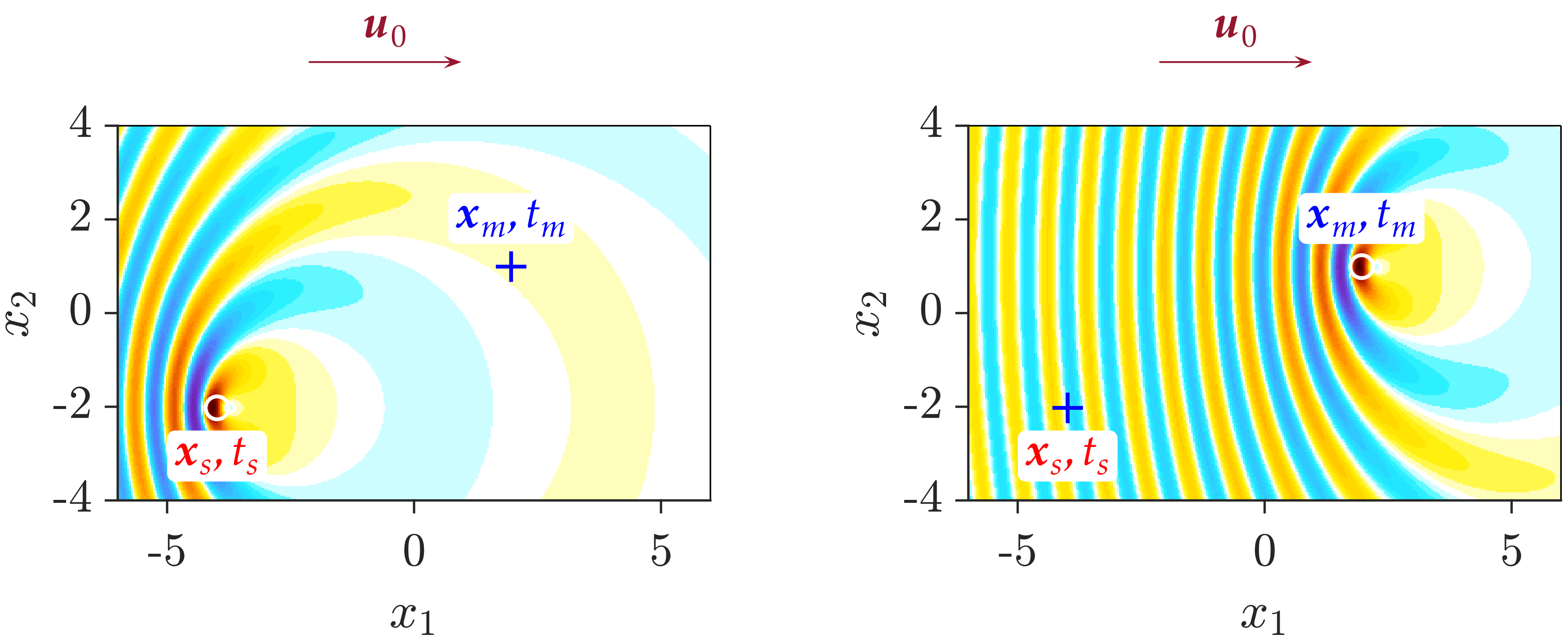
Symmetry in wave transmission between two points in space

