Acoustic microstreaming produced by nonspherical oscillations of a gas bubble. II. Case of modes 1 and *m*

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This paper continues a study that was started in our previous paper [A. A. Doinikov *et al.*, Phys. Rev. E **100**, 033104 (2019)]. The overall aim of the study is to develop a theory for modeling the velocity field of acoustic microstreaming produced by nonspherical oscillations of an acoustically driven gas bubble. In the previous paper, general equations were derived that describe the velocity field of acoustic microstreaming produced by modes n and m of bubble oscillations. In the present paper, the above equations are solved analytically in the case that acoustic microstreaming is the result of the interaction of the translational mode (mode 1) with a mode of arbitrary order $m \geqslant 1$. Solutions are expressed in terms of complex mode amplitudes, which means that the mode amplitudes are assumed to be known and serve as input data for the calculation of the velocity field of acoustic microstreaming. No restrictions are imposed on the ratio of the bubble radius to the viscous penetration depth. Analytical results are illustrated by numerical examples.

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I. INTRODUCTION

In Part I of our study [1], equations were derived for the velocity field of acoustic microstreaming that is produced by modes n and m of oscillations of a gas bubble; see Sec. II C of Part I [1]. The aim of the present paper is to apply the above general equations to the case that acoustic microstreaming is produced by the interaction of the translational mode (mode 1) with a mode of arbitrary order $m \geqslant 1$.

The case $1-1$, where only mode 1 is involved, and the case of modes 1 and m with $m > 1$ are shown in Part I [1] to be described by different equations. Therefore, the present calculation is divided into two parts. In Sec. II A, a solution for the case $1-1$ is derived, while the case $1 - m$ with $m > 1$ is considered in Sec. II B.

The case $1-1$ was considered previously by Davidson and Riley [2] and Longuet-Higgins [3]. We consider this case in a different formulation. In Refs. [2] and [3], it is assumed that the bubble is fixed while the liquid oscillates about it. This means that the liquid at infinity has a unidirectional velocity. Conversely, we assume that the bubble is moving while the liquid at infinity is at rest. Our results show that these assumptions lead to different solutions for the streaming. The fact that these two cases are not equivalent as far as acoustic streaming is concerned is also confirmed by results of Wu and Du [4], which are presented in more detail below. Another important distinctive feature of our solutions is that they do not impose any restrictions on the ratio of the bubble radius to the viscous penetration depth, whereas the results obtained in Refs. [2] and [3] are valid only when the bubble radius is much greater than the viscous penetration depth.

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II. THEORY

We consider a gas bubble undergoing axisymmetric oscillations, which include the radial pulsation (mode 0), translation (mode 1), and shape modes of order $m \geqslant 2$. The liquid motion produced by the bubble oscillations is described by spherical coordinates r and θ whose origin is at the equilibrium center of the bubble. The geometry of the problem is depicted by Fig. 1 of Part I [1].

A. Acoustic microstreaming produced by mode 1 alone

According to the theory developed in Part I [1], in the case $1-1$, the Eulerian streaming velocity is represented by

$$\langle \boldsymbol{v}_2^{11} \rangle = \nabla \times \left[\langle \psi_2^{11}(r,\theta) \rangle \boldsymbol{e}_\varepsilon \right], \tag{1}$$

where $\langle\rangle$ denotes the time average, $\boldsymbol{e}_\varepsilon$ is the unit azimuth vector, and $\langle \psi_2^{11} \rangle$ is the amplitude of the vector potential of the streaming velocity that is calculated from Eq. (33) of Part I, in which n is set equal to 1, giving the following result:

$$\left(\Delta_{r\theta} - \frac{1}{r^2 \sin^2\theta} \right)^2 \langle \psi_2^{11} \rangle = \frac{\mu\sqrt{1-\mu^2}}{\nu r^2} \mathrm{Re}\left\{ k_1^2 a_1 b_1^* \left(\frac{R_0}{r} \right)^2 \left[x_1 h_1^{(1)\prime}(x_1) - h_1^{(1)}(x_1) \right]^* - k_1^2 b_1 b_1^* x_1 h_1^{(1)\prime}(x_1) h_1^{(1)*}(x_1) \right\}. \tag{2}$$

In Eq. (2), $\Delta_{r\theta}$ denotes the $r\theta$ part of the Laplace operator (see Appendix A), $\mu = \cos\theta$, $x_1 = k_1 r$, $k_1 = (1+i)/\delta_1$, $\delta_1 = \sqrt{2\nu/\omega_1}$, ν is the kinematic liquid viscosity, ω_1 is the frequency of mode 1, R_0 is the bubble radius at rest, a_1 and b_1 are linear scattering coefficients (see Appendix A), $h_n^{(1)}$

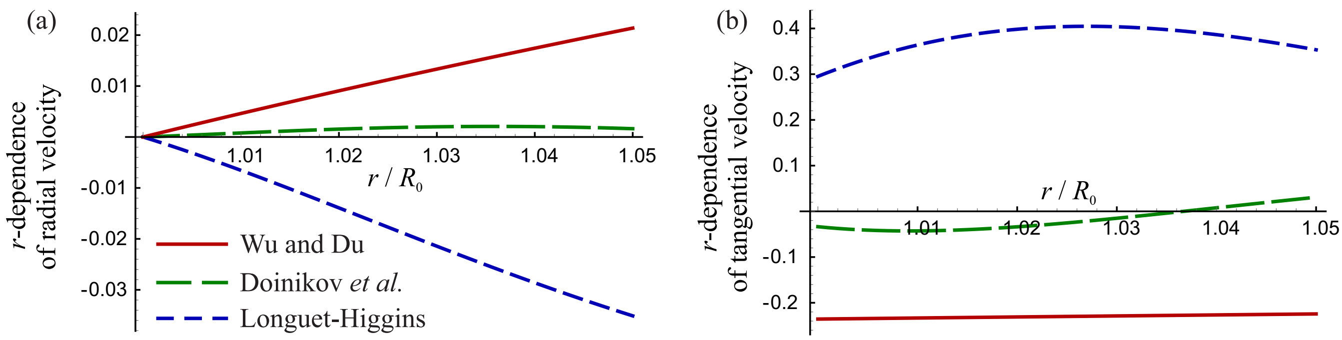

FIG. 1. Case 1−1. Comparison of the streaming velocity components given by different theories inside the viscous boundary layer.

is the spherical Hankel function of the first kind, $h_n^{(1)\prime}(x_1) = dh_n^{(1)}(x_1)/dx_1$, and the asterisk denotes complex conjugate.

A solution to Eq. (2) (see Appendix A) is given by

$$\langle \psi_2^{11} \rangle = \mu\sqrt{1-\mu^2}\frac{|b_1|^2}{6\nu}\mathrm{Re}\{F(x_1)\}, \qquad (3)$$

where the function $F(x_1)$ is defined by Eq. (A12). Substituting Eq. (3) into Eq. (1) yields the following expressions for the radial and tangential components of the streaming velocity:

$$\langle v_{2r}^{11} \rangle = \frac{|b_1|^2}{3\nu r}\mathrm{Re}\{F(x_1)\}P_2(\mu), \qquad (4)$$

$$\langle v_{2\theta}^{11} \rangle = -\frac{|b_1|^2}{6\nu r}\mathrm{Re}\{F(x_1) + x_1F'(x_1)\}\mu\sqrt{1-\mu^2}, \qquad (5)$$

where P_2 is the Legendre polynomial of order 2 and the function $F'(x_1)$ is defined by Eq. (A21).

It should be emphasized that Eqs. (4) and (5) give the components of the Eulerian streaming velocity, the functions F and F' specifying the dependence of these components on distance. To calculate the Lagrangian streaming velocity, Eqs. (4) and (5) are added with the components of the Stokes drift velocity, which are given by Eqs. (A28) and (A29). A MATLAB code for the calculation of the Eqs. (4), (5), (A28), and (A29) is provided as Supplemental Material [5].

As said in the Introduction, we consider a case different from that considered by Davidson and Riley [2] and Longuet-Higgins [3]. We assume that the bubble is moving and the liquid at infinity is at rest, whereas the above authors assume that the bubble is fixed and the liquid at infinity is moving. The streaming velocity, as a nonlinear effect, is different in these two cases. This inference follows from our results and is corroborated by results of Wu and Du [4].

Wu and Du [4] derived approximate solutions for the streaming velocity within the thin viscous boundary layer at the outer and inner surface of a gas bubble undergoing the monopole and dipole vibrations. They assumed that the gas inside the bubble was viscous and used the non-slip boundary conditions. Therefore, their main solutions cannot be correctly compared to those of Davidson and Riley [2] and Longuet-Higgins [3], as well as our solutions. However, we can use limiting equations obtained by Wu and Du [4] in the case that the gas viscosity tends to zero, Eqs. (26′) and (28′) in their paper, which give the streaming velocity in the case 1−1 within the boundary layer outside the bubble. We cannot perform an exact quantitative comparison as Wu and Du [4] use a quantity u_0 called by them "the velocity amplitude of a sound source." It is not clear how to correctly recalculate this quantity to the translational amplitude used in our theory and in the theories of Davidson and Riley [2] and Longuet-Higgins [3]. However, we can compare the sign of the components of the streaming velocity inside the viscous boundary layer.

It follows from the theory of Wu and Du [4] that u_0 can be treated as the amplitude of the liquid velocity generated by the incident acoustic wave at the center of the bubble as if the bubble were absent. Then the following relation between u_0 and the magnitude of the translational velocity of the bubble, v_b, can be written: $v_b = \omega_1|s_1| = 3u_0$ [6], where s_1 is the complex amplitude of mode 1 used in our theory. Substituting this relation into Eqs. (26′) and (28′) of Ref. [4], we can write the components of the streaming velocity derived by Wu and Du in the following form:

$$v_{\mathrm{WD}r} = -\frac{\omega_1^2|s_1|^2R_0^3\rho_g}{2\eta r^2}\left(1-\frac{r}{R_0}\right)P_2(\mu), \qquad (6)$$

$$v_{\mathrm{WD}\theta} = -\frac{\omega_1^2|s_1|^2R_0^2\rho_g}{4\eta r}\mu\sqrt{1-\mu^2}, \qquad (7)$$

where η is the dynamic liquid viscosity and ρ_g is the equilibrium gas density.

