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Short Note

# On the application of explicit spatial filtering to the variables or fluxes of linear equations

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## 1. Introduction

The need for filtering high-frequency waves is a recurrent issue in numerical simulations. These waves might indeed lead to instability, and they are in general not calculated accurately by the discretization algorithms. Selective filters have therefore been designed in order to damp high-frequency waves without affecting significantly low-frequency disturbances [1–6]. These filters are particularly used in computational aeroacoustics, but they appear also suitable for Large-Eddy Simulations (LES), in which only the scales larger than the grid size are computed, and whose equations are derived formally by applying a filter operator to the Navier–Stokes equations [7]. Moreover LES based specially on explicit filtering have been also developed [8–10].

In practice, the flow variables are usually filtered explicitly after each time step. Consider for example the time integration of the following differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \tag{1}$$

where $u$ is the unknown variable and the flux operator $F$ is a function of $u$. The solution at $(n+1)$th iteration at time $t + \Delta t$ is obtained from solution $u_n$ at $n$th iteration at time $t$, where $\Delta t$ is the time step, in the following way: the numerical integration of Eq. (1) provides $u_{n+1}$ at time $t + \Delta t$, which is then filtered, yielding the solution $\bar{u}_{n+1}$ at $(n+1)$th iteration (the bar denotes the filtering). In this case, the dissipative effects of the filtering on the large scales depend on the filters applied.

In order to minimize undesirable damping for any filter, the explicit filtering of the flow fluxes has been proposed instead of that of the variables [3]. The following equation is then solved:

$$\frac{\partial u}{\partial t} + \frac{\partial \overline{F}(u)}{\partial x} = 0 \tag{2}$$

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The fluxes are thus filtered before derivation during the time integration. This method has been used for LES [11] and for Direct Numerical Simulations [12], and enables to avoid the cumulative dissipative effects that might result from the multiple filterings of the flow variables after each time step. Unfortunately other spurious negative effects are likely to be produced.

In this note, the influence of filtering the variables or the fluxes is investigated for a linear operator $F(u)$. Results are shown for standard explicit centered high-order schemes, namely the 10th-order finite differences and the 6th, 8th, 10th and 12th-order filters, whose coefficients can be found in the Appendix and in Ref. [5] for instance. The accuracies in phase and in amplitude are presented in the wave number space, and they are illustrated by the solutions of a test case.

## 2. Effects of spatial filtering in the wave number space

### 2.1. Filtering of variables

For simplicity, the one-dimensional wave equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0 \tag{3}$$

discretized on a mesh with uniform spacing $\Delta x$ is considered.

The spatial derivative is approximated by centered, $(2N+1)$-point, finite differences, yielding

$$\frac{\partial u}{\partial x}(x)=\frac{1}{\Delta x}\sum_{j=-N}^{N}a_j u(x+j\Delta x) \tag{4}$$

where the scheme coefficients are such as $a_j=-a_{-j}$, ensuring no dissipation. By applying spatial Fourier transform to (4) as in [13] for instance, the effective wave number $k^{fd}$ of the scheme is obtained

$$k^{fd}\Delta x=2\sum_{j=1}^{N}a_j\sin(jk\Delta x) \tag{5}$$

The spatial derivation thus leads to a numerical wave number $k^{fd}$ that differs from the exact wave number $k$. The phase errors $E_k=(k\Delta x-k^{fd}\Delta x)/\pi$ obtained for the 10th-order finite differences are represented in Fig. 1a as a function of the wave number $k\Delta x$. They are negligible for low wave numbers, but significant for high wave numbers. More quantitatively, the accuracy limit of the scheme, estimated from the arbitrary criterium $E_k\leqslant 5\times10^{-4}$ and expressed in term of number of points per wave length, is $\lambda_k/\Delta x=5.25$.