With a uniform medium at rest,



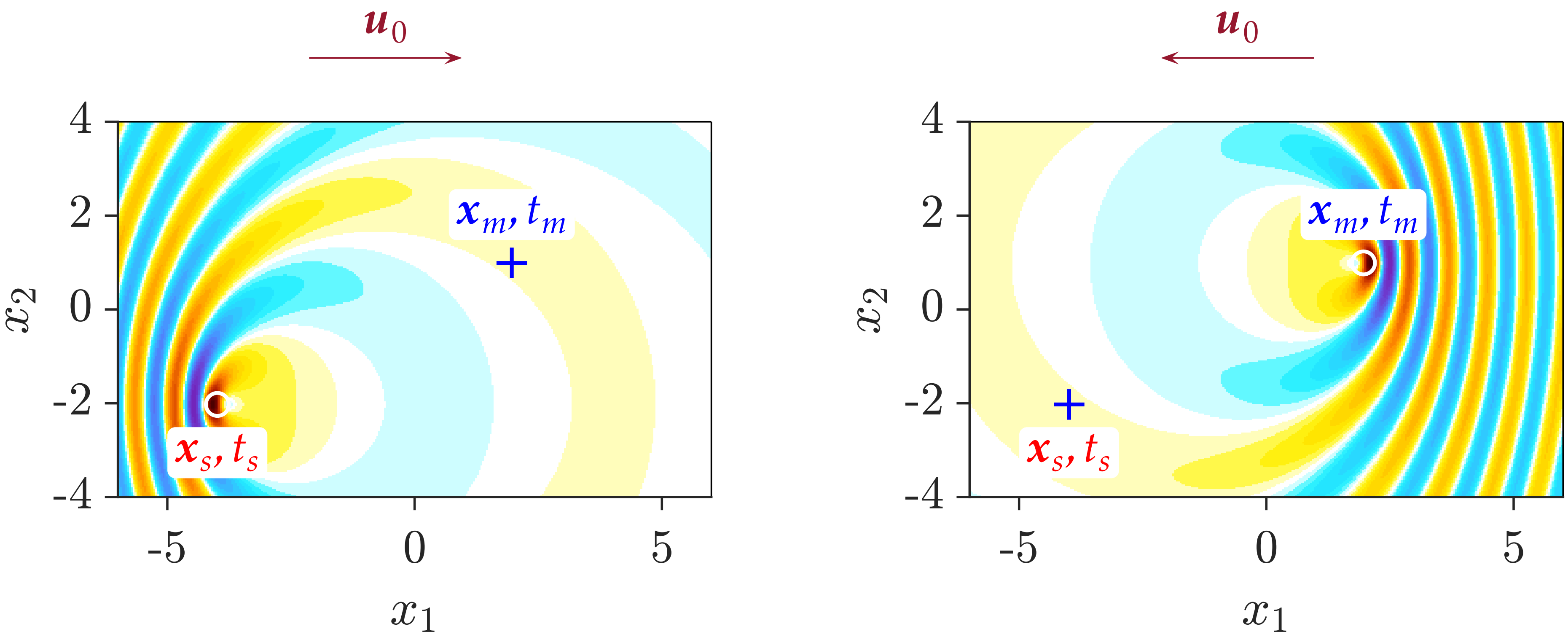
- **Reciprocity principle**

With a uniform medium at velocity $\boldsymbol{u}_0$



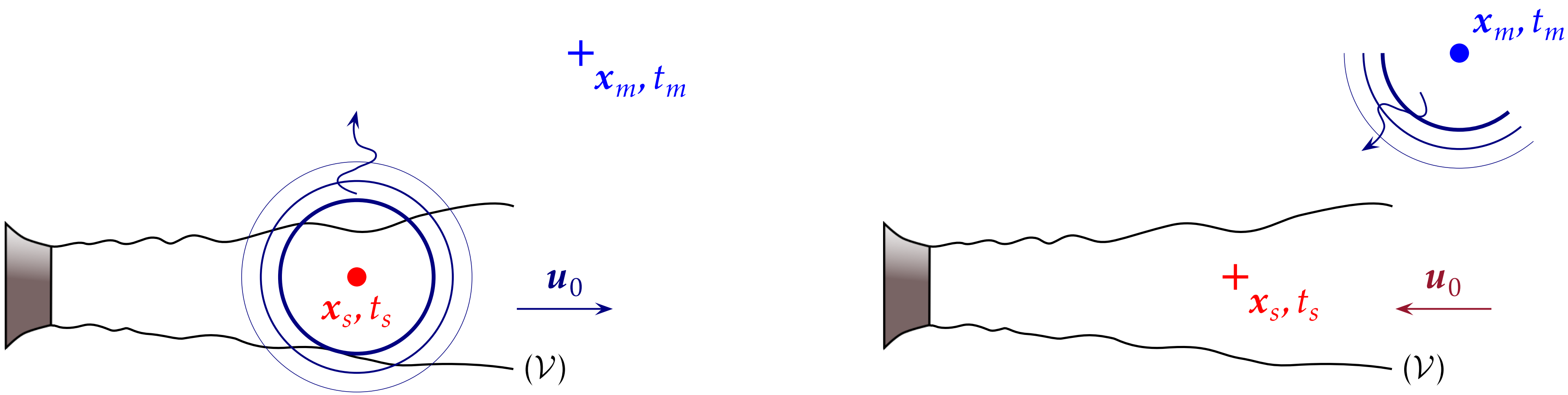
Reciprocity principle

With a uniform medium at velocity $\boldsymbol{u}_0$



Reciprocity principle

This approach – just by reversing the flow direction – is (unfortunately) not correct for a non uniform mean flow



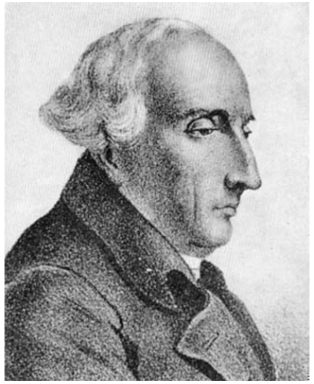
Adjoint method : Green–Lagrange's identity

Given a linear operator $\mathcal{L}_0$ with $\mathcal{L}_0 p = s$ (no boundary terms to simplify, not an issue) and a scalar product $< , >$, there is a unique $\mathcal{L}_0^\dagger$ adjoint operator, such that for any adjoint field $p^\dagger$

$$< p^\dagger, \mathcal{L}_0 p > = < \mathcal{L}_0^\dagger p^\dagger, p >$$

With $s^\dagger = \mathcal{L}_0^\dagger p^\dagger$,

Green–Lagrange's identity is recast as $\boxed{< p^\dagger, s > = < s^\dagger, p >}$



Joseph Louis Lagrange (1736–1813)

George Green (1793–1841)

Lazarus Fuchs (1833–1902)

Adjoint method : the representation theorem

The direct problem

$G_{\boldsymbol{x}_s,t_s}(\boldsymbol{x},t)$ Green's function

$$\mathcal{L}_0 G_{\boldsymbol{x}_s,t_s}(\boldsymbol{x},t) = \delta_{\boldsymbol{x}_s}\delta_{t_s}$$

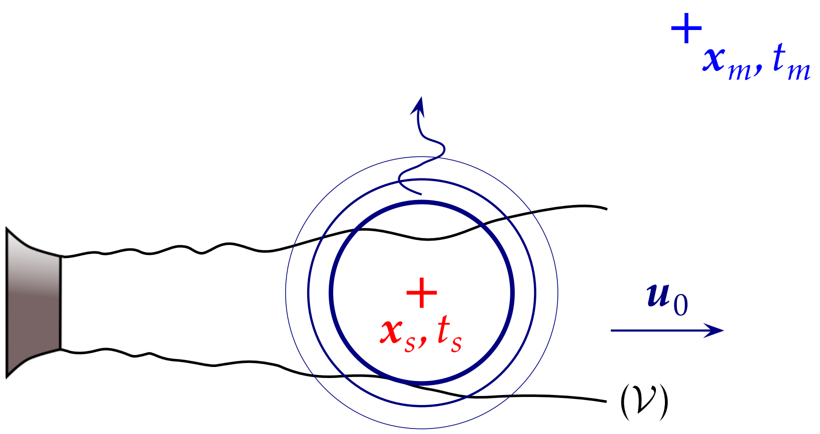
$$p(\boldsymbol{x},t) \; = \; < G_{\boldsymbol{x}_s,t_s}, s >$$

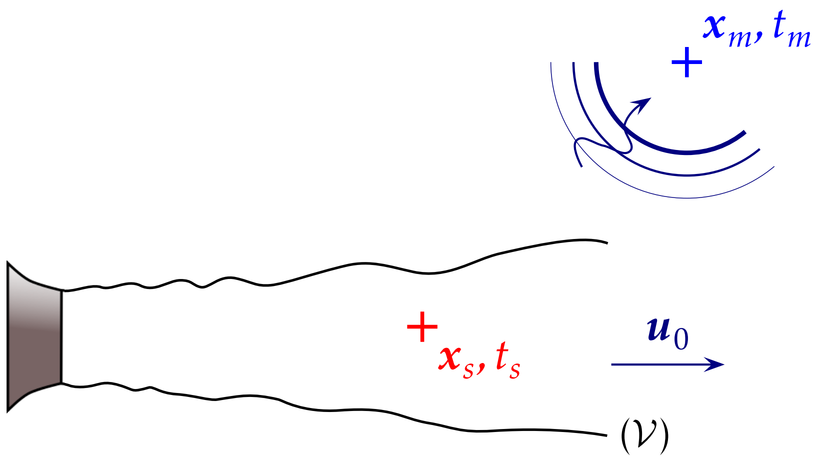
The adjoint problem

Using $< p^\dagger, s > = < s^\dagger, p >$ applied to the adjoint Green function $p^\dagger = G^\dagger_{\boldsymbol{x}_m,t_m}$

$$< G^\dagger_{\boldsymbol{x}_m,t_m}, s > = < \; \delta_{\boldsymbol{x}_m}\delta_{t_m}, \; p >$$

$$\Longrightarrow \quad p(\boldsymbol{x}_m,t_m) \; = \; < G^\dagger_{\boldsymbol{x}_m,t_m}, s >$$





(Afsar, Dowling, Karabasov, 2007)

Adjoint problem : the reciprocity principle

Using again $< p^\dagger, s > = < s^\dagger, p >$ with now $p^\dagger = G^\dagger_{\boldsymbol{x}_m, t_m}$ and $p = G_{\boldsymbol{x}_s, t_s}$

$$\Longrightarrow \quad G_{\boldsymbol{x}_s, t_s}(\boldsymbol{x}_m, t_m) = G^\dagger_{\boldsymbol{x}_m, t_m}(\boldsymbol{x}_s, t_s)$$

The direct problem is causal, $G_{\boldsymbol{x}_s, t_s}(\boldsymbol{x}_m, t_m) = 0$ for $t_m < t_s$,
the adjoint problem is thus anticausal, $G^\dagger_{\boldsymbol{x}_m, t_m}(\boldsymbol{x}_s, t_s) = 0$ for $t_m < t_s$,
the source acts as a sink, waves move toward the source as time advances

As a result, no self-adjoint formulation exists for time-dependent problem ...
The definition of a self-adjoint problem is usually relaxed, with

$$G^\dagger_{\boldsymbol{x}_m, t_m}(\boldsymbol{x}_s, t_s) = G_{\boldsymbol{x}_s, -t_s}(\boldsymbol{x}_m, -t_m) \qquad \mathcal{L}_0^\dagger = \pm \mathcal{L}_0$$

$\mathcal{L}_0 = \partial_t + \boldsymbol{u}_0 \cdot \nabla \qquad \mathcal{L}_0^\dagger = -\mathcal{L}_0 \qquad$ (antisymmetric)
$\mathcal{L}_0 = (1/c_0^2)\partial_{tt}^2 - \nabla^2 \qquad \mathcal{L}_0^\dagger = \mathcal{L}_0 \qquad$ (symmetric)