According to the theory of Longuet-Higgins [3], in case 1−1, the components of the Lagrangian streaming velocity inside the viscous boundary layer are calculated by

$$v_{\mathrm{LH}r} = \frac{18\omega_1|s_1|^2\delta_1^2}{R_0r^2} \times\left(e^{-\xi}\cos\xi - 1 + \frac{1}{2}\xi e^{-\xi}\cos\xi - \frac{3}{20}\xi\right)P_2(\mu), \qquad (8)$$

$$v_{\mathrm{LH}\theta} = \frac{9\omega_1|s_1|^2\delta_1}{4R_0r}\left[2e^{-\xi}(\cos\xi + 2\sin\xi) + 2\xi e^{-\xi}(\cos\xi+\sin\xi) + \frac{3}{5}\right]\mu\sqrt{1-\mu^2}, \qquad (9)$$

where $\xi = (r-R_0)/\delta_1$. These equations follow from Eq. (6.5) of Ref. [3].

In Fig. 1, we compare the dependence on r given by Eqs. (6)–(9) to the results of our theory. Since Longuet-Higgins [3] states that his results for the case 1−1 are identical to those of Davidson and Riley [2], we only provide the results of Longuet-Higgins [3] in Fig. 1. The simulations were made at the following values of the physical parameters: $R_0 = 50\,\mu$m, $f = \omega_1/2\pi = 50$ kHz, the liquid density $\rho = 1000\,\mathrm{kg/m^3}$, $\eta = 0.001$ Pa s, and $\rho_g = 1.2\,\mathrm{kg/m^3}$. The components of the streaming velocity were normalized by

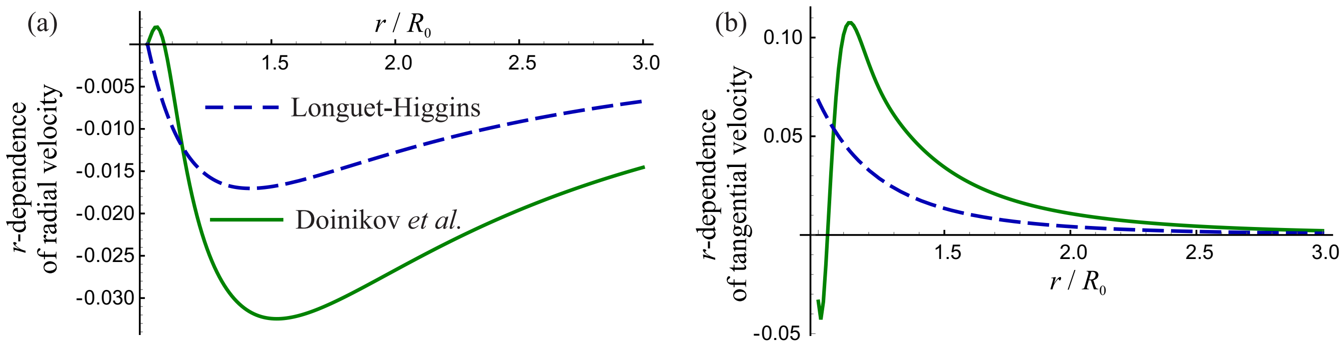

FIG. 2. Case 1−1. Comparison of the streaming velocity components given by Longuet-Higgins' and our theories outside the viscous boundary layer.

the factor $\omega_1|s_1|^2/R_0$. The comparison is carried out within the viscous boundary layer, from $r/R_0 = 1$ up to $r/R_0 = 1 + \delta_1/R_0$, where $\delta_1/R_0 = 0.05$ for the above mentioned parameters. As one can see, the velocity components of Wu and Du [4] are of the same sign as those predicted by our theory inside the viscous boundary layer, whereas the velocity components of Longuet-Higgins [3] are of opposite sign.

Outside the viscous boundary layer, the theory of Longuet-Higgins [3] gives

$$v_{\mathrm{LH}r} = \frac{27\omega_1|s_1|^2\delta_1}{20R_0^2}\left(\frac{R_0^4}{r^4} - \frac{R_0^2}{r^2}\right)P_2(\mu), \tag{10}$$

$$v_{\mathrm{LH}\theta} = \frac{27\omega_1|s_1|^2R_0^2\delta_1}{20r^4}\mu\sqrt{1-\mu^2}. \tag{11}$$

These equations follow from Eq. (6.7) of Ref. [3]. Figure 2 compares the dependence on r given by Eqs. (10) and (11) to the results of our theory. The parameters are the same as in Fig. 1. The velocity components are normalized by the factor $\omega_1|s_1|^2/R_0$. As one can see, our theory predicts a greater velocity magnitude. However, it should be emphasized once again that Fig. 2 compares two different physical cases.

B. Acoustic microstreaming produced by modes 1 and m with $m > 1$

In the case $1 - m$, the Eulerian streaming velocity is represented by

$$\langle \boldsymbol{v}_2^{1m}\rangle = \nabla \times \left[\langle \psi_2^{1m}(r,\theta)\rangle \boldsymbol{e}_\varepsilon\right], \tag{12}$$

where $\langle\psi_2^{1m}\rangle$ is calculated from Eq. (32) of Part I [1], in which n is set equal to 1, leading to

$$\begin{aligned}
&\left(\Delta_{r\theta} - \frac{1}{r^2\sin^2\theta}\right)^2\langle\psi_2^{1m}\rangle \\
&= \frac{1}{\nu r^2}\mu P_m^1(\mu)\mathrm{Re}\left\{k_1^2 a_1 b_m^*\left(\frac{R_0}{r}\right)^2\left[2h_m^{(1)}(x_1) - x_1 h_m^{(1)\prime}(x_1)\right]^* - k_1^2 b_1 b_m^*\left[x_1 h_1^{(1)\prime}(x_1)h_m^{(1)*}(x_1) + x_1^* h_1^{(1)}(x_1)h_m^{(1)\prime *}(x_1)\right]\right\} \\
&\quad - \frac{m+1}{2\nu r^2}\sqrt{1-\mu^2}P_m(\mu)\mathrm{Re}\left\{k_1^2 a_m b_1^*\left(\frac{R_0}{r}\right)^{m+1}\left[(m+1)h_1^{(1)}(x_1) - x_1 h_1^{(1)\prime}(x_1)\right]^*\right. \\
&\quad \left. - m k_1^2 b_m b_1^*\left[x_1 h_m^{(1)\prime}(x_1)h_1^{(1)*}(x_1) + x_1^* h_m^{(1)}(x_1)h_1^{(1)\prime *}(x_1)\right]\right\} \\
&\quad + \frac{\sqrt{1-\mu^2}}{2\nu r^2}\left[\sqrt{1-\mu^2}P_m^1(\mu)\right]'\mathrm{Re}\left\{k_1^2 h_m^{(1)*}(x_1)\left[a_1 b_m^*\left(\frac{R_0}{r}\right)^2 - b_1 b_m^*\left[h_1^{(1)}(x_1) + x_1 h_1^{(1)\prime}(x_1)\right]\right]\right. \\
&\quad \left. + k_1^2 h_1^{(1)*}(x_1)\left[a_m b_1^*\left(\frac{R_0}{r}\right)^{m+1} - b_m b_1^*\left[h_m^{(1)}(x_1) + x_1 h_m^{(1)\prime}(x_1)\right]\right]\right\}.
\end{aligned} \tag{13}$$

Here, the scattering coefficients a_m and b_m are defined by Eqs. (B1) and (B2), P_m is the Legendre polynomial of order m, and P_m^1 is the associated Legendre polynomial of the first order and of degree m.

A solution to Eq. (13) (see Appendix B) is given by

$$\langle\psi_2^{1m}\rangle = \frac{1}{\nu}\mathrm{Re}\left\{b_1^* b_m\left[\mu P_m^1(\mu)F_1(x_1) + \sqrt{1-\mu^2}P_m(\mu)F_2(x_1)\right]\right\}, \tag{14}$$

where the functions $F_1(x_1)$ and $F_2(x_1)$ are defined by Eqs. (B35) and (B36). Substituting Eq. (14) into Eq. (12) yields the following expressions for the radial and tangential components of the streaming velocity:

$$\begin{aligned}\langle v_{2r}^{1m}\rangle = -\frac{1}{\nu r}\mathrm{Re}\left\{b_1^* b_m\left[\mu P_m(\mu)[m(m+1)F_1(x_1) - 2F_2(x_1)]\right.\right. \\ \left.\left. + \sqrt{1-\mu^2}P_m^1(\mu)[F_1(x_1) - F_2(x_1)]\right]\right\},\end{aligned} \tag{15}$$

$$\langle v_{2\theta}^{1m}\rangle = -\frac{1}{\nu r}\mathrm{Re}\{b_1^* b_m[\mu P_m^1(\mu)[F_1(x_1) + x_1 F_1'(x_1)] + \sqrt{1-\mu^2}P_m(\mu)[F_2(x_1) + x_1 F_2'(x_1)]]\}, \quad (16)$$

where the functions $F_1'(x_1)$ and $F_2'(x_1)$ are defined by Eqs. (B41) and (B42).

It should be emphasized that Eqs. (15) and (16) give the components of the Eulerian streaming velocity. As one can see, the dependence of these components on distance is determined by the functions F_1, F_2 and their derivatives. Note that the above functions are independent of the phase shift between the modes. The phases of the modes are included in the coefficients b_1 and b_m, which, as Eqs. (A3) and (B2) show, are proportional to the complex amplitudes of the modes, s_1 and s_m. These amplitudes are defined as $s_m = |s_m| \exp(i\phi_m)$, where $|s_m|$ and ϕ_m are the magnitude and the phase of mode m, respectively.

