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Occurrence of Caustics for High-Frequency Acoustic Waves Propagating Through Turbulent Fields[1]

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Abstract. A numerical technique for simulating the propagation of high-frequency acoustic waves through turbulent fields is introduced. The technique involves two elements: the generation of a random isotropic scalar or vectorial field in terms of a superposition of discrete random Fourier modes; and the integration of the ray-trace equations of geometrical acoustics to describe the trajectories of rays and their distortion. For each realization we compute the ray trajectories and the evolution of the cross section of an elementary ray tube. We then accumulate statistics over an ensemble of realizations to estimate the probability of occurrence of the first caustic. Our results demonstrate that the position of caustics is governed by universal parameters related to the derivatives of the correlation function of the fluctuating components of the turbulent medium.

1. Introduction

The propagation of acoustic waves through turbulent media is a topic of great interest with a wide variety of applications in underwater or in atmospheric acoustics. For example, fluctuations appear in the phase pattern of the transmitted waves leading to noise in the imaging process of arrays. Intensity fluctuations are also present and all the perturbations grow substantially as the waves propagate further in the medium. Even if the turbulent fluctuations are weak in the random medium, the cumulative effects over long distances may be great. Similar problems exist in optical propagation of laser beams in atmospheric turbulence.

The theory of wave propagation in random media has been developed by a number of authors and extensive reviews are given in Tatarski (1971), Strohbehn (1978), and Ishimaru (1978). The usual procedure relies on a statistical approach. As a first step, making use of several hypotheses, a wave equation (Helmholtz or parabolic) is obtained in which the interaction between the wave and the medium is expressed in terms of a random refractive index related to the fluctuations of the medium. In the case of a monochromatic incident sound field propagating in a time-independent medium with temperature and velocity fluctuations, the pressure field is governed by the Helmholtz equation:

$$\{\Delta + k_0^2(1 + \varepsilon(\vec{x}))\}p(\vec{x}) = 0, \tag{1}$$

$$\varepsilon(\vec{x}) = -\frac{T'(\vec{x})}{T_0} - \frac{2v_1'(\vec{x})}{c_0}, \tag{2}$$

[1] Dedicated to Professor J.L. Lumley on the occasion of his 60th birthday.

where $p(\vec{x})$ denotes the space dependence of the pressure, k_0 is the acoustic wave number, and $\varepsilon(\vec{x})$ is the fluctuation of the index of refraction. Here $\varepsilon(\vec{x})$ is related to the fluctuation T' of the temperature and to the component v_1' of the velocity fluctuation in the direction of the propagation of the incident wave. T_0 and c_0 are respectively the temperature and the sound speed without turbulence. Indeed, it can be shown that for small turbulent Mach numbers, and monochromatic waves, the forward propagation of sound in turbulent fluid can be adequately represented by (1) if the acoustic wavelength λ does not exceed a typical length scale of the turbulent field (Neubert and Lumley, 1970; Candel, 1979). The second step of this statistical approach is to generate equations which govern the second and higher moments of the transmitted sound pressure field. Then, using a closure hypothesis for the index fluctuation $\varepsilon(\vec{x})$, solutions may be obtained either by some analytical developments or by numerical integration (Whitman and Beran, 1985; Uscinski, 1985).

It is only recently that studies have made use of computer-generated fields to simulate wave propagation in random media, but they are still limited. Hesselink and Sturtevant (1988) computed the ray trajectories of weak shock waves in a random Gaussian correlated field obtained by the filtering of a white noise. Martin and Flatte (1988) characterize the intensity statistics using a phase-screen description of the medium. In these works the index of refraction exhibited only the scalar part of the random field. However, in acoustics, the presence of velocity components introduces additional effects of wave convection which cannot be described *a priori* by such a simplified approach. Truman and Lee (1990), who used a direct numerical simulation to generate the index fluctuation ε, have studied the propagation of optical beams through a homogeneous turbulent shear flow for which the paraxial approximation is well suited. Juvé *et al.* (1990) made use of a similar technique to investigate the transmission of acoustic waves through mixing layers and two-dimensional isotropic turbulence.