Illustration with Lilley's equation

We consider a parallel shear flow $\boldsymbol{u}_0 = u_0\boldsymbol{e}_1$ and $\rho_0(x_{1\perp})$. Lilley's equation (1972) is known to exactly take into account the propagation effects for a sheared and stratified mean flow. $D_{\boldsymbol{u}_0} = \partial_t + \boldsymbol{u}_0 \cdot \nabla$ denoting the material derivative along the mean flow,

$$\mathcal{L}_0\, p = D_{\boldsymbol{u}_0}\Big(D^2_{\boldsymbol{u}_0}(p) - \nabla\cdot(c_0^2\nabla p)\Big) + 2c_0^2\nabla u_0\cdot\nabla\left(\frac{\partial p}{\partial x_1}\right)$$

$$\mathcal{L}_0^\dagger\, p^\dagger = -D_{\boldsymbol{u}_0}\Big(D^2_{\boldsymbol{u}_0}(p^\dagger) - \nabla\cdot(c_0^2\nabla p^\dagger)\Big) + 4c_0^2\nabla u_0\cdot\nabla\left(\frac{\partial p^\dagger}{\partial x_1}\right) + 3\nabla\cdot\left(c_0^2\nabla u_0\right)\frac{\partial p^\dagger}{\partial x_1}$$

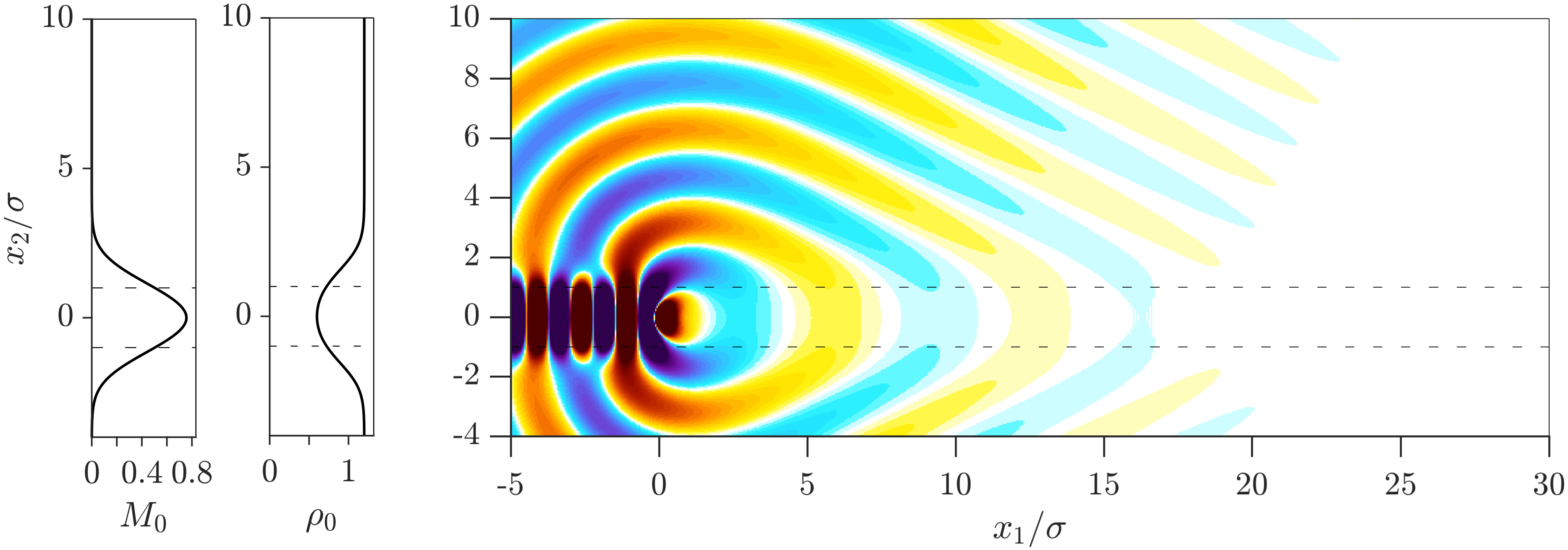
Sir Geoffrey Lilley (1919–2015)

$\mathcal{L}_0$ is not self-adjoint ...

Lilley's equation : solution to the direct problem

4th CAA workshop (Dahl, NASA-TR, 2004)
$M_j = 0.756$, $T_j = 600$ K, $St_{2\sigma} = 0.60$
$\mathcal{L}_0\, p = s_{\text{Lilley}}$, quadrupole source term from LEE (Bailly *et al.*, 2010)

Pressure field $\mathfrak{Re}(p)$



Lilley's equation

In the frequency domain,

$$f(\boldsymbol{x},\omega) \equiv \int_{-\infty}^{\infty} dt\; f(\boldsymbol{x},t)e^{+i\omega t} \qquad D_{\boldsymbol{u}_0} = -i\omega + \boldsymbol{u}_0 \cdot \nabla$$

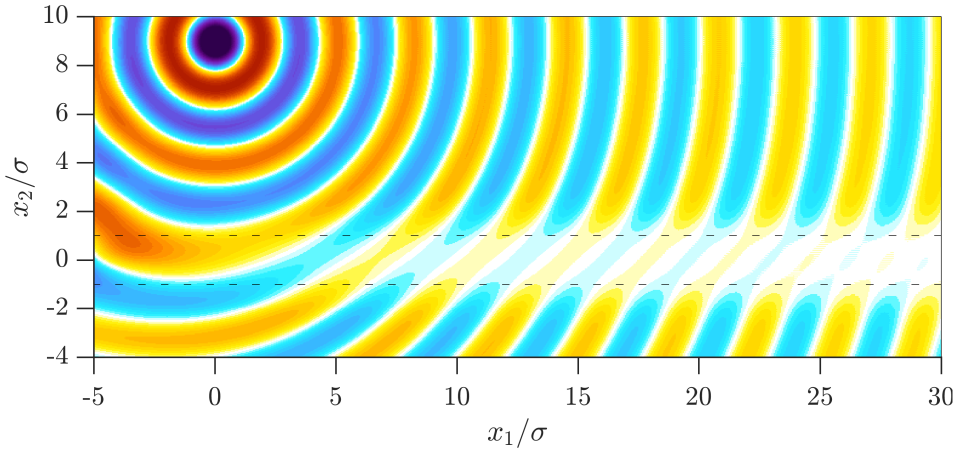
$\rightsquigarrow$ For a self-adjoint operator, $G^{\dagger}_{\boldsymbol{x}_m}(\boldsymbol{x}_s) = G^{-\star}_{\boldsymbol{x}_s}(\boldsymbol{x}_m)$ $\qquad \star$ complexe conjugate

In-house solver in Fourier space (É. Spieser)