To calculate the Lagrangian streaming velocity, Eqs. (15) and (16) are added with the components of the Stokes drift velocity, which are given by Eqs. (B52) and (B53). A MATLAB code for the calculation of Eqs. (15), (16), (B52), and (B53) is provided as Supplemental Material [5].

The case $1-m$ with $m > 1$ was considered previously by Spelman and Lauga [7]. Just as in the case of the microstreaming produced by modes 0 and m discussed in Part I [1], the difference between their theoretical model and ours is that they assume that the bubble is fixed while the liquid oscillates around it, whereas we assume that the bubble is moving while the liquid at infinity is at rest. This means that at infinity the first-order liquid velocity tends to a nonzero value in their case. Hence our model and the one of Spelman and Lauga describe two different physical situations. Figure 3 compares the radial and tangential components of the Lagrangian velocity given by the theory of Spelman and Lauga [7] and the present model, for the microstreaming produced by modes 1 and 4. The simulations were made at the following values of physical parameters: $\rho = 1000\,\mathrm{kg/m^3}$, $\eta = 0.001$ Pa s, $f = 50$ kHz, and $R_0 = 50\,\mu m$. To eliminate from consideration the magnitudes of the modes $|s_1|$ and $|s_4|$, the streaming velocity was normalized by the factor $R_0/(\omega_0|s_1||s_4|)$. The phase shift between modes 1 and 4 is set to 0. Because of the relative complexity of the equations for the Eulerian or Stokes drift velocities [Eqs. (15), (16), (B52), and (B53)] containing terms with different dependence on theta, we plot the streaming velocities for particular angles: $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. Both models show similar behavior for the evolution of the Lagrangian streaming velocities. However, as in the case $0-m$ [1], a difference in the amplitudes of the streaming velocities is observed, which, however, decreases with increasing r.

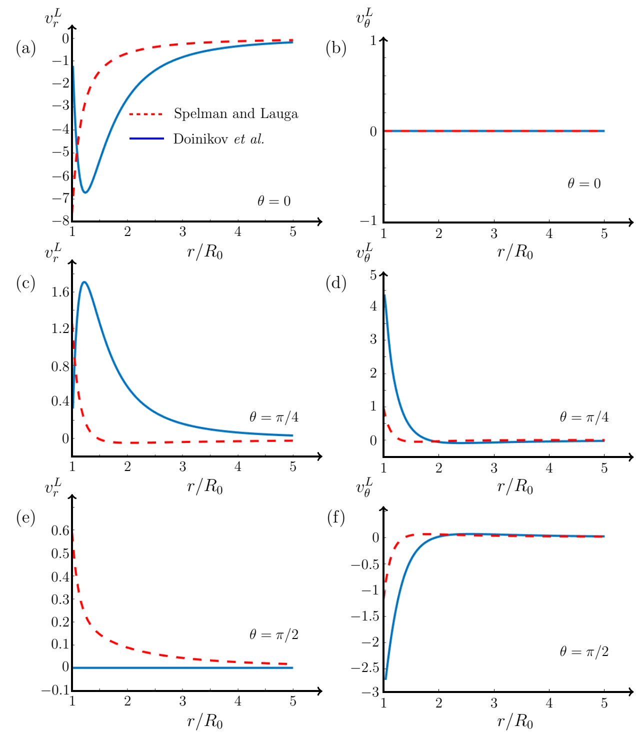

FIG. 3. Evolution of the radial (left column) and tangential (right column) components of the Lagrangian streaming velocity given by the theory of Spelman and Lauga [7] and the present model for modes 1 and 4. The velocity components are plotted for three angles: (a), (b) $\theta = 0$; (c), (d) $\theta = \pi/4$; (e), (f) $\theta = \pi/2$.

III. NUMERICAL RESULTS

Numerical simulations were made at the following values of physical parameters: $\rho = 1000\,\mathrm{kg/m^3}$, $\eta = 0.001$ Pa s, $f = 50$ kHz, and $R_0 = 100\,\mu$m. The streaming velocity was normalized by the factor $\omega_1|s_1||s_m|/R_0$.

Figure 4 exemplifies Lagrangian streamline patterns produced by modes $1-1$, $1-2$, $1-3$, and $1-4$. The phase shift between the modes was set zero. As one can see, the main vortices have a form of lobes. The numerical examples presented in Fig. 4 show that for modes 1 and m with $m > 1$, the number of lobes is equal to $2(m-1)$. It is interesting to note that the streamline patterns in the cases $1-1$ and $1-3$ look identical.

Figure 5 shows the dependence of the normalized magnitude of the Eulerian streaming velocity on the distance from the bubble surface at various values of the phase shift between modes. The case of modes 1 and 3 is presented. The variation of the streaming velocity along three directions is shown: $\theta = 0, \theta = \pi/4$, and $\theta = \pi/2$. As one can see, a change in the phase shift leads to a considerable change in the magnitude of the streaming velocity. As the phase shift increases, the magnitude of the streaming velocity first decreases but then again increases. The bend of the $\phi = \pi/6$ curve in Fig. 5(c) results from the fact that the sign of the velocity changes at this spatial point.

IV. CONCLUSIONS

In the present paper, a general theory developed in our previous paper [1] has been applied to the case that acoustic microstreaming is produced by the interaction between the bubble translation (mode 1) and a mode of arbitrary order $m \geqslant 1$. Since the case $1-1$, where only mode 1 is involved, and the case of modes 1 and m with $m > 1$ are described by different equations [1], solutions were obtained separately for these cases. Analytical results were then used to carry out numerical simulations. The simulations have shown that in the

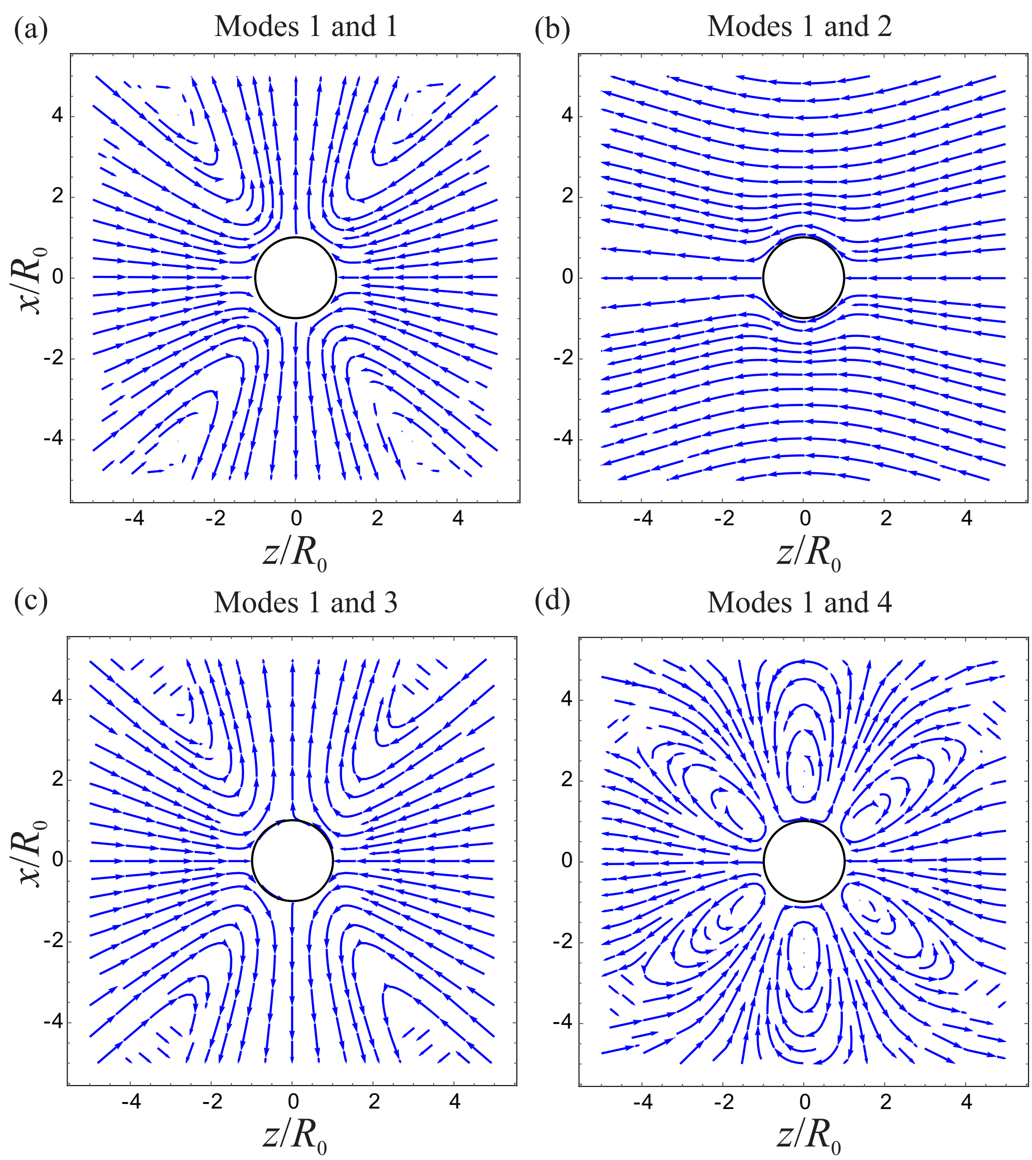

FIG. 4. Numerical examples of streamline patterns produced by various mode pairs.

case $1-m$ with $m>1$ streamlines form lobes whose number is equal to $2(m-1)$.

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APPENDIX A: SOLUTION OF EQ. (2)

Let us first define the operator $\Delta_{r\theta}$ and the constants a_1 and b_1 that appear in Eq. (2).