In Fig. 1a, waves for $k\Delta x=\pi$ are shown not to be properly calculated. They can be damped by applying a central, $(2N+1)$-point filter to variable $u$ after each time step, providing

$$\bar{u}(x)=u(x)-\sigma_d[D_p(u)](x) \tag{6}$$

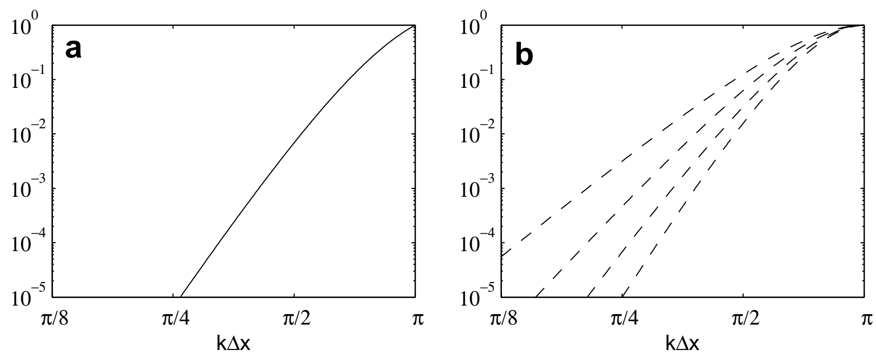


Fig. 1. Representation in logarithmic scales, as a function of the wave number $k\Delta x$ of: (a) phase error $(k\Delta x-k^{fd}\Delta x)/\pi$ obtained for the standard 10th-order finite differences, (b) damping functions $\widehat{D}_p(k\Delta x)$ of the standard 6th, 8th, 10th and 12th-order filters.

with

$$[D_p(u)](x) = \sum_{j=-N}^{N} d_j u(x + j\Delta x) \tag{7}$$

where the filter coefficients are such as $d_j = d_{-j}$, ensuring no phase error, and $\sigma_d$ is a constant between 0 and 1. The application of Fourier transform to (7) gives the damping function of the filter

$$\widehat{D}_p(k\Delta x) = d_0 + 2\sum_{j=1}^{N} d_j \cos(jk\Delta x) \tag{8}$$

in the wave number space. The damping functions obtained for the 6th, 8th, 10th and 12th-order filters are presented in Fig. 1b. Grid-to-grid oscillations are found to be removed by the filters, whereas low wave numbers are weakly affected. To determine an accuracy limit for each filter, the criterium $\sigma_d\widehat{D}_p \leqslant 5 \times 10^{-4}$ is used. The variables being filtered after each iteration, the effects of filtering are cumulative, and it is not necessary to set $\sigma_d = 1$. A value of $\sigma_d = 0.2$ is therefore chosen here. The limits obtained for the filters are reported in Table 1 in terms of number of points per wave length. They range from $\lambda_d/\Delta x = 8.33$ for the 6th-order filter down to $\lambda_d/\Delta x = 4.82$ for the 12th-order filter.

When centered schemes are used and that filtering is applied explicitly to the flow variables, the spatial discretization thus generates phase errors due to the derivation and damping due to the filtering, which can be evaluated separately as in Table 1.

### 2.2. Filtering of fluxes

In the case of flux filtering, the flow fluxes are filtered before spatial derivation in order to remove grid-to-grid oscillations, so that the frequency content of the solution is controlled. The following equation is then solved:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[u - D_p(u)] = 0 \tag{9}$$

The application of spatial Fourier transform, denoted by a hat, yields

$$\frac{\partial \hat{u}}{\partial t} + \mathrm{i}k^{fd}[1 - \widehat{D}_p(k\Delta x)]\hat{u} = 0 \tag{10}$$

where the expressions of $k^{fd}$ and $\widehat{D}_p$ are given by (5) and (8). The numerical wave number associated with the spatial discretization is now $k^\star = k^{fd}[1 - \widehat{D}_p(k\Delta x)]$, which suggests that the explicit filtering of the fluxes does not introduce additional dissipation of the variable, but modifies the wave number calculated by the finite differences.

These effects of flux filtering are shown in Fig. 2a and b, where, respectively, the approximated wave number $k^\star\Delta x$ and the phase errors $E_k = (k\Delta\, x - k^\star\Delta x)/\pi$, obtained for 10th-order finite differences used in combination with 6th, 8th, 10th and 12th-order filters, are presented as functions of the exact wave number $k\Delta x$. Filtering the fluxes is clearly found to increase the phase errors, both for low and high wave numbers. The deterioration in phase accuracy is however less important when higher order filters are used. These observa-