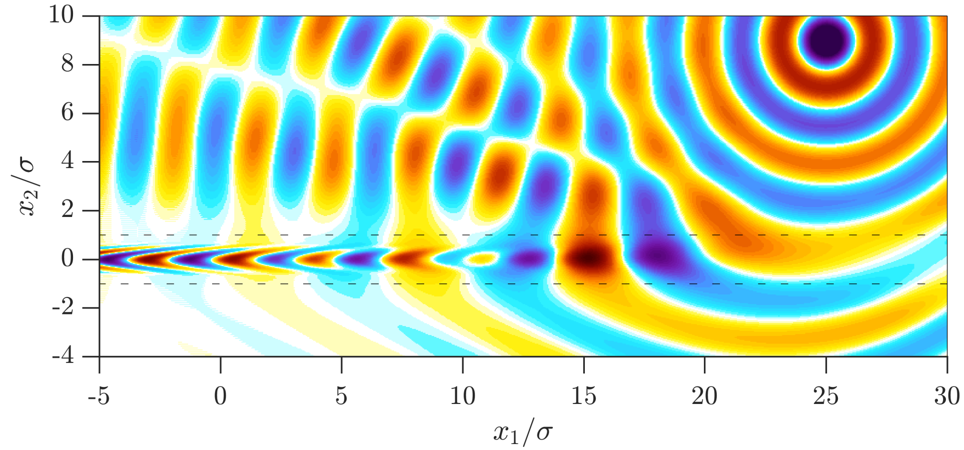
MATLAB environment with UMFPACK solver
Bogey & Bailly (2004), Berland *et al.* (2007) for space discretization
PML boundary condition by Hu (2008)
Anti-radiating boundary conditions $p^{\dagger}(-\omega) = p^{\dagger\star}(\omega)$ since $p^{\dagger} \in \mathbb{R}$

Lilley's equation : solution to the adjoint problem

$\mathcal{L}_0^\dagger G^\dagger_{\boldsymbol{x}_m} = \delta_{\boldsymbol{x}_m}$ with $\boldsymbol{x}_m/\sigma = (0, 9)$



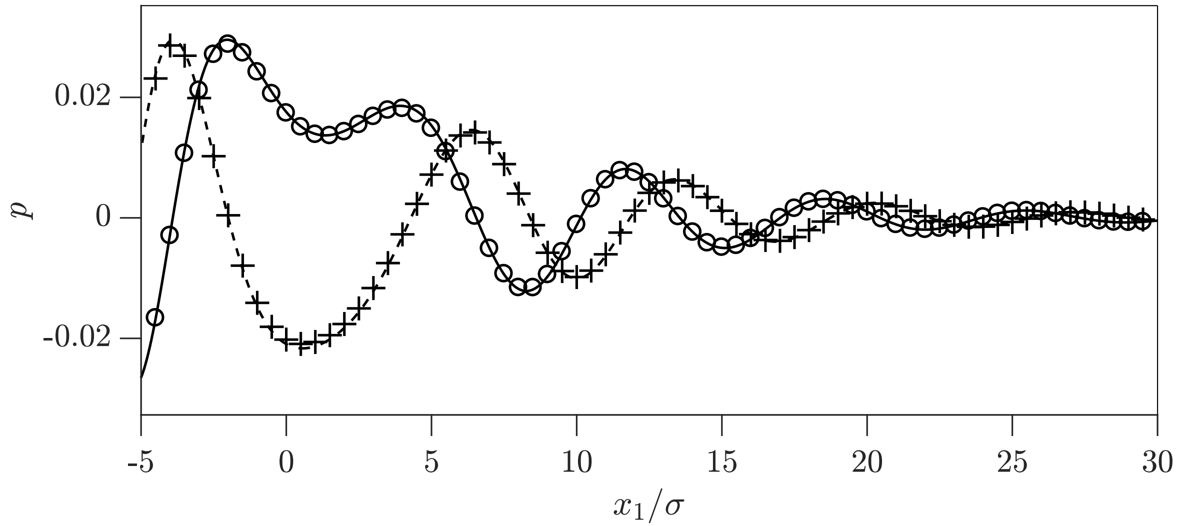
with $\boldsymbol{x}_m/\sigma = (25, 9)$



Lilley's equation : solution to the adjoint problem (cont.)

At a given observer point $\boldsymbol{x}_m$, the solution is computed from the representation theorem (simple scalar product with the source term)

$$p(\boldsymbol{x}_m,\omega) = < G^{\dagger}_{\boldsymbol{x}_m}, s_{\text{Lilley}} > = \int_{\mathcal{V}} d\boldsymbol{x}\; G^{\dagger\star}_{\boldsymbol{x}_m}(\boldsymbol{x})\; s_{\text{Lilley}}(\boldsymbol{x})$$



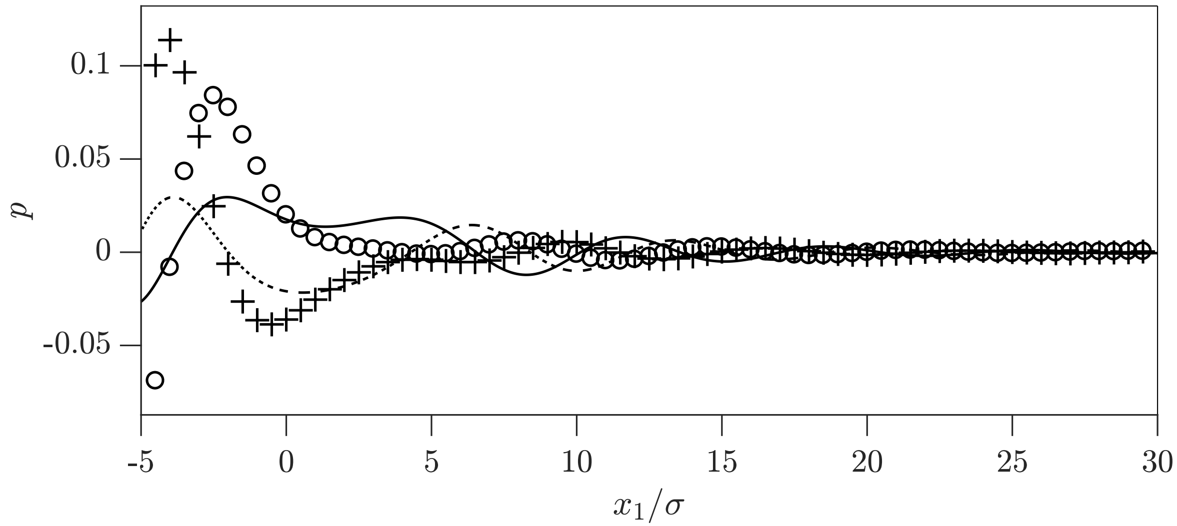
Direct problem —— $\mathfrak{Re}(p)$, - - - $\mathfrak{Im}(p)$ $(x_2/\sigma = 9)$

Reconstruction from adjoint formulation $\circ$ $\mathfrak{Re}(p)$, $+$ $\mathfrak{Im}(p)$

Lilley's equation : reversing the flow

The solution is now computed as $p(\boldsymbol{x}_m, \omega) = < G^{-\star}_{\boldsymbol{x}_m}(\boldsymbol{x}_s), s_{\text{Lilley}} >$

where $\mathcal{L}_0^- G^-_{\boldsymbol{x}_m} = \delta_{\boldsymbol{x}_m}$ and $\mathcal{L}_0^-$ is $\mathcal{L}_0$ by reversing the flow $\boldsymbol{u}_0 \longrightarrow -\boldsymbol{u}_0$



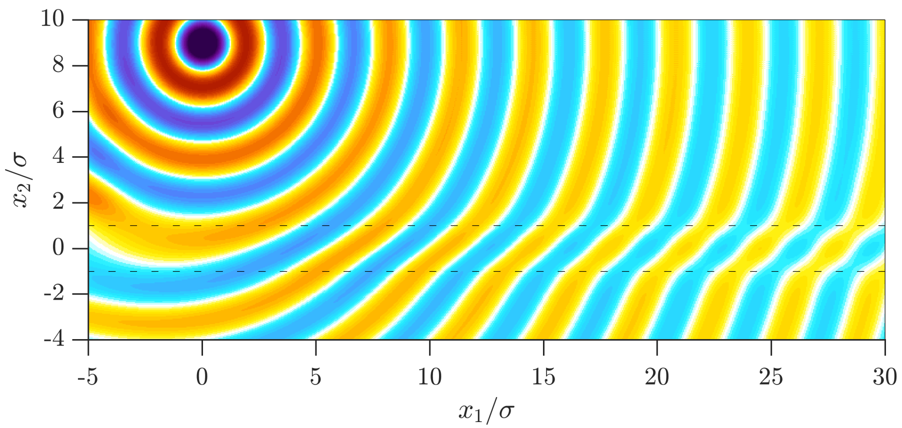
Direct problem —— $\Re(\hat{p})$, - - - $\Im(\hat{p})$ $\quad (x_2/\sigma = 9)$