According to Eq. (A9) of Part I [1], $\Delta_{r\theta}$ is given by

$$\Delta_{r\theta}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right). \tag{A1}$$

The constants a_1 and b_1 are known as the linear scattering coefficients of, respectively, the potential and the vortical parts of the scattered wave from the bubble. According to Eqs. (18) and (19) of Part I [1], they are calculated by

$$a_1=\frac{iR_0\omega_1 s_1\bar{x}_1^2 h_1^{(1)\prime\prime}(\bar{x}_1)}{2\left[\bar{x}_1^2 h_1^{(1)\prime\prime}(\bar{x}_1)+6h_1^{(1)}(\bar{x}_1)\right]}, \tag{A2}$$

$$b_1=\frac{3iR_0\omega_1 s_1}{\bar{x}_1^2 h_1^{(1)\prime\prime}(\bar{x}_1)+6h_1^{(1)}(\bar{x}_1)}, \tag{A3}$$

where s_1 is the complex amplitude of mode 1 and $\bar{x}_1=k_1R_0$.

Making use of Eqs. (A1)–(A3) to express $\Delta_{r\theta}$ in terms of x_1 and μ and a_1 in terms of b_1, Eq. (2) is transformed to

$$\begin{aligned}D^2\langle\psi_2^{11}\rangle &= \mu\sqrt{1-\mu^2}\frac{k_1^4|b_1|^2}{6\nu x_1^4}\\ &\quad\times\mathrm{Re}\left\{\bar{x}_1^4 h_1^{(1)\prime\prime}(\bar{x}_1)\left[x_1 h_1^{(1)\prime}(x_1)-h_1^{(1)}(x_1)\right]^*\right.\\ &\quad\left.-6x_1^3 h_1^{(1)\prime}(x_1)h_1^{(1)*}(x_1)\right\},\end{aligned} \tag{A4}$$

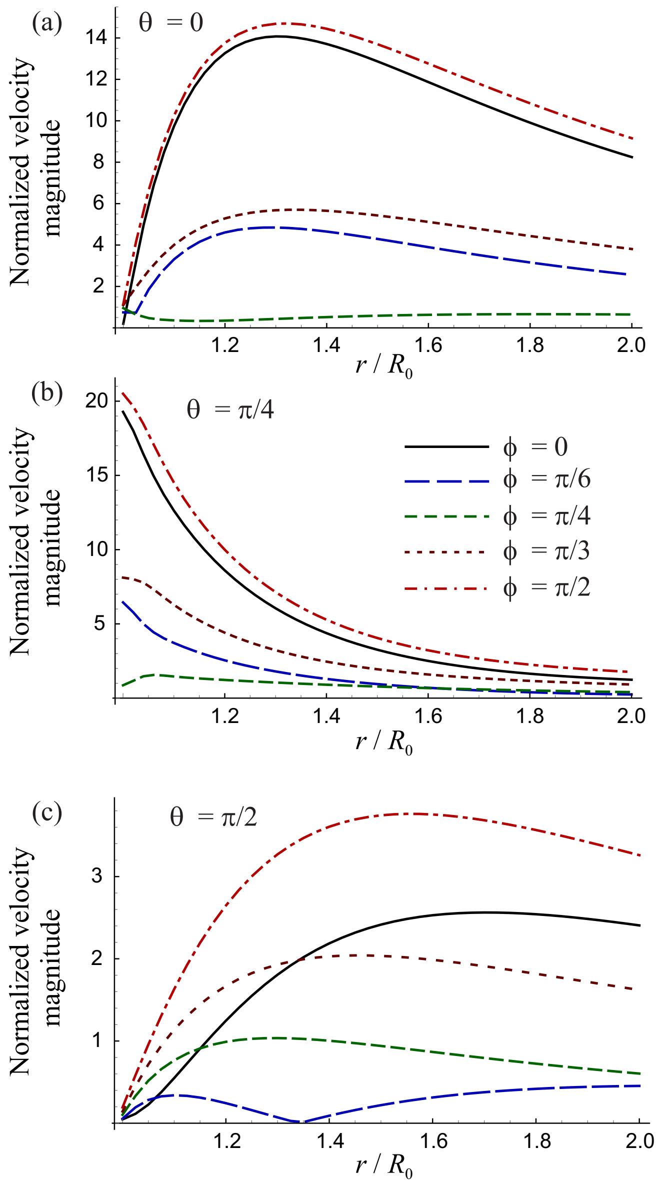

FIG. 5. Case 1 – 3. Dependence of the magnitude of the Eulerian streaming velocity on the distance from the bubble surface at various values of the phase shift ϕ between the modes. The velocity variation along three directions is shown: (a) $\theta = 0$, (b) $\theta = \pi/4$, (c) $\theta = \pi/2$.

where the operator D is given by

$$D = \frac{k_1^2}{x_1^2}\left[\frac{\partial}{\partial x_1}\left(x_1^2\frac{\partial}{\partial x_1}\right) + (1-\mu^2)\frac{\partial^2}{\partial\mu^2} - 2\mu\frac{\partial}{\partial\mu} - \frac{1}{1-\mu^2}\right]. \tag{A5}$$

Taking $\langle\psi_2^{11}\rangle$ in the form

$$\langle\psi_2^{11}\rangle = \mu\sqrt{1-\mu^2}\frac{|b_1|^2}{6\nu}\mathrm{Re}\{F(x_1)\} \tag{A6}$$

and substituting into Eq. (A4), one obtains the following equation for $F(x_1)$:

$$\frac{d^4F}{dx_1^4} + \frac{4}{x_1}\frac{d^3F}{dx_1^3} - \frac{12}{x_1^2}\frac{d^2F}{dx_1^2} + \frac{24F}{x_1^4} = G(x_1), \tag{A7}$$

where $G(x_1)$ is given by

$$G(x_1) = \frac{1}{x_1^4}\{\bar{x}_1^4 h_1^{(1)\prime\prime}(\bar{x}_1)[x_1 h_1^{(1)\prime}(x_1) - h_1^{(1)}(x_1)]^* - 6x_1^3 h_1^{(1)\prime}(x_1)h_1^{(1)*}(x_1)\}. \tag{A8}$$

Equation (A7) is solved by the method of variation of parameters [8], which means that we first solve a homogeneous equation that corresponds to Eq. (A7),

$$\frac{d^4F}{dx_1^4} + \frac{4}{x_1}\frac{d^3F}{dx_1^3} - \frac{12}{x_1^2}\frac{d^2F}{dx_1^2} + \frac{24F}{x_1^4} = 0. \tag{A9}$$

Assuming that partial solutions to Eq. (A9) are given by x^λ and substituting them into Eq. (A9), one obtains a polynomial for λ,

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 4\lambda(\lambda-1)(\lambda-2) - 12\lambda(\lambda-1) + 24 = 0. \tag{A10}$$

The roots of Eq. (A10) are –3, –1, 2, and 4, which means that the general solution of Eq. (A9) is given by

$$\frac{C_1}{x_1^3} + \frac{C_2}{x_1} + C_3x_1^2 + C_4x_1^4, \tag{A11}$$

and hence the solution of Eq. (A7) can be written as

$$F(x_1) = \frac{C_1(x_1)}{x_1^3} + \frac{C_2(x_1)}{x_1} + C_3(x_1)x_1^2 + C_4(x_1)x_1^4, \tag{A12}$$

where $C_n(x_1)$ should obey the following system of equations:

$$\begin{aligned} C_1'y_1 + C_2'y_2 + C_3'y_3 + C_4'y_4 &= 0,\\ C_1'y_1' + C_2'y_2' + C_3'y_3' + C_4'y_4' &= 0,\\ C_1'y_1'' + C_2'y_2'' + C_3'y_3'' + C_4'y_4'' &= 0,\\ C_1'y_1''' + C_2'y_2''' + C_3'y_3''' + C_4'y_4''' &= G(x_1). \end{aligned} \tag{A13}$$

Here, the prime denotes the derivative with respect to x_1 and the functions y_n are given by

$$y_1 = x_1^{-3}, \quad y_2 = x_1^{-1}, \quad y_3 = x_1^2, y_4 = x_1^4, \tag{A14}$$

Solving system (A13) for C_n' and integrating the solutions, one obtains

$$C_1(x_1) = C_{10} - \frac{1}{70}\int_{\bar{x}_1}^{x_1} s^6G(s)ds, \tag{A15}$$

$$C_2(x_1) = C_{20} + \frac{1}{30}\int_{\bar{x}_1}^{x_1} s^4G(s)ds, \tag{A16}$$

$$C_3(x_1) = C_{30} - \frac{1}{30}\int_{\bar{x}_1}^{x_1} sG(s)ds, \tag{A17}$$

$$C_4(x_1) = C_{40} + \frac{1}{70}\int_{\bar{x}_1}^{x_1} \frac{G(s)}{s}ds, \tag{A18}$$

where C_{n0} are constants to be determined by boundary conditions.

To apply the boundary conditions, we first calculate the components of the Eulerian streaming velocity, using

Eq. (A6),

$$\langle v_{2r}^{11}\rangle = -\frac{1}{r}\frac{\partial}{\partial\mu}\left(\langle\psi_2^{11}\rangle\sqrt{1-\mu^2}\right) = \frac{|b_1|^2}{6\nu r}(3\mu^2-1)\mathrm{Re}\{F(x_1)\}, \quad \text{(A19)}$$

$$\langle v_{2\theta}^{11}\rangle = -\frac{1}{r}\frac{\partial}{\partial x_1}\left(x_1\langle\psi_2^{11}\rangle\right) = -\frac{|b_1|^2}{6\nu r}\mu\sqrt{1-\mu^2}\mathrm{Re}\{F(x_1)+x_1F'(x_1)\}, \quad \text{(A20)}$$

where, as follows from Eqs. (A12) and (A13),

$$F'(x_1) = -\frac{3C_1(x_1)}{x_1^4} - \frac{C_2(x_1)}{x_1^2} + 2C_3(x_1)x_1 + 4C_4(x_1)x_1^3. \quad \text{(A21)}$$

From the condition $\langle \boldsymbol{v}_2^{11}\rangle \to 0$ for $r\to\infty$, it follows that

$$C_{30} = \frac{1}{30}\int_{\bar{x}_1}^{\infty} sG(s)ds, \quad \text{(A22)}$$

$$C_{40} = -\frac{1}{70}\int_{\bar{x}_1}^{\infty}\frac{G(s)}{s}ds. \quad \text{(A23)}$$

To apply boundary conditions at the bubble surface, we need the Lagrangian streaming velocity, which is defined by [3]

$$\boldsymbol{v}_L^{11} = \langle\boldsymbol{v}_2^{11}\rangle + \boldsymbol{v}_S^{11}, \quad \text{(A24)}$$

where $\boldsymbol{v}_S^{11}$, called the Stokes drift velocity, is calculated by [3]

$$\boldsymbol{v}_S^{11} = \left\langle\left(\left(\int \boldsymbol{v}_1^{11}dt\cdot\nabla\right)\boldsymbol{v}_1^{11}\right\rangle = \frac{1}{2\omega_1}\mathrm{Re}\{i(\boldsymbol{v}_1^{11}\cdot\nabla)\boldsymbol{v}_1^{11*}\}, \quad \text{(A25)}$$

$\boldsymbol{v}_1^{11}$ being the linear liquid velocity produced by mode 1.