In this paper we describe a different approach, which takes into account fluctuations in velocity fields and avoids purely statistical theory and not well-founded hypotheses. Assuming that the turbulent field is frozen during the transit time of the acoustic wave, the medium can be modeled by a sequence of independent realizations of a random field. Each realization of the field is generated by a superposition of a finite number of discrete random Fourier modes. The amplitude of these modes is chosen in order to obtain a predefined energy spectum. Isotropic fields with scalar or vectorial fluctuations can be obtained with roughly the same technique. Then we consider the deterministic propagation of acoustic waves (in the geometrical approximation) through individual realizations of the simulated turbulent field. We integrate the equations of geometrical acoustics to describe the trajectories of rays as well as the evolution of the cross section of an elementary ray tube. The points where this section vanishes define the position of caustics. We illustrate the focusing and defocusing effects of the inhomogeneous media by plotting the distorted rays. We accumulate statistics over an ensemble of realizations to estimate the probability density function (p.d.f.) of occurrence of the first caustic at a given distance from the source. Using initially plane waves propagating in two-dimensional Gaussian correlated temperature or velocity random fields, we interpret the formation of caustics and the differences between scalar and vectorial inhomogeneous media. The extension to a three-dimensional case presents no essential difficulties except the increase in computation time.

2. Simulation of a Two-Dimensional Isotropic Random Velocity Field

During the transit time of the acoustic wave the turbulent field is, as usual, considered to be frozen. The medium can then be modeled by a sequence of independent realizations of a random field. Following Kraichnan (1970), the velocity $\vec{v}'$ at a given point $\vec{x}$ is simulated as a sum of N random incompressible Fourier modes:

$$\vec{v}'(\vec{x}) = \sum_{l=1}^{N} \vec{\mathscr{U}}(\vec{K}^l)\cos(\vec{K}^l\cdot\vec{x} + \psi^l), \tag{3}$$

$$\vec{\mathscr{U}}(\vec{K}^l)\cdot\vec{K}^l = 0. \tag{4}$$

The direction of the wave vector $\vec{K}^l$ and the phase ψ^l are independent random variables with uniform distributions. The amplitude $\|\vec{\mathscr{U}}(\vec{K}^l)\|$ is a deterministic variable whose value is set according to a given two-dimensional kinetic energy spectrum $E(K)$ with $K = \|\vec{K}\|$. In this paper we have used a field with a Gaussian correlation function $f(r) = \exp(-r^2/L^2)$. This simple form is often chosen for comparison with analytical developments (Kulkarny and White, 1982). Here L is related to the longitudinal integral length scale L_f by $L_f = L\sqrt{\pi}/2$. In two dimensions the spectrum $E(K)$ is related to $f(r)$ by the formula

$$E(K) = \frac{v'^2}{2} K \int_0^\infty \frac{\partial}{\partial r}(r^2 f(r)) J_0(Kr)\, dr, \tag{5}$$

where J_0 is the Bessel function of the first kind of order zero and v'^2 is the mean square of the velocity fluctuations ($v'^2 = \overline{v_1'^2} = \overline{v_2'^2}$). The Gaussian correlation function $f(r)$ for the spectrum $E(K)$ gives

$$E(K) = \frac{v'^2}{8} K^3 L^4 \exp\left(-\frac{K^2 L^2}{4}\right). \tag{6}$$

In our simulations this spectrum has been sampled with $N = 50$ modes linearly distributed between $K_{\min} = 0.1/L$ and $K_{\max} = 10/L$. The minimum number of realizations of the field is around 500 to get reasonable properties of the generated field as far as homogeneity, isotropy, and correlation lengths are concerned. However, for acoustic quantities the global trends are obtained with only around 100 realizations, because of a further mean along the propagation path.