Reconstruction by reversing the flow ∘ $\Re(p)$, + $\Im(p)$

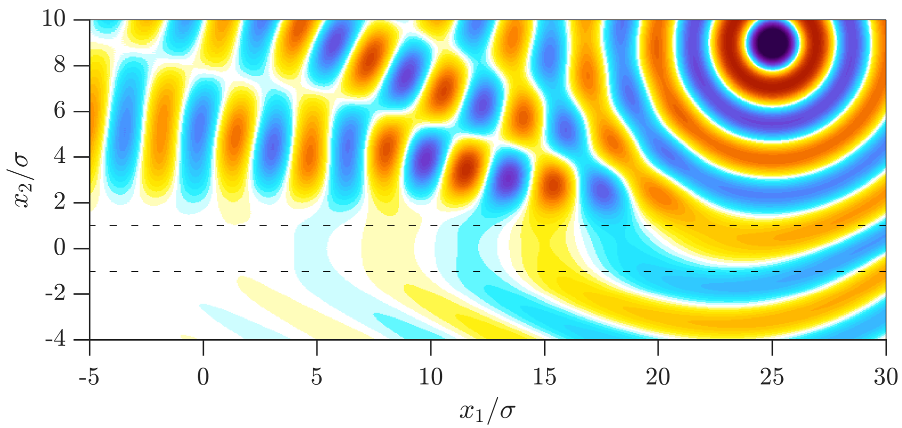
$\Longrightarrow \quad G^{-\star}_{\boldsymbol{x}_m}(\boldsymbol{x}_s) \neq G^{\dagger}_{\boldsymbol{x}_m}(\boldsymbol{x}_s)$ not a self-adjoint problem

Lilley's equation

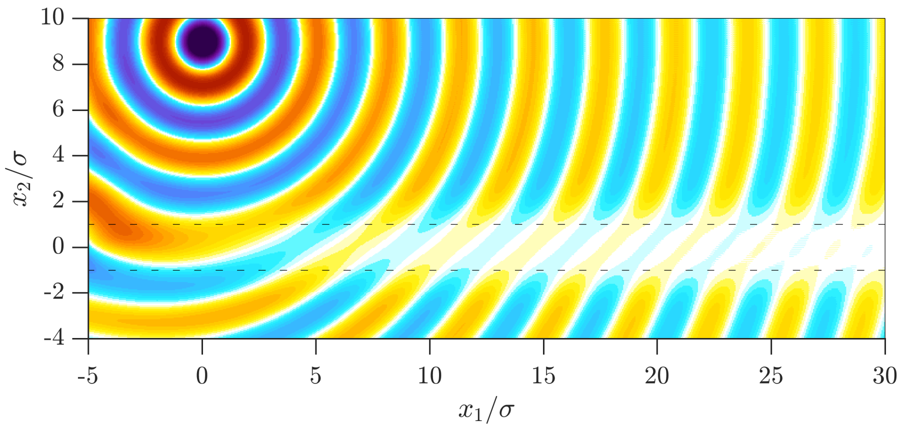
$G^{-\star}_{\boldsymbol{x}_m}$ with $\boldsymbol{x}_m/\sigma = (0,9)$



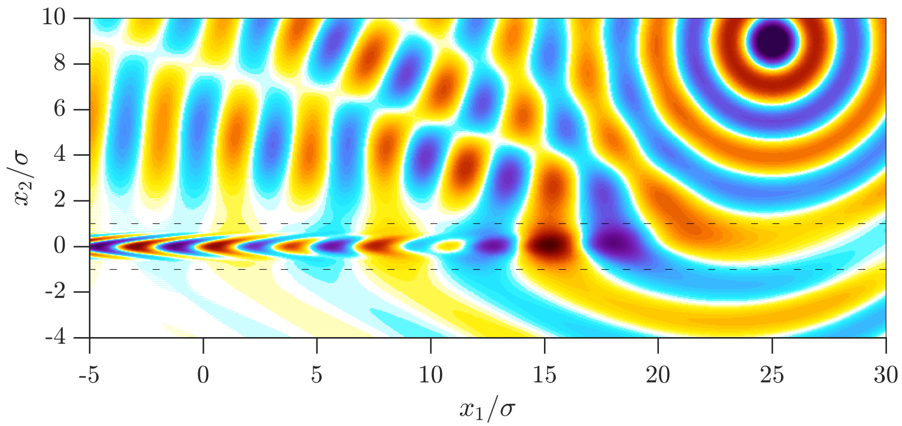
$G^{-\star}_{\boldsymbol{x}_m}$ with $\boldsymbol{x}_m/\sigma = (25,9)$



$G^{\dagger}_{\boldsymbol{x}_m}$



$G^{\dagger}_{\boldsymbol{x}_m}$



Connection with the Flow Reversal Theorem

Lyamshev (1961), Howe (1975), Godin (1997), Möhring (1999)

FRT and adjoint are fully equivalent for self-adjoint operators

$$G_{\mathbf{x}_s,t_s}(\mathbf{x}_m,t_m) = G^{\dagger}_{\mathbf{x}_m,t_m}(\mathbf{x}_s,t_s) = G^{-}_{\mathbf{x}_m,-t_m}(\mathbf{x}_s,-t_s)$$

An energy conservating law can be derived for the acoustic field (Möhring, 1999), which prevents the development of Kelvin-Helmholtz's instability waves in aeroacoustics or internal gravity waves for infrasound propagation for example.

Lilley's equation and Linearized Euler Equations (LEE) are not self-adjoint wave operators ...

What is the best self-adjoint approximation of LEE ?

Willi Möhring
(Lyon, 1985)

Self-adjoint wave equations for sound propagation

- Lilley's equation (1972)

$$\mathcal{L}_0\, p = D_{\boldsymbol{u}_0}\left(D^2_{\boldsymbol{u}_0}(p) - \nabla\cdot(c_0^2\nabla p)\right) + 2c_0^2\nabla u_0\cdot\nabla\left(\frac{\partial p}{\partial x_1}\right) \qquad \mathcal{L}_0^\dagger \neq \mathcal{L}_0$$

- Helmholtz equation with a variable index of refraction,
  low Mach number approximation (outdoor propagation, combustion)
  Bergmann (1946), Phillips (1960)

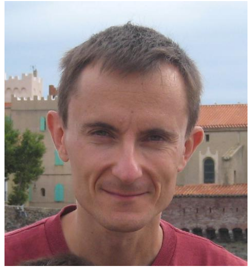
$$\mathcal{L}_0\, p = D^2_{\boldsymbol{u}_0}(p) - \nabla\cdot(c_0^2\nabla p) \qquad \mathcal{L}_0^\dagger = \mathcal{L}_0$$

- Many other significant contributions in the literature

  However, after investigation, one formulation stood out in terms of precision to best approximate LEE ...

- Lyon is a great place to do research!