From Eqs. (11) and (12) of Part I [1], it follows that

$$v_{1r}^{11} = -e^{-i\omega_1 t}\frac{k_1b_1}{3}\frac{\mu}{x_1^3}\left[\bar{x}_1^4h_1^{(1)''}(\bar{x}_1)+6x_1^2h_1^{(1)}(x_1)\right], \quad \text{(A26)}$$

$$v_{1\theta}^{11} = -e^{-i\omega_1 t}\frac{k_1b_1}{6}\frac{\sqrt{1-\mu^2}}{x_1^3} \times\left[\bar{x}_1^4h_1^{(1)''}(\bar{x}_1)-6x_1^3h_1^{(1)'}(x_1)-6x_1^2h_1^{(1)}(x_1)\right]. \quad \text{(A27)}$$

Substituting Eqs. (A26) and (A27) into Eq. (A25) yields

$$v_{Sr}^{11} = \frac{|b_1|^2}{6\omega_1 r^3}(1-3\mu^2)\mathrm{Re}\left\{6ix_1h_1^{(1)*}(x_1)h_1^{(1)'}(x_1) - i\frac{\bar{x}_1^4h_1^{(1)''}(\bar{x}_1)}{x_1^2}\left[2h_1^{(1)}(x_1)+x_1h_1^{(1)'}(x_1)\right]^*\right\}, \quad \text{(A28)}$$

$$v_{S\theta}^{11} = \frac{|b_1|^2}{6\omega_1 r^3}\mu\sqrt{1-\mu^2}\mathrm{Re}\left\{6ix_1^2h_1^{(1)}(x_1)h_1^{(1)''*}(x_1) + i\frac{\bar{x}_1^4h_1^{(1)''}(\bar{x}_1)}{x_1^2}\left[6h_1^{(1)}(x_1)-x_1^2h_1^{(1)''}(x_1)\right]^*\right\}. \quad \text{(A29)}$$

The boundary conditions at the bubble surface are written as (see Part I [1] for more detail)

$$v_{Lr}^{11} = 0 \quad \text{at} \quad r = R_0, \quad \text{(A30)}$$

$$\frac{1}{r}\frac{\partial v_{Lr}^{11}}{\partial\theta} + \frac{\partial v_{L\theta}^{11}}{\partial r} - \frac{v_{L\theta}^{11}}{r} = 0 \quad \text{at} \quad r = R_0. \quad \text{(A31)}$$

Substituting Eqs. (A19), (A20), (A28), and (A29) into Eqs. (A30) and (A31) yields

$$C_{10} + \bar{x}_1^2C_{20} = A, \quad \text{(A32)}$$

$$16C_{10} + 6\bar{x}_1^2C_{20} = B, \quad \text{(A33)}$$

where

$$A = -C_{30}\bar{x}_1^5 - C_{40}\bar{x}_1^7 + \bar{x}_1^3h_1^{(1)''*}(\bar{x}_1)\left[2h_1^{(1)}(\bar{x}_1)+\bar{x}_1h_1^{(1)'}(\bar{x}_1)\right] - 6\bar{x}_1^2h_1^{(1)'}(\bar{x}_1)h_1^{(1)*}(\bar{x}_1), \quad \text{(A34)}$$

$$B = -6C_{30}\bar{x}_1^5 - 16C_{40}\bar{x}_1^7 + \bar{x}_1^4h_1^{(1)'''}(\bar{x}_1)\left[\bar{x}_1^2h_1^{(1)''*}(\bar{x}_1)-6h_1^{(1)*}(\bar{x}_1)\right] - 2\bar{x}_1^3h_1^{(1)''}(\bar{x}_1)\left[2\bar{x}_1^2h_1^{(1)''*}(\bar{x}_1)+3\bar{x}_1^*h_1^{(1)'*}(\bar{x}_1)-6h_1^{(1)*}(\bar{x}_1)\right] + 48\bar{x}_1^3h_1^{(1)''*}(\bar{x}_1)h_1^{(1)}(\bar{x}_1) - 36\bar{x}_1^2h_1^{(1)*}(\bar{x}_1)h_1^{(1)'}(\bar{x}_1). \quad \text{(A35)}$$

It follows from Eqs. (A32) and (A33) that

$$C_{10} = \frac{B-6A}{10}, \quad \text{(A36)}$$

$$C_{20} = \frac{16A-B}{10\bar{x}_1^2}. \quad \text{(A37)}$$

APPENDIX B: SOLUTION OF EQ. (13)

From Eqs. (18) and (19) of Part I [1], it follows that

$$a_m = \frac{\bar{x}_m^2h_m^{(1)''}(\bar{x}_m)-(m-1)(m+2)h_m^{(1)}(\bar{x}_m)}{2(m+2)}b_m \text{ for } m\geqslant 1, \quad \text{(B1)}$$

$$b_m = \frac{2iR_0(m+2)\omega_ms_m}{(m+1)\left[\bar{x}_m^2h_m^{(1)''}(\bar{x}_m)+(m^2+3m+2)h_m^{(1)}(\bar{x}_m)\right]}, \quad \text{(B2)}$$

where $\bar{x}_m = k_mR_0$, $k_m = (1+i)/\delta_m$, $\delta_m = \sqrt{2\nu/\omega_m}$, ω_m is the frequency of the mth mode, and s_m is the complex amplitude of the mth mode. In the case under consideration, it is assumed that modes 1 and m oscillate at the same frequency ω_1, so $\omega_m = \omega_1$, $k_m = k_1$, and $\bar{x}_m = \bar{x}_1$.

By using Eqs. (B1) and (B2) and the transformation

$$\sqrt{1-\mu^2}\left[\sqrt{1-\mu^2}P_m^1(\mu)\right]' = m(m+1)\sqrt{1-\mu^2}P_m(\mu), \quad \text{(B3)}$$

Eq. (13) is rearranged to

$$D^2\langle\psi_2^{1m}\rangle = \frac{k_1^4}{\nu}\mathrm{Re}\left\{b_1^*b_m\left[\mu P_m^1(\mu)G_1(x) + \sqrt{1-\mu^2}P_m(\mu)G_2(x)\right]\right\}, \quad \text{(B4)}$$

where, for convenience, we denote $x_1 = x$ and $\bar{x}_1 = \bar{x}$, the operator D is given by Eq. (A5), and $G_1(x)$ and $G_2(x)$ are calculated by

$$G_1(x) = \frac{\bar{x}^4}{6x^4} h_1^{(1)\prime\prime *}(\bar{x})\big[2h_m^{(1)}(x) - x h_m^{(1)\prime}(x)\big] + \frac{1}{x} h_1^{(1)*}(x) h_m^{(1)\prime}(x) - \frac{1}{x^*} h_1^{(1)\prime *}(x) h_m^{(1)}(x), \tag{B5}$$

$$G_2(x) = m(m+1) h_m^{(1)}(x)\left[\frac{\bar{x}^4}{12x^4} h_1^{(1)\prime\prime}(\bar{x}) - \frac{1}{x} h_1^{(1)\prime}(x)\right]^* + \frac{(m+1)\bar{x}^{m+1}}{4(m+2)x^{m+5}}\big[\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})\big]\big[x^2 h_1^{(1)}(x) - x^3 h_1^{(1)\prime}(x)\big]^*. \tag{B6}$$

A solution to Eq. (B4) is sought in the following form:

$$\langle \psi_2^{1m} \rangle = \frac{1}{\nu} \mathrm{Re}\big\{b_1^* b_m \big[\mu P_m^1(\mu) F_1(x) + \sqrt{1-\mu^2} P_m(\mu) F_2(x)\big]\big\}. \tag{B7}$$

Substitution of Eq. (B7) into Eq. (B4) yields

$$D^2\big[\mu P_m^1(\mu) F_1(x)\big] + D^2[\sqrt{1-\mu^2} P_m(\mu) F_2(x)] = k_1^2 \mu P_m^1(\mu) G_1(x) + k_1^2 \sqrt{1-\mu^2} P_m(\mu) G_2(x). \tag{B8}$$

By using Eqs. (C10) and (C14) from Appendix C, one obtains

$$\mu P_m^1\left[F_1^{IV} + \frac{4}{x}F_1''' - \frac{2m(m+1)}{x^2}F_1'' + \frac{m(m+1)(m^2+m+2)}{x^4}F_1 + \frac{4}{x^2}F_2'' - \frac{4m(m+1)}{x^4}F_2\right] + \sqrt{1-\mu^2}P_m\left[F_2^{IV} + \frac{4}{x}F_2''' - \frac{2(m^2+m+2)}{x^2}F_2'' + \frac{m(m+1)(m^2+m+6)}{x^4}F_2 + \frac{4m(m+1)}{x^2}F_1'' - \frac{4m^2(m+1)^2}{x^4}F_1\right] = \mu P_m^1 G_1 + \sqrt{1-\mu^2}P_m G_2. \tag{B9}$$