3. Simulation of a Two-Dimensional Isotropic Random Temperature Field

For a temperature random field we proceed in a similar way. The temperature T' is represented by

$$T'(\vec{x}) = \sum_{j=1}^{N} \mathscr{T}(\vec{K}^j) \cos(\vec{K}^j \cdot \vec{x} + \varphi^j), \tag{7}$$

where the direction of $\vec{K}^j$ and the phase φ^j are independent random variables with uniform distributions. The amplitude $\mathscr{T}(\vec{K}^j)$ is a deterministic variable whose value is set according to the two-dimensional spectrum $G(K)$ of the temperature fluctuation T':

$$\theta^2 = \int_0^\infty G(K)\, dK, \tag{8}$$

again with $K = \|\vec{K}\|$. The spectrum $G(K)$ is connected to the two-point temperature correlation $m(r)$ by

$$G(K) = \theta^2 K \int_0^\infty r m(r) J_0(Kr)\, dr \tag{9}$$

which, with $m(r) = \exp(-r^2/L^2)$, leads to

$$G(K) = \frac{\theta^2}{2} K L^2 \exp\left(-\frac{K^2 L^2}{4}\right). \tag{10}$$

4. Ray-Trace Equations in the Geometrical Acoustic Approximation

The geometrical acoustic approximation gives a clear visualization of the focusing or defocusing properties of an inhomogeneous medium. It is well suited to compute the ray trajectories and the exact position of caustics along the ray path. In this high-frequency approximation the acoustic pressure is written in the form

$$p(\vec{x}, t) = A(\vec{x}) e^{iS(\vec{x})} e^{-i\omega t}. \tag{11}$$

The amplitude $A(\vec{x})$ and the local wave vector $\vec{k}(\vec{x}) = \vec{\nabla}(S)$ are assumed to vary slowly on the scale of a wavelength $\lambda = 2\pi\, c_0/\omega$. An asymptotic expansion for $\omega \to \infty$ of the exact linearized fluid mechanics equations gives the dispersion relation of acoustic waves propagating in an inhomogeneous medium in steady motion (Candel, 1977):

$$\omega = kc + \vec{k} \cdot \vec{V}, \tag{12}$$

where k is the modulus of the acoustic wave number $\vec{k}$, $\vec{V}$ is the total velocity field, and c is the local speed of sound in the medium.

The rays are the lines tangent to the group velocity $\vec{c}_g$ ($\vec{c}_g = c\vec{v} + \vec{V}$; $\vec{v} = \vec{k}/k$). They can be determined as the characteristic lines of the dispersion relation through the following Hamiltonian system (Whitham, 1974):

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{c_0}{N}\frac{p_i}{N - p_i M_j} + V_i, \\ \frac{dp_i}{dt} &= \frac{c_0}{N}\frac{\partial N}{\partial x_i} - \frac{1}{N} p_j \frac{\partial V_j}{\partial x_i}, \\ p &= \frac{N}{1 + \vec{M}\cdot\vec{v}}, \end{aligned} \tag{13}$$

where $\vec{p}$ is a nondimensional wave vector ($\vec{p} = p\vec{v}$), N is the index of refraction c_0/c, and $\vec{M}$ is the Mach number $\vec{V}/c_0$. The rays have been parametrized by the transit time t from the source to a given point. The position vector $\vec{x} = (x, y)$ and the wave vector $\vec{p}$ at a current point on the ray trajectory are completely determined by the value of t and the initial position along the incident wave front. For an incident plane wave the initial conditions are

$$\vec{x}(t = 0) = \begin{pmatrix} 0 \\ y^0 \end{pmatrix}, \qquad \vec{p}(t = 0) = \frac{N}{1 + \vec{M}\cdot\vec{v}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{14}$$