Acoustic team @ Lyon

(25 July 2019, Ampère museum at Poleymieux)

Pierce's wave equation (1990)

From work done by Bergmann (1946), Blokhintzev (1946)
Wave equation derived from LEE including gravity (*e.g.* infrasound propagation)

Eq. for the acoustic potential velocity ϕ

$$\mathcal{L}_0\,\phi = D^2_{\boldsymbol{u}_0}(\phi) - \nabla\cdot(c_0^2\nabla\phi) \qquad \mathcal{L}_0^\dagger = \mathcal{L}_0$$

where the primary fields are defined as
$\rho_0\boldsymbol{u} = \nabla\phi$ and $p = -D_{\boldsymbol{u}_0}(\phi)$

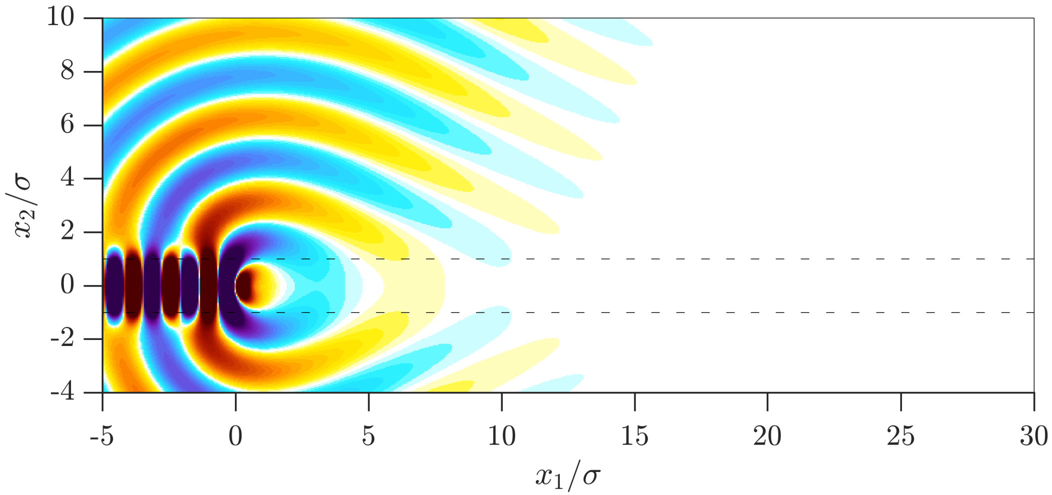
Not to be confused with Goldstein's Eq. (1978),

$$D_{\boldsymbol{u}_0}\Big[(1/c_0^2)D_{\boldsymbol{u}_0}(\phi)\Big] - (1/\rho_0)\nabla\cdot(\rho_0\nabla\phi) = 0 \qquad \boldsymbol{u} = \nabla\phi$$

Allan Pierce
(ASA Meeting, 2005)

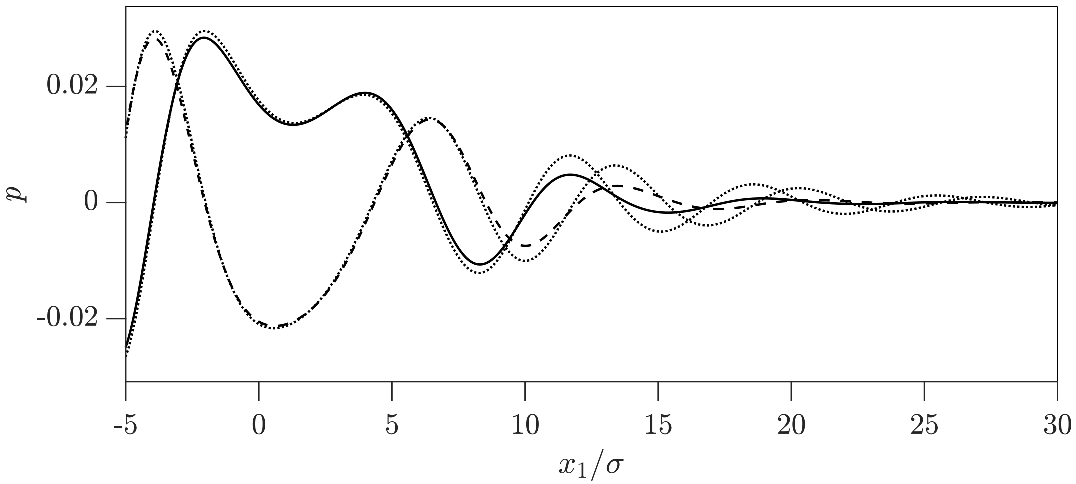
J. Acoust. Soc. Am. (1990), Eq. (27)

Pierce's wave equation



$\mathcal{L}_0\,\phi = s_{\text{Pierce}}$
(expression derived from LEE)

$\mathfrak{Re}(p)$ from $p = -D_{\boldsymbol{u}_0}(\phi)$



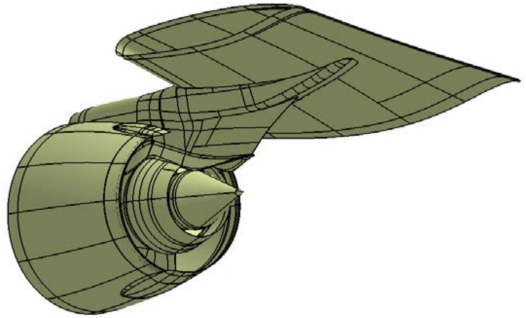
Pierce —— $\mathfrak{Re}(p)$,
- - - $\mathfrak{Im}(p)$ along $x_2/\sigma = 9$
······ $p(\boldsymbol{x},\omega)$ Lilley (reference)

Behaviour of a high frequency approximation : some differences in the shadow zone only

Noise prediction of installed modern aircraft engines

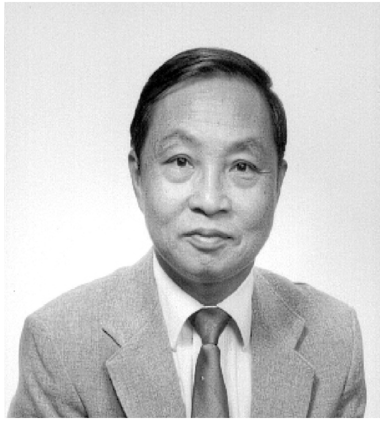
It does not seem reasonable to directly implement the volume source term – two-point statistics $S_{pp}(\boldsymbol{x}_m, \omega)$ – in a finite element formulation, the method must be competitive in industrial context

e.g. Tam & Auriault mixing noise model (1998)
(see also companion paper by Morris & Farassat, 2002)



(EXEJET program)

Using an adjoint formulation, the scalar product between the volume source term $\mathcal{V}$ and the adjoint Green function $G^{\dagger}_{\boldsymbol{x}_m}$ is performed outside of the FE acoustic solver : of interest for jet noise, when the number of microphones $\boldsymbol{x}_m$ is small with respect to the "number of sources" $\boldsymbol{x}_s \in \mathcal{V}$



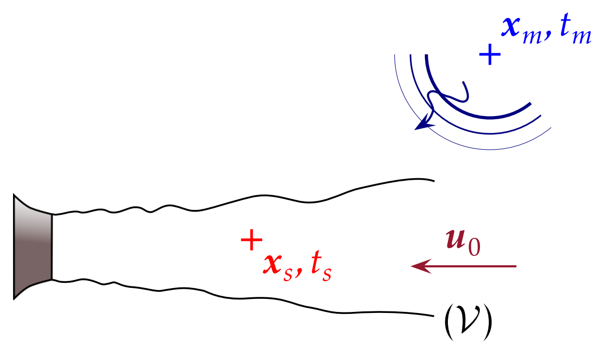
Christopher Tam

Use of Pierce's wave equation

Actran TM's hijacking ...