Equation (B9) is divided into the following two equations:

$$F_1^{IV} + \frac{4}{x}F_1''' - \frac{2m(m+1)}{x^2}F_1'' + \frac{m(m+1)(m^2+m+2)}{x^4}F_1 + \frac{4}{x^2}F_2'' - \frac{4m(m+1)}{x^4}F_2 = G_1, \tag{B10}$$

$$F_2^{IV} + \frac{4}{x}F_2''' - \frac{2(m^2+m+2)}{x^2}F_2'' + \frac{m(m+1)(m^2+m+6)}{x^4}F_2 + \frac{4m(m+1)}{x^2}\left[F_1'' - \frac{m(m+1)}{x^2}F_1\right] = G_2. \tag{B11}$$

To solve Eqs. (B10) and (B11), we need to transform them into a system of ordinary differential equations of first order [8]. To this end, we introduce the following functions:

$$Y_1 = F_1, \quad Y_2 = xY_1', \quad Y_3 = xY_2', \quad Y_4 = xY_3', \tag{B12}$$

$$Y_5 = F_2, \quad Y_6 = xY_5', \quad Y_7 = xY_6', \quad Y_8 = xY_7'. \tag{B13}$$

Equation (B12) gives the equations

$$Y_2 = xF_1', \quad Y_3 = x^2F_1'' + xF_1', \quad Y_4 = x^3F_1''' + 3x^2F_1'' + xF_1', \quad Y_4' = x^3F_1^{IV} + 6x^2F_1''' + 7xF_1'' + F_1', \tag{B14}$$

from which we obtain

$$F_1' = \frac{1}{x}Y_2, \quad F_1'' = \frac{1}{x^2}(Y_3 - Y_2), \quad F_1''' = \frac{1}{x^3}(Y_4 - 3Y_3 + 2Y_2), \quad F_1^{IV} = \frac{1}{x^3}Y_4' - \frac{1}{x^4}(6Y_4 - 11Y_3 + 6Y_2). \tag{B15}$$

In a similar way, from Eq. (B13) we find

$$F_2' = \frac{1}{x}Y_6, \quad F_2'' = \frac{1}{x^2}(Y_7 - Y_6), \quad F_2''' = \frac{1}{x^3}(Y_8 - 3Y_7 + 2Y_6), \quad F_2^{IV} = \frac{1}{x^3}Y_8' - \frac{1}{x^4}(6Y_8 - 11Y_7 + 6Y_6). \tag{B16}$$

Substitution of Eqs. (B15) and (B16) into Eqs. (B10) and (B11) yields

$$Y_4' = -\frac{m(m+1)(m^2+m+2)}{x}Y_1 - \frac{2m(m+1)+2}{x}Y_2 + \frac{2m(m+1)+1}{x}Y_3 + \frac{2}{x}Y_4 + \frac{4m(m+1)}{x}Y_5 + \frac{4}{x}Y_6 - \frac{4}{x}Y_7 + x^3G_1, \tag{B17}$$

$$Y_8' = \frac{4m(m+1)}{x}[m(m+1)Y_1 + Y_2 - Y_3] - \frac{m(m+1)(m^2+m+6)}{x}Y_5 - \frac{2(m^2+m+2)+2}{x}Y_6 + \frac{2(m^2+m+2)+1}{x}Y_7 + \frac{2}{x}Y_8 + x^3G_2. \tag{B18}$$

Equations (B12), (B13), (B17), and (B18) are combined into the following system:

$$\begin{aligned}
&Y_1' = \frac{1}{x}Y_2, \quad Y_2' = \frac{1}{x}Y_3, \quad Y_3' = \frac{1}{x}Y_4,\\
&Y_4' = -\frac{m(m+1)(m^2+m+2)}{x}Y_1 - \frac{2(m^2+m+1)}{x}Y_2 + \frac{2m^2+2m+1}{x}Y_3 + \frac{2}{x}Y_4 + \frac{4m(m+1)}{x}Y_5 + \frac{4}{x}Y_6 - \frac{4}{x}Y_7 + x^3G_1,\\
&Y_5' = \frac{1}{x}Y_6, \quad Y_6' = \frac{1}{x}Y_7, \quad Y_7' = \frac{1}{x}Y_8,\\
&Y_8' = \frac{4m(m+1)}{x}[m(m+1)Y_1 + Y_2 - Y_3] - \frac{m(m+1)(m^2+m+6)}{x}Y_5 - \frac{2(m^2+m+3)}{x}Y_6\\
&\quad + \frac{2m^2+2m+5}{x}Y_7 + \frac{2}{x}Y_8 + x^3G_2.
\end{aligned} \tag{B19}$$

The system of Eqs. (B19) is solved by the method of variation of parameters [8]. We first seek solutions to homogeneous equations corresponding to system Eqs. (B19), setting $G_1 = G_2 = 0$. Partial solutions are sought as $Y_n = \gamma_n x^\lambda$ $(n = 1,2,\ldots,8)$ with γ_n being a constant. Substituting this expression into system Eqs. (B19) with $G_1 = G_2 = 0$, one obtains a system of equations for γ_n,

$$\begin{aligned}
&\lambda\gamma_1 - \gamma_2 = 0, \quad \lambda\gamma_2 - \gamma_3 = 0, \quad \lambda\gamma_3 - \gamma_4 = 0,\\
&m(m+1)(m^2+m+2)\gamma_1 + 2(m^2+m+1)\gamma_2 - (2m^2+2m+1)\gamma_3 + (\lambda-2)\gamma_4 - 4m(m+1)\gamma_5 - 4\gamma_6 + 4\gamma_7 = 0,\\
&\lambda\gamma_5 - \gamma_6 = 0, \quad \lambda\gamma_6 - \gamma_7 = 0, \quad \lambda\gamma_7 - \gamma_8 = 0,\\
&4m(m+1)[m(m+1)\gamma_1 + \gamma_2 - \gamma_3] - m(m+1)(m^2+m+6)\gamma_5 - 2(m^2+m+3)\gamma_6 + (2m^2+2m+5)\gamma_7 + (2-\lambda)\gamma_8 = 0.
\end{aligned} \tag{B20}$$

This system has nonzero solutions for γ_n only if its determinant is equal to zero. This condition gives an equation for λ,

$$(\lambda^2 - \lambda - m - m^2)^2[(\lambda^2-\lambda)^2 - 2(4+m+m^2)(\lambda^2-\lambda) + 12 - 8m - 7m^2 + 2m^3 + m^4] = 0. \tag{B21}$$

This equation is transformed to

$$(\lambda^2 - \lambda - m - m^2)^2(\lambda^2 - \lambda - m^2 - 5m - 6)(\lambda^2 - \lambda - m^2 + 3m - 2) = 0. \tag{B22}$$

It is easy to check that the roots of Eq. (59) are given by

$$\lambda_1 = -m-2, \quad \lambda_2 = m+3, \quad \lambda_3 = 2-m, \quad \lambda_4 = m-1, \quad \lambda_{5,6} = -m, \quad \lambda_{7,8} = m+1. \tag{B23}$$

When we substitute λ_1 into system Eqs. (B20), we get equations that give the values of γ_n corresponding to the root λ_1. Let us denote these values as γ_{n1} $(n = 1,2,\ldots,8)$. Since the determinant of system Eqs. (B20) is equal to zero, only seven equations are independent. This means that one of the unknowns γ_{n1} should be taken as an arbitrary constant and then the other unknowns γ_{n1} can be expressed in terms of this constant. Let us set γ_{11} as an arbitrary constant. Then, substituting λ_1 into system Eqs. (B20) and solving this latter for γ_{n1} with $n = 2,3,\ldots,8$, we obtain

$$\begin{aligned}
&\gamma_{21} = -(m+2)\gamma_{11}, \quad \gamma_{31} = (m+2)^2\gamma_{11}, \quad \gamma_{41} = -(m+2)^3\gamma_{11}, \quad \gamma_{51} = -(m+1)\gamma_{11},\\
&\gamma_{61} = (m+1)(m+2)\gamma_{11}, \quad \gamma_{71} = -(m+1)(m+2)^2\gamma_{11}, \quad \gamma_{81} = (m+1)(m+2)^3\gamma_{11}.
\end{aligned} \tag{B24}$$

In an analogous way, for the roots λ_2, λ_3, and λ_4, one finds

$$\begin{aligned}
&\gamma_{22} = (m+3)\gamma_{12}, \quad \gamma_{32} = (m+3)^2\gamma_{12}, \quad \gamma_{42} = (m+3)^3\gamma_{12}, \quad \gamma_{52} = -(m+1)\gamma_{12},\\
&\gamma_{62} = -(m+1)(m+3)\gamma_{12}, \quad \gamma_{72} = -(m+1)(m+3)^2\gamma_{12}, \quad \gamma_{82} = -(m+1)(m+3)^3\gamma_{12},
\end{aligned} \tag{B25}$$

$$\begin{aligned}
&\gamma_{23} = (2-m)\gamma_{13}, \quad \gamma_{33} = (2-m)^2\gamma_{13}, \quad \gamma_{43} = (2-m)^3\gamma_{13}, \quad \gamma_{53} = m\gamma_{13},\\
&\gamma_{63} = m(2-m)\gamma_{13}, \quad \gamma_{73} = m(2-m)^2\gamma_{13}, \quad \gamma_{83} = m(2-m)^3\gamma_{13},
\end{aligned} \tag{B26}$$

$$\begin{aligned}
&\gamma_{24} = (m-1)\gamma_{14}, \quad \gamma_{34} = (m-1)^2\gamma_{14}, \quad \gamma_{44} = (m-1)^3\gamma_{14}, \quad \gamma_{54} = m\gamma_{14},\\
&\gamma_{64} = m(m-1)\gamma_{14}, \quad \gamma_{74} = m(m-1)^2\gamma_{14}, \quad \gamma_{84} = m(m-1)^3\gamma_{14},
\end{aligned} \tag{B27}$$

where γ_{12}, γ_{13}, and γ_{14} are arbitrary constants.