Plotting the rays permits a clear visualization of the trajectories followed by the acoustic energy radiating from a source. The spatial distribution of rays is a qualitative indicator of the local intensity of the field, since the square root of the amplitude is inversely proportional to the cross-sectional area of a ray tube. In order to determine more precisely the caustics, which are defined as the envelopes of families of rays where the ray-tube section vanishes, we need additional differential equations. In two dimensions, for a plane wave, the two geodesic elements $\vec{R} = (\partial\vec{x}/\partial y^0)_t$ and $\vec{Q} = (\partial\vec{p}/\partial y^0)_t$ govern the evolution of the wave front along each ray and they permit the evaluation of the cross-sectional area of an infinitesimal ray tube:

$$\begin{aligned} \frac{dR_i}{dt} &= \frac{c_0}{pN}(Q_i - v_i v_j Q_j) - \frac{c_0}{N^2}(M_i + v_i)R_j\frac{\partial N}{\partial x_j} + \frac{c_0}{N}R_j\frac{\partial M_i}{\partial x_j}, \\ \frac{dQ_i}{dt} &= \frac{c_0}{N}\left(R_j\frac{\partial^2 N}{\partial x_j\,\partial x_i} - R_j\frac{\partial^2 N}{\partial x_j\,\partial x_i}M_k p_k - Q_j\frac{\partial M_j}{\partial x_i}\right) - \frac{c_0}{N^2}R_j\frac{\partial N}{\partial x_j}\left(\frac{\partial N}{\partial x_i} - p_k\frac{\partial M_k}{\partial x_i}\right). \end{aligned} \tag{15}$$

These differential equations require appropriate initial conditions. If we expand $\vec{x}(t, y^0)$ and $\vec{p}(t, y^0)$ using a Taylor series near the origin ($t \to 0$) for an initial plane wave, we get

$$\vec{R}(t = 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \vec{Q}(t = 0) = \frac{\partial p(0)}{\partial y^0}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{16}$$

In this work we have considered two different random fields:

(i) The case of an inhomogeneous medium exhibiting only temperature fluctuations, we called it the scalar case, hence (13) and (15) can be simplified making use of $N^2 = 1 + T'/T_0$, $\vec{M} = 0$.
(ii) The case of a medium exhibiting only velocity fluctuations and (13) and (15) can be simplified making use of $N = 1$, we called it the vectorial case.

In both cases we solved the differential system ((13)–(15)) by a Runge–Kutta fourth-order scheme. The time step is $dt = 1.0/c_0 K_{\max}$. It is important to note that the description of the temperature $T'(\vec{x})$ and the velocity $\vec{v}'(\vec{x})$ in terms of Fourier modes, allows us to obtain analytically all the spatial derivatives needed in the resolution of the differential system. Numerical errors are then reduced and computation time is saved in comparison with the usual finite-difference approximations. More details can be found in Blanc-Benon *et al.* (1990) and Karweit *et al.* (1991).

5. Results for the Random Temperature Field

Figure 1 shows a typical example of ray-tracing through a random temperature field with a scale L of 0.1 m and a fluctuating temperature level θ/T_0 of 1.176×10^{-2}. The initially linear trajectories are launched at regular-space intervals in the transversal direction ($\Delta_y = 0.015$ m; 161 rays). The rays are plotted in terms of the nondimensional variables x/L and y/L. During the first correlation lengths the rays are only slightly distorted and the initial plane wave structure is still apparent. Afterwards the deflections of the rays are amplified, neighboring rays begin to cross, and strong concentrations occur at different transverse positions y. Caustics appear in a concentrated zone x/L, $15 \leq x/L \leq 25$. After passing through a focus the families of rays have large expansion fans in the y-direction. Locally the wave behaves like a spherical wave, and the initial plane wave structure is totally destroyed. It is possible to distinguish a second zone of ray crossing around $x/L = 50$, defining a new region of emergence of caustics. The process of formation of caustics can be clearly visualized by launching a family of rays greater by a factor of 15 (i.e., 2401 rays in the same realization). For each ray the exact position of occurrence of a caustic (where the ray-tube section vanishes) is computed. In Figure 2 we have plotted the first set of caustics by marking the first point of zero cross-sectional area along the rays. The cusps which characterize the two-dimensional caustics are clearly visible. At increasing x/L the structure of the field becomes more complex with overlaping of caustics.