Table 1
Accuracy limits in phase and in amplitude, $\lambda_k$ and $\lambda_d$, of the algorithm combining the 10th order finite differences with the 6th, 8th, 10th and 12th filtering of the variables and of the fluxes

| | Variables ($\sigma_d = 0.2$) | | Fluxes | |
|---|---|---|---|---|
| | $\lambda_k/\Delta x$ | $\lambda_d/\Delta x$ | $\lambda_k/\Delta x$ | $\lambda_d/\Delta x$ |
| 6th order | 5.25 | 8.33 | 8.57 | – |
| 8th order | 5.25 | 6.38 | 6.79 | – |
| 10th order | 5.25 | 5.40 | 5.94 | – |
| 12th order | 5.25 | 4.82 | 5.54 | – |

Note that $\lambda_d/\Delta x = 5.61$ when a 12th filtering of the variables is used with $\sigma_d = 1$.

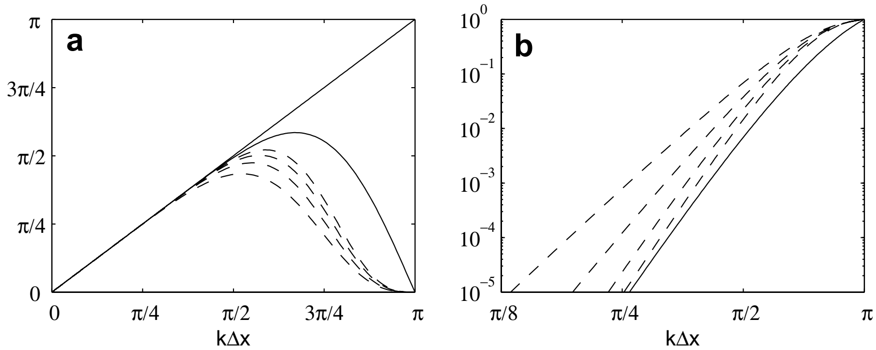


Fig. 2. Representation as a function of the wave number $k\Delta x$ of: (a) approximated wave number $k^{\star}\Delta x$, in linear scales, and (b) phase error $E_k = (k\Delta x - k^{\star}\Delta x)/\pi$, in logarithmic scales, obtained using the standard 10th-order finite differences, —— alone, and ---- in combination with explicit 6th, 8th, 10th and 12th-order filtering of the fluxes.

tions are supported by the accuracy limits reported in Table 1, which are estimated from $E_k \leqslant 5 \times 10^{-4}$ as previously. The limits obtained using flux filtering are indeed for instance $\lambda_k/\Delta x = 8.57$ for the 6th-order filter and $\lambda_k/\Delta x = 5.54$ for the 12th-order filter, which is higher than the limit $\lambda_k/\Delta x = 5.25$ obtained without filtering.

In summary, the numerical errors due to the filtering of the flow fluxes are added to those of the spatial derivation, which decreases the accuracy in phase.

## 3. Test case

### 3.1. Definition and reference solutions

A one-dimensional problem is considered to illustrate the influence of spatial filtering. The wave equation (3) is solved on a mesh with uniform spacing $\Delta x$. The disturbances at $t = 0$ are defined as

$$u(x) = \sin\left(\frac{2\pi x}{8\Delta x}\right)\exp\left[-\ln(2)\left(\frac{x}{3\Delta x}\right)^2\right]$$

In the wave number space, as shown in [5], they are characterized by a dominant component for $k\Delta x = \pi/4$, corresponding to a wave length of $8\Delta x$, and also by significant components for $0 < k\Delta x < \pi/2$. To emphasize the errors resulting from the spatial discretization, the perturbations are propagated over $800\Delta x$, and Eq. (3) is integrated in time by a low-storage 6-stage Runge–Kutta algorithm [5], using the small time step $\Delta t = 0.2\Delta x$. The spatial derivation is taken into account by 10th-order explicit finite differences.

The solution calculated without filtering is displayed in Fig. 3. Compared to the exact solution, it is slightly distorted by the finite differences. The error rate evaluated between the exact and the calculated solutions as

$$e_{\text{num}} = \left(\sum(u_{\text{cal}} - u_{\text{exact}})^2 \Big/ \sum u_{\text{exact}}^2\right)^{1/2} \tag{11}$$

is in this case $e_{\text{num}} = 0.307$.