We have two pairs of repeated roots: $\lambda_5 = \lambda_6$ and $\lambda_7 = \lambda_8$. Partial solutions corresponding to these roots are written as [8]

$$Y_n = (\gamma_{n5} + \gamma_{n6} \ln x) x^{\lambda_5}, \quad Y_n = (\gamma_{n7} + \gamma_{n8} \ln x) x^{\lambda_7}, \quad n = 1, 2, ...8. \tag{B28}$$

To find γ_{n5} and γ_{n6}, we substitute the partial solutions given by the first expression of Eq. (B28) into system Eqs. (B19) with $G_1 = G_2 = 0$. As a result, we obtain

$$\begin{aligned}
&m\gamma_{15} - \gamma_{16} + \gamma_{25} + (m\gamma_{16} + \gamma_{26}) \ln x = 0,\\
&m\gamma_{25} - \gamma_{26} + \gamma_{35} + (m\gamma_{26} + \gamma_{36}) \ln x = 0,\\
&m\gamma_{35} - \gamma_{36} + \gamma_{45} + (m\gamma_{36} + \gamma_{46}) \ln x = 0,\\
&m(m+1)(m^2+m+2)(\gamma_{15} + \gamma_{16} \ln x) + 2(m^2+m+1)(\gamma_{25} + \gamma_{26} \ln x) - (2m^2+2m+1)(\gamma_{35} + \gamma_{36} \ln x)\\
&\quad - (m+2)(\gamma_{45} + \gamma_{46} \ln x) + \gamma_{46} - 4m(m+1)(\gamma_{55} + \gamma_{56} \ln x) - 4(\gamma_{65} + \gamma_{66} \ln x) + 4(\gamma_{75} + \gamma_{76} \ln x) = 0,\\
&m\gamma_{55} - \gamma_{56} + \gamma_{65} + (m\gamma_{56} + \gamma_{66}) \ln x = 0,\\
&m\gamma_{65} - \gamma_{66} + \gamma_{75} + (m\gamma_{66} + \gamma_{76}) \ln x = 0,\\
&m\gamma_{75} - \gamma_{76} + \gamma_{85} + (m\gamma_{76} + \gamma_{86}) \ln x = 0,\\
&4m(m+1)[m(m+1)(\gamma_{15} + \gamma_{16} \ln x) + \gamma_{25} - \gamma_{35} + (\gamma_{26} - \gamma_{36}) \ln x] - m(m+1)(m^2+m+6)(\gamma_{55} + \gamma_{56} \ln x)\\
&\quad - 2(m^2+m+3)(\gamma_{65} + \gamma_{66} \ln x) + (2m^2+2m+5)(\gamma_{75} + \gamma_{76} \ln x) + (m+2)(\gamma_{85} + \gamma_{86} \ln x) - \gamma_{86} = 0.
\end{aligned} \tag{B29}$$

From these equations, it follows that all the constants $\gamma_{n6} = 0$ and the constants γ_{n5} are defined by

$$\gamma_{25} = -m\gamma_{15}, \quad \gamma_{35} = m^2\gamma_{15}, \quad \gamma_{45} = -m^3\gamma_{15}, \quad \gamma_{65} = -m\gamma_{55}, \quad \gamma_{75} = m^2\gamma_{55}, \quad \gamma_{85} = -m^3\gamma_{55}, \tag{B30}$$

where γ_{15} and γ_{55} are arbitrary constants.

In a similar way, one obtains for the roots $\lambda_7 = \lambda_8$ that $\gamma_{n8} = 0$ and the constants γ_{n7} are defined by

$$\begin{aligned}
&\gamma_{27} = (m+1)\gamma_{17}, \quad \gamma_{37} = (m+1)^2\gamma_{17}, \quad \gamma_{47} = (m+1)^3\gamma_{17},\\
&\gamma_{67} = (m+1)\gamma_{57}, \quad \gamma_{77} = (m+1)^2\gamma_{57}, \quad \gamma_{87} = (m+1)^3\gamma_{57},
\end{aligned} \tag{B31}$$

where γ_{17} and γ_{57} are arbitrary constants.

The general homogeneous solutions are written as

$$Y_n = \gamma_{n1}x^{-m-2} + \gamma_{n2}x^{m+3} + \gamma_{n3}x^{2-m} + \gamma_{n4}x^{m-1} + \gamma_{n5}x^{-m} + \gamma_{n7}x^{m+1}, \quad n = 1, 2, ..., 8. \tag{B32}$$

Recall that eight of the constants γ_{nk} are arbitrary.

To find solutions to the inhomogeneous system Eqs. (B19), we assume that eight arbitrary constants (γ_{11}, γ_{12}, γ_{13}, γ_{14}, γ_{15}, γ_{17}, γ_{55}, γ_{57}) are functions of x. With this assumption, the substitution of the general solutions given by Eq. (B32) into system Eqs. (B19) yields

$$\begin{aligned}
&\gamma'_{11}x^{-m-2} + \gamma'_{12}x^{m+3} + \gamma'_{13}x^{2-m} + \gamma'_{14}x^{m-1} + \gamma'_{15}x^{-m} + \gamma'_{17}x^{m+1} = 0,\\
&\quad (m+2)\gamma'_{11}x^{-m-2} - (m+3)\gamma'_{12}x^{m+3} + (m-2)\gamma'_{13}x^{2-m} - (m-1)\gamma'_{14}x^{m-1} + m\gamma'_{15}x^{-m} - (m+1)\gamma'_{17}x^{m+1} = 0,\\
&(m+2)^2\gamma'_{11}x^{-m-2} + (m+3)^2\gamma'_{12}x^{m+3} + (m-2)^2\gamma'_{13}x^{2-m} + (m-1)^2\gamma'_{14}x^{m-1} + m^2\gamma'_{15}x^{-m} + (m+1)^2\gamma'_{17}x^{m+1} = 0,\\
&(m+2)^3\gamma'_{11}x^{-m-2} - (m+3)^3\gamma'_{12}x^{m+3} + (m-2)^3\gamma'_{13}x^{2-m} - (m-1)^3\gamma'_{14}x^{m-1} + m^3\gamma'_{15}x^{-m}\\
&\quad - (m+1)^3\gamma'_{17}x^{m+1} = -x^3G_1,\\
&(m+1)\gamma'_{11}x^{-m-2} + (m+1)\gamma'_{12}x^{m+3} - m\gamma'_{13}x^{2-m} - m\gamma'_{14}x^{m-1} - \gamma'_{55}x^{-m} - \gamma'_{57}x^{m+1} = 0,\\
&(m+1)(m+2)\gamma'_{11}x^{-m-2} - (m+1)(m+3)\gamma'_{12}x^{m+3} - m(m-2)\gamma'_{13}x^{2-m} + m(m-1)\gamma'_{14}x^{m-1} - m\gamma'_{55}x^{-m}\\
&\quad + (m+1)\gamma'_{57}x^{m+1} = 0,\\
&(m+1)(m+2)^2\gamma'_{11}x^{-m-2} + (m+1)(m+3)^2\gamma'_{12}x^{m+3} - m(m-2)^2\gamma'_{13}x^{2-m} - m(m-1)^2\gamma'_{14}x^{m-1}\\
&\quad - m^2\gamma'_{55}x^{-m} - (m+1)^2\gamma'_{57}x^{m+1} = 0,\\
&(m+1)(m+2)^3\gamma'_{11}x^{-m-2} - (m+1)(m+3)^3\gamma'_{12}x^{m+3} + m(2-m)^3\gamma'_{13}x^{2-m} + m(m-1)^3\gamma'_{14}x^{m-1}\\
&\quad - m^3\gamma'_{55}x^{-m} + (m+1)^3\gamma'_{57}x^{m+1} = x^3G_2.
\end{aligned} \tag{B33}$$

From these equations, one obtains that

$$\begin{aligned}
\gamma_{11}(x) &= \bar{\gamma}_{11} + \frac{1}{2(2m+1)(2m+3)(2m+5)} \int_{\bar{x}}^{x} [G_2(s) - mG_1(s)]s^{m+5} ds,\\
\gamma_{12}(x) &= \bar{\gamma}_{12} + \frac{1}{2(2m+1)(2m+3)(2m+5)} \int_{\bar{x}}^{x} [mG_1(s) - G_2(s)]s^{-m} ds,\\
\gamma_{13}(x) &= \bar{\gamma}_{13} + \frac{1}{2(2m+1)(2m-1)(2m-3)} \int_{\bar{x}}^{x} [(m+1)G_1(s) + G_2(s)]s^{m+1} ds,\\
\gamma_{14}(x) &= \bar{\gamma}_{14} - \frac{1}{2(2m+1)(2m-1)(2m-3)} \int_{\bar{x}}^{x} [(m+1)G_1(s) + G_2(s)]s^{4-m} ds,\\
\gamma_{15}(x) &= \bar{\gamma}_{15} - \frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{x} [3G_1(s) + 2G_2(s)]s^{m+3} ds,\\
\gamma_{17}(x) &= \bar{\gamma}_{17} + \frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{x} [3G_1(s) + 2G_2(s)]s^{2-m} ds,\\
\gamma_{55}(x) &= \bar{\gamma}_{55} - \frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{x} [2m(m+1)G_1(s) + G_2(s)]s^{m+3} ds,\\
\gamma_{57}(x) &= \bar{\gamma}_{57} + \frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{x} [2m(m+1)G_1(s) + G_2(s)]s^{2-m} ds,
\end{aligned} \tag{B34}$$

where $\bar{\gamma}_{11}$, $\bar{\gamma}_{12}$, etc. are constants to be determined by boundary conditions.