To quantify this phenomena, we evaluate the probability of occurrence of the first caustic for a family of 81 rays over an ensemble of 200 realizations. With these 16,200 samples we evaluate the p.d.f., using 256 classes. In this calculation we have used three scalar fields with the same scale L and three different values of θ/T_0 (2.352×10^{-2}; 1.176×10^{-2}; 5.882×10^{-3}). The corresponding p.d.f.'s are plotted in Figure 3 as a function of the arc length s. For each curve we notice that up to a certain length s along the ray there is no caustic. Then we observe a sharp peak at a fixed distance and an exponential decay at larger distances. If the level of fluctuations is higher, the first caustic appears at a shorter distance.

These results confirm the theoretical analysis of Kulkarny and White (1982), and these also extend significantly the numerical simulation of Hesselink and Sturtevant (1988). These authors have studied the propagation of a plane wave through an isotropic random medium with weak fluctuations in the index of refraction ($N = 1 + \mu$; $\mu = \varepsilon/2$), assuming that the fluctuations μ are predominantly due to the inhomogeneity of the medium. Using a geometrical acoustic approach, they demonstrate that caustics occur along every ray. For the first caustic, the p.d.f. of occurrence is given by a universal

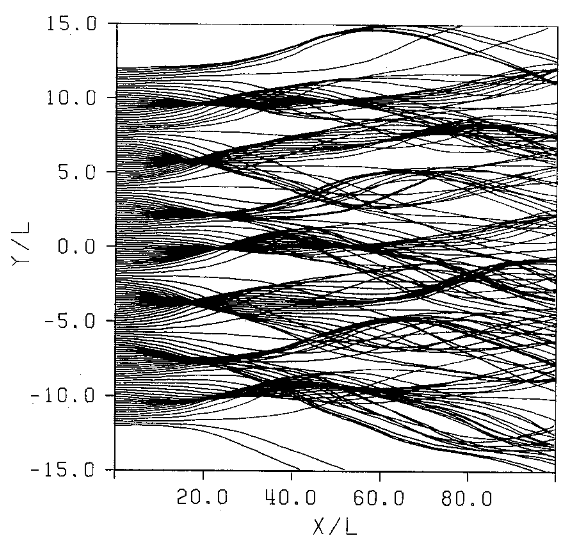

Figure 1. Ray-tracing through a single realization of a two-dimensional isotropic scalar field ($\theta/T_0 = 1.176 \times 10^{-2}$, $L =$ 0.1 m).

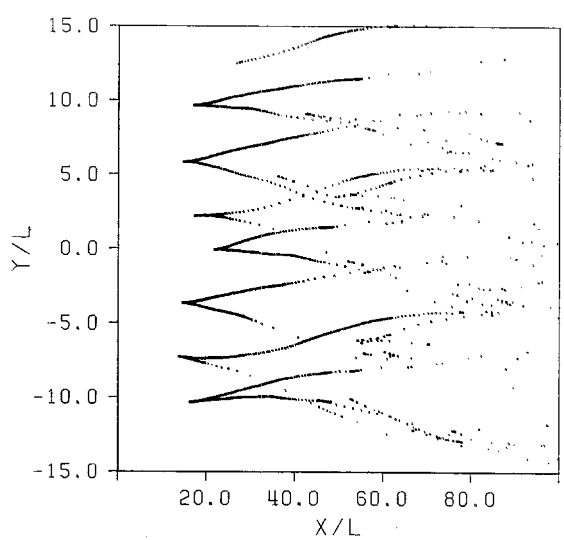

Figure 2. Caustics formation in a single realization of a two-dimensional isotropic scalar field ($\theta/T_0 = 1.176 \times 10^{-2}$, $L =$ 0.1 m).