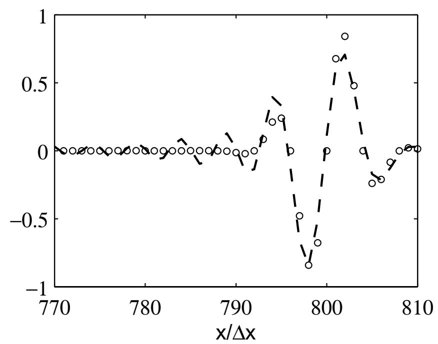


Fig. 3. Test case: ○ exact solution at $t = 800$, ---- numerical solution obtained without filtering.

### 3.2. Solutions obtained using filterings

The solutions obtained using filtering of the variables after each time step with $\sigma_d = 0.2$ are presented in Fig. 4, and compared with the solution calculated without filtering. The dissipative effects of the filtering are visible, especially with the filters of 6th and 8th order in Fig. 4a and b, but they are significantly reduced with filters of higher order. The error rates with respect to the exact solution are thus observed in Table 2 to decrease from 0.814 with the 6th-order filter down to 0.245 for the 12th-order filter. The error obtained with the 12th-order filter is moreover smaller than that without filtering, which indicates a higher numerical accuracy. In this case, the filtering affects only wave numbers that are not properly calculated by the finite differences, as shown in Table 1.

The solutions obtained using flux filtering are displayed in Fig. 5. They are calculated by solving Eq. (9). Flux filtering is therefore applied at each stage of the Runge–Kutta algorithm. With respect to the solution without filtering, the solutions show a larger dispersion of the disturbances, with the strengthening of the tail of the wave packet. This is observed with the 6th-order filter in Fig. 5a, as well as with the 12th-order filter in Fig. 5d. The additional errors in phase due to flux filtering however decrease using filters of higher order. The error rates between the numerical and the exact solutions in Table 2 are thus between 1.234 and 0.404, which is still larger than the error of 0.307 obtained without filtering. As the order of the filter increases, one can indeed

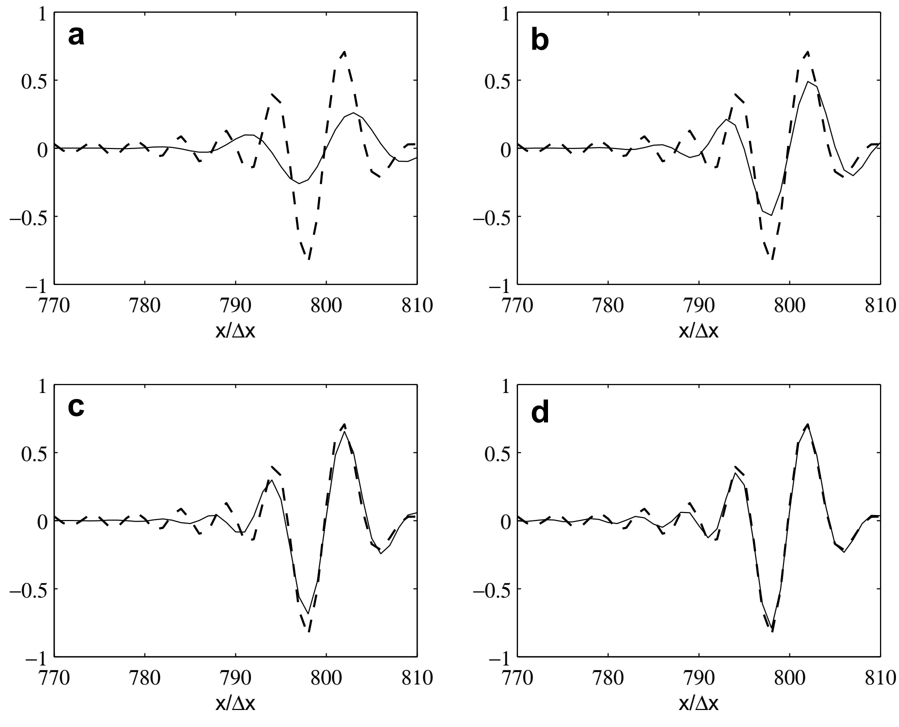


Fig. 4. Test case: ---- numerical solution obtained without filtering, —— solutions using explicit filtering of the variables ($\sigma_d = 0.2$). Filters of: (a) 6th order, (b) 8th order, (c) 10th order, (d) 12th order.