Application of the FRT to compute

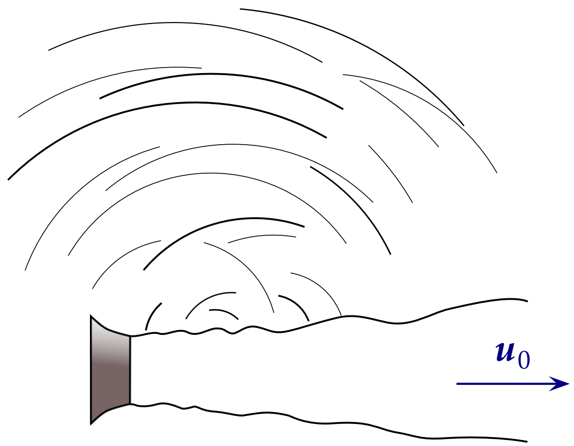
$G^{\dagger}_{\boldsymbol{x}_m, t_m}(\boldsymbol{x}_s, t_s) = G^{-}_{\boldsymbol{x}_m, -t_m}(\boldsymbol{x}_s, -t_s)$



From Möhring's Eq. written for the fluctuating stagnation enthalpy b to Pierce's Eq.
(C. Legendre - FFT)

$$p(\boldsymbol{x}_m, \omega) = \int_{\mathcal{V}} d\boldsymbol{x}_s \, G^{-}_{\boldsymbol{x}_m}(\boldsymbol{x}_s) \, s(\boldsymbol{x}_s, \omega)$$

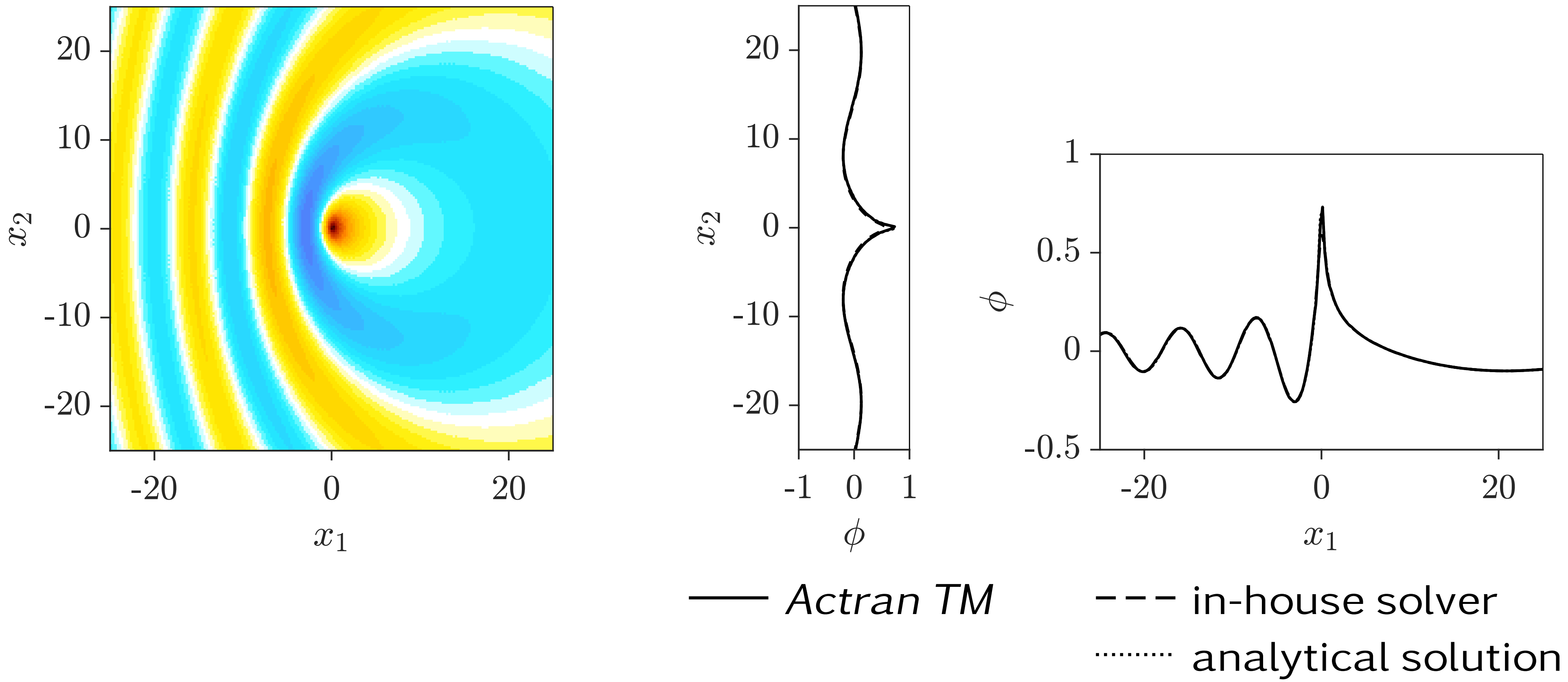
$$\frac{\partial}{\partial t}\left[\frac{\rho_0}{\rho_{t0}^2 c_0^2} D_{\boldsymbol{u}_0}(b)\right] + \nabla \cdot \left[\frac{\rho_0 \boldsymbol{u}_0}{\rho_{t0}^2 c_0^2} D_{\boldsymbol{u}_0}(b) - \frac{\rho_0}{\rho_{t0}^2} \nabla b\right] = 0$$



Change of variables without affecting $\boldsymbol{u}_0$ and c_0 to finally solve Pierce's Eq. (including expression of the source term)

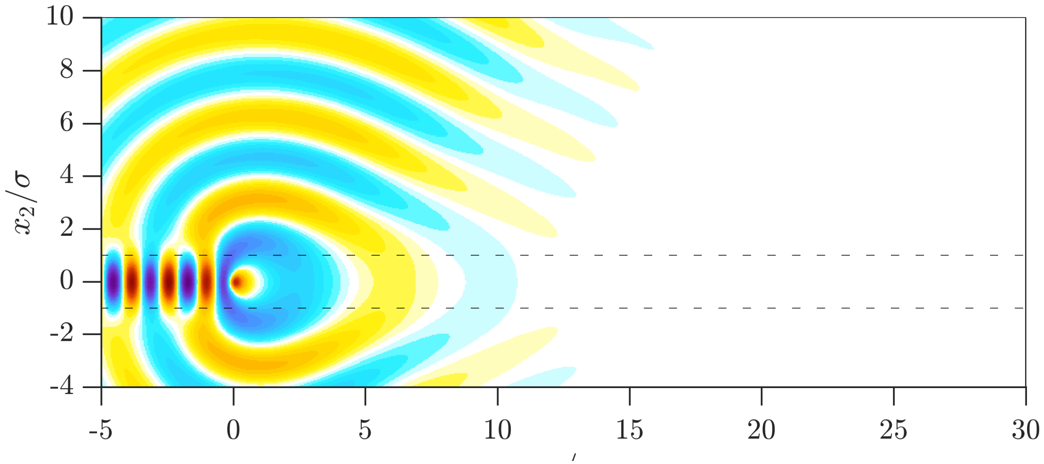
- **Validation of the implementation of Pierce's Eq. (1)**

Radiation of a point source in a uniform flow : acoustic potential field $\mathfrak{Re}(\phi)$ computed with *Actran TM*