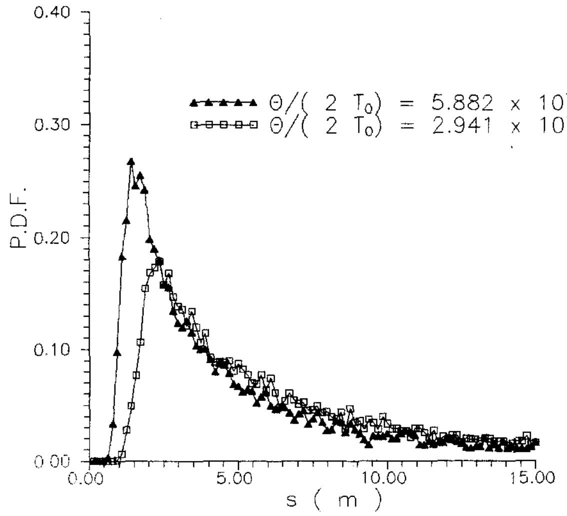

Figure 3. Probability densities of distances to the first caustic for two-dimensional scalar fields ($L = 0.1$ m).

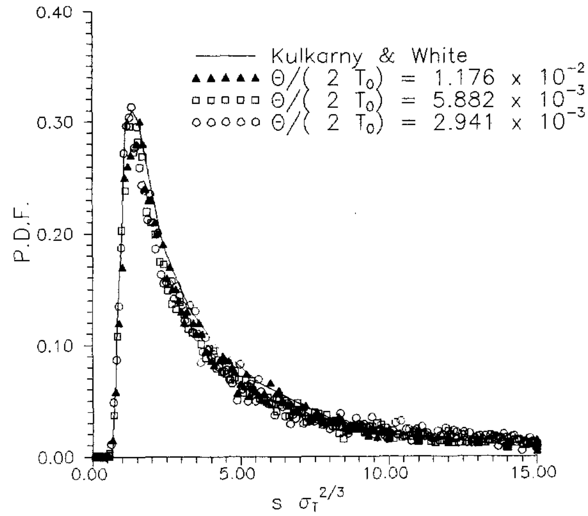

Figure 4. Rescaling of the probability densities curves in terms of the Kulkarny and White parameter σ_T for two-dimensional scalar fields.

curve characterized by a single scaling factor related to the space correlation function of the medium:

$$R_\mu(\vec{r}) = \langle \mu(\vec{x})\mu(\vec{x} + \vec{r})\rangle. \tag{17}$$

In Figure 4 we report this universal curve using the theoretical scaling factor $\sigma_T^{2/3}$ suggested by Kulkarny and White (1982):

$$\sigma_T^2 = \int_{-\infty}^{+\infty} \frac{\partial^4}{\partial r_2^4} R_\mu(r_1, 0)\, dr_1. \tag{18}$$

For a Gaussian correlation function $R_\mu(r) = \langle \mu^2 \rangle \exp(-r^2/L^2)$, it can be be shown that $\sigma_T^{2/3}$ takes the form

$$\sigma_T^{2/3} = \frac{12^{1/3}\langle \mu^2 \rangle^{1/3} \pi^{1/6}}{L} \tag{19}$$

with $\sqrt{\langle \mu^2 \rangle} = \theta/2T_0$. The data from our numerical experiments are in good agreement with Kulkarny and White's (1982) predictions. We note that the peak of the p.d.f. is obtained for a value of $s\sigma_T^{2/3}$ equal to 1.3, which permits an estimation of the distance at which the formation of the first caustic is most probable.

6. Results for the Random Velocity Field

Now we consider the case of an initial plane wave propagating through a turbulent velocity field ($\vec{V} = \vec{v}\,'$), in order to develop a comparison with the previous scalar field and to examine the possible extension of the results from a scalar to a vectorial field. In our simulation we chose the following values of the length scale L ($L = 0.1$ m) and of the r.m.s. velocity v' (1 m/s; 2 m/s; 4 m/s). At this point we must recall that the fluctuations of the index $\varepsilon(\vec{x})$ in the Helmholtz equation may be obtained by the same value of $T'(\vec{x})/2T_0$ and $v_1'(\vec{x})/c_0$. Our choice of L, T', and v' satisfies this requirement.