Table 2
Test case: errors $e_{\mathrm{num}}$ between the exact solution and the numerical solutions obtained using explicit 6th, 8th, 10th and 12th-order filtering of the variables ($\sigma_d = 0.2$) and of the fluxes; without filtering, $e_{\mathrm{num}} = 0.307$

| | Variables | Fluxes |
|---|---|---|
| 6th order | 0.814 | 1.234 |
| 8th order | 0.532 | 0.860 |
| 10th order | 0.315 | 0.558 |
| 12th order | 0.245 | 0.404 |

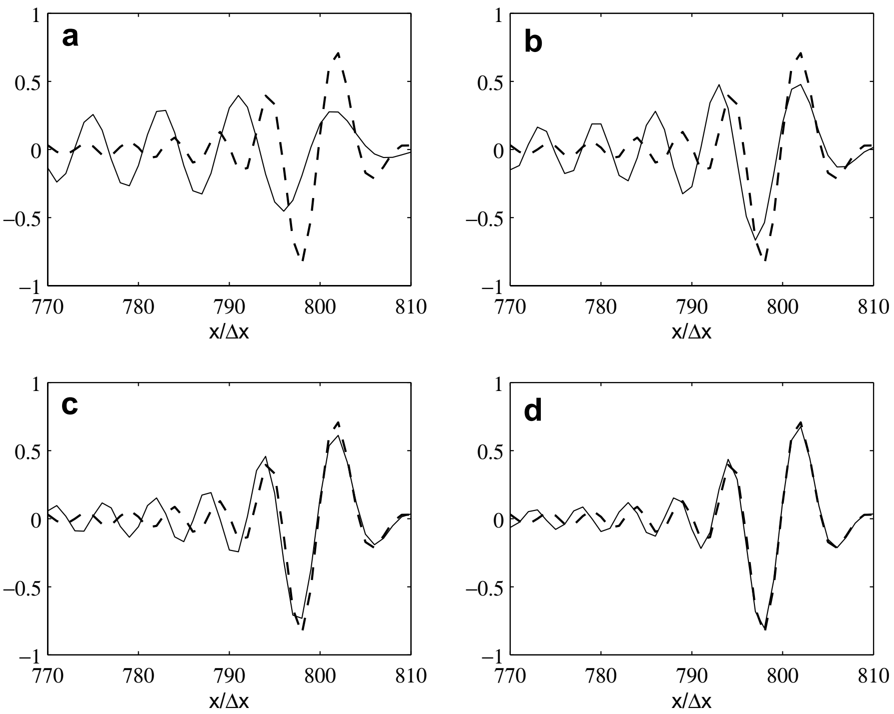


Fig. 5. Test case: ––– numerical solution obtained without filtering, —— solutions using explicit filtering of the fluxes. Filters of: (a) 6th order, (b) 8th order, (c) 10th order, (d) 12th order.

expect the solution to progressively collapse the solution obtained without filtering, and consequently the error rate would tend to 0.307 while remaining larger.

## 4. Concluding remarks

The applications of explicit spatial filtering to the flow variables and to the fluxes in simulations might both generate numerical artifacts, which are not of the same nature. For linear equations, using centered schemes, the filtering of the variables might lead to additional dissipation of the solution, whereas filtering the fluxes might decrease the accuracy in phase. These unwanted effects can be minimized by using filters of high-order, namely at least of the order of the spatial differentiation. Filtering the variables may however appear more relevant than filtering the fluxes because it can enable to remove high-frequency waves that are not properly calculated, and consequently to improve the numerical solutions. The computational cost of filtering the variables is also smaller, because the variables can be filtered every time step, or even every *n*th time step, whereas flux filtering must be applied at each stage of the time integration algorithm. Finally, there might be a problem for numerical stability using flux filtering, because this method does not directly remove grid-to-grid oscillations, but only prevents their generation during the simulation.

## Appendix

The coefficients of the 6th-order centered explicit filter are $d_0 = 5/16$, $d_1 = -15/64$, $d_2 = 3/32$ and $d_3 = -1/64$, and $d_{-j} = d_j$. The coefficients of the other schemes used in this note can be found in Ref. [5] for instance.

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