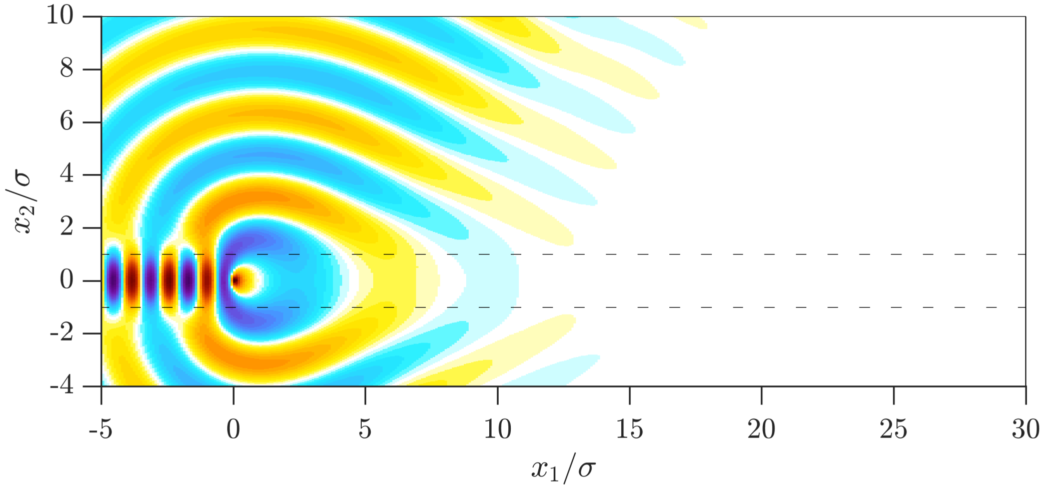


- **Validation of the implementation of Pierce's Eq. (2)**

4th CAA workshop (point source term in s_{Pierce})



Acoustic potential field $\mathfrak{Re}(\phi)$ from our in-house solver

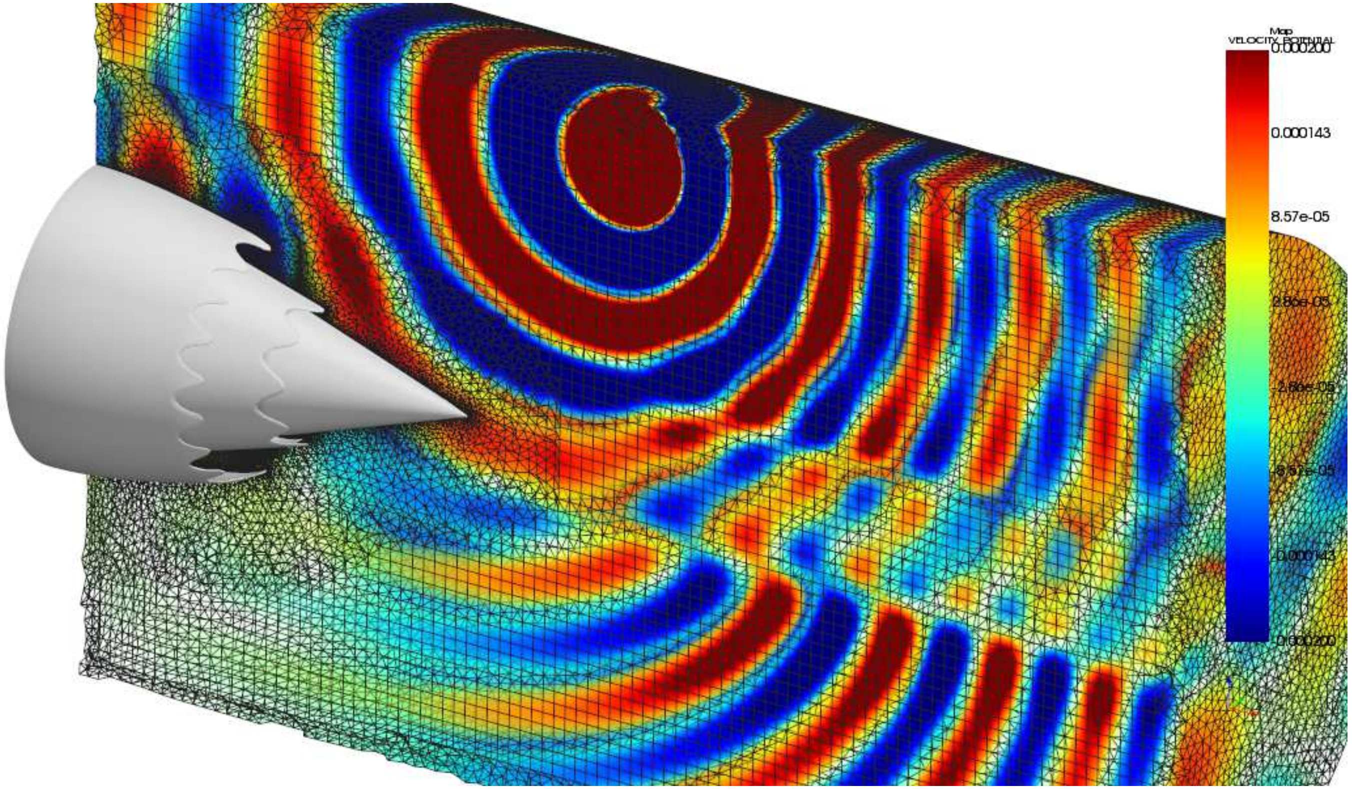


Acoustic potential field $\mathfrak{Re}(\phi)$ from *Actran TM*

(some differences can be observed, not well explained for the moment ...)

Dual stream engine with chevrons

Adjoint Green function $G^{\dagger}_{\boldsymbol{x}_m}(\boldsymbol{x}_s)$ computed with the FRT



data from the EXEJET project

$T_p/T_s \simeq 2.6$
$M_p \simeq 0.67$
$M_s \simeq 0.84$
$M_f = 0.23$

$St_s \simeq 2$

Take home message

Green-Lagrange's identity including the definition of a specific scalar product : good mathematical framework to deal with adjoint

Flow Reversal Theorem : a smart way to compute adjoint Green's function, for self-adjoint problems only (implementation of anti-causal boundary conditions is bypassed)

Pierce's equation has been selected as a good compromise : self-adjoint, energy conservation, possible to implement in existing solvers, good approximation of the exact (not self-adjoint) problem. Application : statiscal models formulated as a volume integral (mainly jet noise)

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Expression of the source terms

Linearized Euler Equations (LEE) for a stratified mean flow (Lilley, $\boldsymbol{u}_0 = u_0 \boldsymbol{e}_1$)

$$\begin{cases} D_{\boldsymbol{u}_0}\rho + \nabla \cdot (\rho_0 \boldsymbol{u}) = 0 \\ D_{\boldsymbol{u}_0}(\rho_0 \boldsymbol{u}) + \rho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u}_0 + \nabla p = \rho_0 \boldsymbol{S}_{\boldsymbol{u}} \\ D_{\boldsymbol{u}_0} p + \gamma p_0 \nabla \cdot \boldsymbol{u} = S_p \end{cases}$$

Lilley

$$s_{\text{Lilley}} = D_{\boldsymbol{u}_0}\left[D_{\boldsymbol{u}_0}(S_p) - \gamma p_0 \nabla \cdot \boldsymbol{S}_{\boldsymbol{u}} \right] + 2\gamma p_0 \nabla u_0 \cdot \frac{\partial \boldsymbol{S}_{\boldsymbol{u}}}{\partial x_1}$$

Pierce

$$s_{\text{Pierce}} = D_{\boldsymbol{u}_0}(S_m) - S_p \qquad \nabla^2 S_m = \nabla \cdot (\rho_0 \boldsymbol{S}_{\boldsymbol{u}})$$