Figure 5 shows the behavior of a family of 161 rays propagating through one realization of a simulated velocity field with $v' = 2$ m/s. The ray distortion seems to be notably higher compared with a scalar field. In Figure 6, which concerns the same realization, we have reported all the points at which caustics occur. We note that the structure of the picture is again more complex than for the scalar field, and that the distance of formation of caustics is much shorter.

Again using 200 realizations of a family of 81 rays, we have computed the p.d.f. of occurrence of the first caustic. In Figure 7 we compare the p.d.f. obtained with a scalar field and a vectorial field

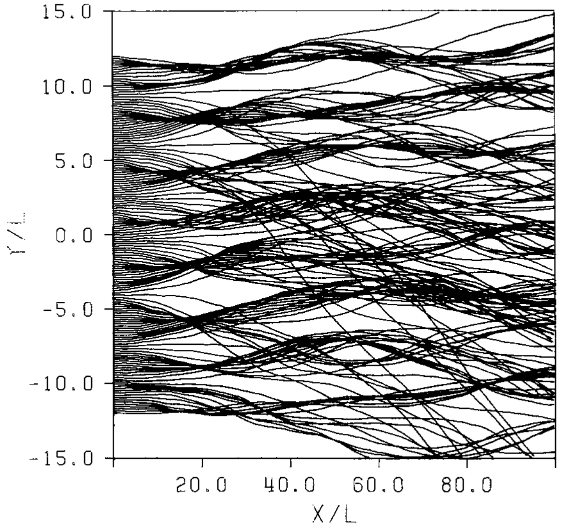

Figure 5. Ray-tracing through a single realization of a two-dimensional isotropic vectorial field ($v'/c_0 = 5.882 \times 10^{-3}$, $L = 0.1$ m).

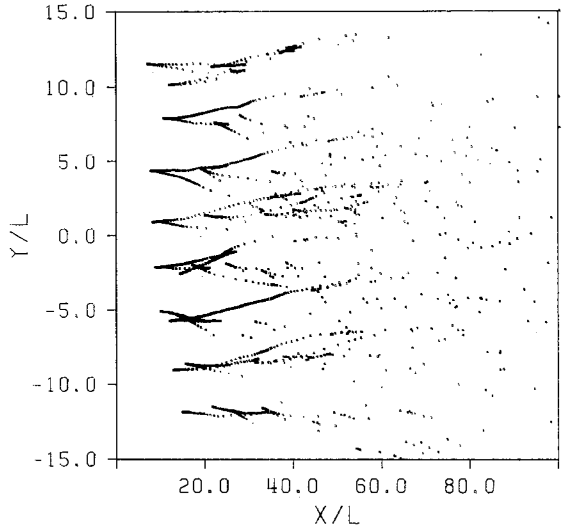

Figure 6. Caustics formation in a single realization of a two-dimensional isotropic vectorial field ($v'/c_0 = 5.882 \times 10^{-3}$, $L = 0.1$ m).

with the same level for the index fluctuation, $\sqrt{\langle \mu^2 \rangle} = 5.582 \times 10^{-3}$, with either $\theta = 3.447$ K or $v' = 2$ m/s. The two curves exhibit the same trend, but the distance of formation of a caustic appears shorter for the velocity field.

In Figure 8 we have introduced the scaling factor $\sigma_V^{2/3}$, which deals with the component of the velocity field $v_1'(\vec{x})$ in the direction of propagation. According to the Helmholtz equation, this component gives the most important contribution to the fluctuation of the index of refraction. We therefore define σ_V^2 as

$$\sigma_V^2 = \frac{1}{c_0^2} \int_{-\infty}^{+\infty} \frac{\partial^4}{\partial r_2^4} R_{11}(r_1, 0)\, dr_1, \tag{20}$$

where R_{11} is the correlation function of v_1' in the direction of propagation. With the Gaussian

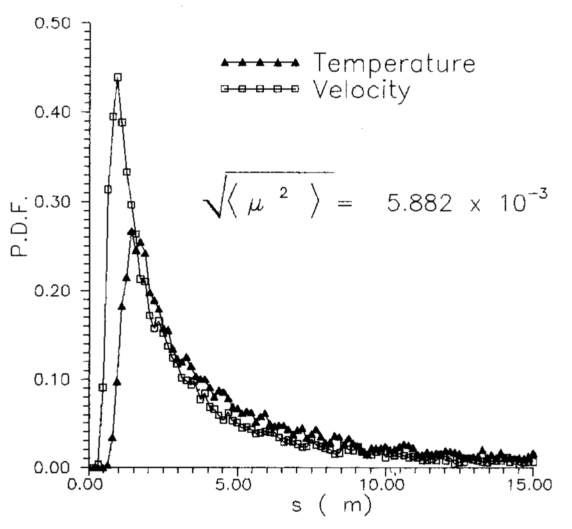

Figure 7. Probability density of distance to the first caustic: comparison between a scalar case and a vectorial case with the same r.m.s. value of the index of fluctuation $\sqrt{\langle \mu^2 \rangle}$.

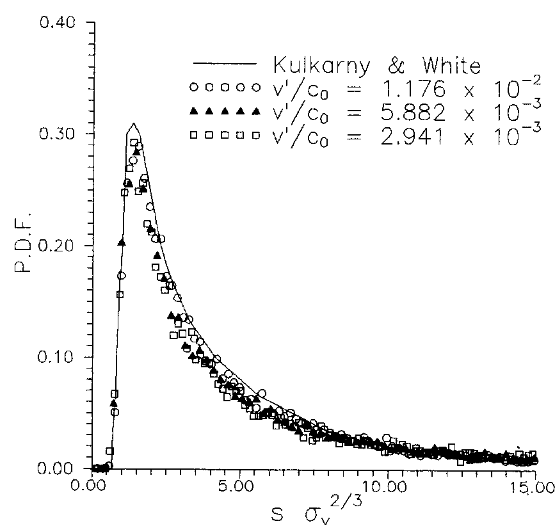

Figure 8. Rescaling of the probability densities curves in terms of the parameter σ_V for two-dimensional vectorial fields.

function used previously,

$$R_{11}(r) = v'^2 \exp\left(\frac{-r^2}{L^2}\right), \tag{21}$$

we obtain, after some computation,

$$\sigma_V^{2/3} = \frac{60^{1/3}\langle\mu^2\rangle^{1/3}\pi^{1/6}}{L} \tag{22}$$

with $\sqrt{\langle\mu^2\rangle} = v'/c_0$. Together with our results, we have plotted in Figure 8 the universal curve of Kulkarny and White (1982). We note that our simulations are in very good agreement with these theoretical results. The scaling parameter $\sigma_V^{2/3}$ therefore seems to be very effective in predicting the occurrence of caustics in a larger variety of situations.

7. Conclusion

In this paper we have simulated the propagation of acoustic plane waves in two-dimensional isotropic random fields with either temperature fluctuations (scalar case) or velocity fluctuations (vectorial case), in the geometric acoustics approximation. We have characterized the region of random caustic formation. In the scalar case, our simulations confirm the theoretical analysis of Kulkarny and White (1982): the probability density of the distance along a ray to the nearest caustic is given by a universal curve with a single distance scale parameter. In the vectorial case we extend the results by defining the appropriate parameter, and we demonstrate that the representation of a velocity field by a refractive index such as in the Helmholtz equation requires special scrutiny.

So far the turbulent fields have been assumed to have Gaussian correlation functions. Extensions of this numerical simulation to more realistic turbulent fields involving several characteristic length scales and three-dimensional aspects are therefore to be considered. They will permit us to analyze the validity limits of the Kulkarny and White parameter. In addition to the geometrical acoustics point of view, the parabolic approximation can be developed using the same numerical process to generate the turbulent fields. The main interest will be to analyse the influence of the wave frequency on the intensity fluctuations of the transmitted pressure field, in connection with the occurrence of caustics.

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