It will be recalled that $Y_1 = F_1$ and $Y_5 = F_2$ so we have

$$F_1(x) = \gamma_{11}(x)x^{-m-2} + \gamma_{12}(x)x^{m+3} + \gamma_{13}(x)x^{2-m} + \gamma_{14}(x)x^{m-1} + \gamma_{15}(x)x^{-m} + \gamma_{17}(x)x^{m+1}, \tag{B35}$$

$$F_2(x) = -(m+1)\gamma_{11}(x)x^{-m-2} - (m+1)\gamma_{12}(x)x^{m+3} + m\gamma_{13}(x)x^{2-m} + m\gamma_{14}(x)x^{m-1} + \gamma_{55}(x)x^{-m} + \gamma_{57}(x)x^{m+1}. \tag{B36}$$

The components of the Eulerian streaming velocity are calculated by

$$\langle v_{2r}^{1m} \rangle = -\frac{1}{r}\frac{\partial}{\partial \mu}\left(\langle \psi_2^{1m} \rangle \sqrt{1-\mu^2}\right), \tag{B37}$$

$$\langle v_{2\theta}^{1m} \rangle = -\frac{1}{r}\frac{\partial}{\partial x}\left(x\langle \psi_2^{1m} \rangle\right). \tag{B38}$$

Substitution of Eq. (B7) into Eqs. (B37) and (B38) yields

$$\langle v_{2r}^{1m} \rangle = -\frac{1}{\nu r}\mathrm{Re}\left\{b_1^* b_m\left[\mu P_m(\mu)[m(m+1)F_1(x) - 2F_2(x)] + \sqrt{1-\mu^2}P_m^1(\mu)[F_1(x) - F_2(x)]\right]\right\}, \tag{B39}$$

$$\langle v_{2\theta}^{1m} \rangle = -\frac{1}{\nu r}\mathrm{Re}\left\{b_1^* b_m\left[\mu P_m^1(\mu)[F_1(x) + xF_1'(x)] + \sqrt{1-\mu^2}P_m(\mu)[F_2(x) + xF_2'(x)]\right]\right\}, \tag{B40}$$

where

$$\begin{aligned}
F_1'(x) = &-(m+2)\gamma_{11}(x)x^{-m-3} + (m+3)\gamma_{12}(x)x^{m+2} + (2-m)\gamma_{13}(x)x^{1-m} + (m-1)\gamma_{14}(x)x^{m-2}\\
&- m\gamma_{15}(x)x^{-m-1} + (m+1)\gamma_{17}(x)x^{m},
\end{aligned} \tag{B41}$$

$$\begin{aligned}
F_2'(x) = &\ (m+1)(m+2)\gamma_{11}(x)x^{-m-3} - (m+1)(m+3)\gamma_{12}(x)x^{m+2} + m(2-m)\gamma_{13}(x)x^{1-m}\\
&+ m(m-1)\gamma_{14}(x)x^{m-2} - m\gamma_{55}(x)x^{-m-1} + (m+1)\gamma_{57}(x)x^{m}.
\end{aligned} \tag{B42}$$

From the condition $\langle \boldsymbol{v}_2^{1m} \rangle \to 0$ for $r \to \infty$, it follows that

$$\bar{\gamma}_{12} = -\frac{1}{2(2m+1)(2m+3)(2m+5)} \int_{\bar{x}}^{\infty} [mG_1(s) - G_2(s)]s^{-m} ds, \tag{B43}$$

$$\bar{\gamma}_{14} = \frac{1}{2(2m+1)(2m-1)(2m-3)} \int_{\bar{x}}^{\infty} [(m+1)G_1(s) + G_2(s)]s^{4-m} ds, \tag{B44}$$

$$\bar{\gamma}_{17} = -\frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{\infty} [3G_1(s) + 2G_2(s)]s^{2-m} ds, \tag{B45}$$

$$\bar{\gamma}_{57} = -\frac{1}{2(2m-1)(2m+1)(2m+3)} \int_{\bar{x}}^{\infty} [2m(m+1)G_1(s) + G_2(s)]s^{2-m} ds. \tag{B46}$$

To find the other constants, we need to calculate the Lagrangian streaming velocity, $\boldsymbol{v}_L^{1m} = \langle \boldsymbol{v}_2^{1m} \rangle + \boldsymbol{v}_S^{1m}$, where $\boldsymbol{v}_S^{1m}$ denotes the Stokes drift velocity, which is calculated by [3]

$$\boldsymbol{v}_S^{1m} = \left\langle \left(\left(\int \boldsymbol{v}_1 dt \cdot \nabla \right) \boldsymbol{v}_1 \right) \right\rangle_{1m} = \frac{1}{2\omega_1} \mathrm{Re}\{i(\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1^*\}_{1m}. \tag{B47}$$

In Eq. (B47), $\boldsymbol{v}_1$ is the linear liquid velocity and the subscript $1m$ means that cross terms produced by modes 1 and m should be only kept. Equation (B47) gives

$$v_{Sr}^{1m} = \frac{1}{2\omega_1} \mathrm{Re}\left\{ i v_{1r}^m \frac{\partial v_{1r}^{1*}}{\partial r} - i v_{1r}^{1*} \frac{\partial v_{1r}^m}{\partial r} + \frac{i v_{1\theta}^m}{r} \frac{\partial v_{1r}^{1*}}{\partial \theta} - \frac{i v_{1\theta}^{1*}}{r} \frac{\partial v_{1r}^m}{\partial \theta} \right\}, \tag{B48}$$

$$v_{S\theta}^{1m} = \frac{1}{2\omega_1} \mathrm{Re}\left\{ i v_{1r}^m \frac{\partial v_{1\theta}^{1*}}{\partial r} - i v_{1r}^{1*} \frac{\partial v_{1\theta}^m}{\partial r} + \frac{i v_{1\theta}^m}{r} \frac{\partial v_{1\theta}^{1*}}{\partial \theta} - \frac{i v_{1\theta}^{1*}}{r} \frac{\partial v_{1\theta}^m}{\partial \theta} + i \frac{v_{1r}^{1*} v_{1\theta}^m - v_{1\theta}^{1*} v_{1r}^m}{r} \right\}, \tag{B49}$$

where v_{1r}^m and $v_{1\theta}^m$ are the radial and tangential components of the linear liquid velocity $\boldsymbol{v}_1^m$ produced by mode m.

From Eqs. (11) and (12) of Part I [1] and Eq. (B1), one obtains

$$v_{1r}^m = -e^{-i\omega_1 t} b_m P_m(\mu) \frac{m+1}{R_0} \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{R_0}{r} \right)^{m+2} + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right], \tag{B50}$$

$$v_{1\theta}^m = e^{-i\omega_1 t} P_m^1(\mu) \frac{b_m}{R_0} \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{R_0}{r} \right)^{m+2} - \frac{\bar{x}}{x} h_m^{(1)}(x) - \bar{x} h_m^{(1)\prime}(x) \right]. \tag{B51}$$

Substitution of Eqs. (B50) and (B51) into Eqs. (B48) and (B49) yields

$$v_{Sr}^{1m} = -\frac{1}{6\nu R_0} \mathrm{Re}\left\{ b_1^* b_m \left[\mu P_m(\mu) S_1(x) + \sqrt{1-\mu^2} P_m^1(\mu) S_2(x) \right] \right\}, \tag{B52}$$

$$v_{S\theta}^{1m} = -\frac{1}{6\nu R_0} \mathrm{Re}\left\{ b_1^* b_m \left[\mu P_m^1(\mu) S_3(x) + \sqrt{1-\mu^2} P_m(\mu) S_4(x) \right] \right\}, \tag{B53}$$

where

$$\begin{aligned} S_1(x) = (m+1) \Bigg\{ & \left[\frac{3\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{x} h_1^{(1)\prime}(x) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \\ & \times \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{\bar{x}}{x} \right)^{m+2} + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right] - \left[\frac{\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \\ & \times \left[\frac{\left[\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x}) \right]}{2} \left(\frac{\bar{x}}{x} \right)^{m+2} - m\bar{x} h_m^{(1)\prime}(x) + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right] \Bigg\}, \end{aligned} \tag{B54}$$

$$\begin{aligned} S_2(x) = \frac{m+1}{2} & \left[\frac{\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{x} h_1^{(1)\prime}(x) - \frac{6}{x^2} h_1^{(1)}(x) \right]^* \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{\bar{x}}{x} \right)^{m+2} + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right] \\ & - \left[\frac{\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{\bar{x}}{x} \right)^{m+2} - \bar{x} h_m^{(1)\prime}(x) - \frac{\bar{x}}{x} h_m^{(1)}(x) \right], \end{aligned} \tag{B55}$$

$$\begin{aligned} S_3(x) = & \left[\frac{2\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{x} h_1^{(1)\prime}(x) \right]^* \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2(m+2)} \left(\frac{\bar{x}}{x} \right)^{m+2} - \bar{x} h_m^{(1)\prime}(x) - \frac{\bar{x}}{x} h_m^{(1)}(x) \right] \\ & - \left[\frac{\bar{x}^4}{x^4} h_1^{(1)\prime\prime}(\bar{x}) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \left[\frac{\bar{x}^2 h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2) h_m^{(1)}(\bar{x})}{2} \left(\frac{\bar{x}}{x} \right)^{m+2} + \bar{x} x h_m^{(1)\prime\prime}(x) + \bar{x} h_m^{(1)\prime}(x) - \frac{\bar{x}}{x} h_m^{(1)}(x) \right], \end{aligned} \tag{B56}$$

$$S_4(x) = \frac{m+1}{2}\left\{\left[\frac{3\bar{x}^4}{x^4}h_1^{(1)\prime\prime}(\bar{x}) + 6h_1^{(1)\prime\prime}(x) + \frac{6}{x}h_1^{(1)\prime}(x) - \frac{6}{x^2}h_1^{(1)}(x)\right]^* \times\left[\frac{\bar{x}^2h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2)h_m^{(1)}(\bar{x})}{2(m+2)}\left(\frac{\bar{x}}{x}\right)^{m+2} + \frac{m\bar{x}}{x}h_m^{(1)}(x)\right] + \left[\frac{\bar{x}^4}{x^4}h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{x}h_1^{(1)\prime}(x) - \frac{6}{x^2}h_1^{(1)}(x)\right]^*\left[\frac{(m+1)[\bar{x}^2h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2)h_m^{(1)}(\bar{x})]}{2(m+2)}\left(\frac{\bar{x}}{x}\right)^{m+2} - m\bar{x}h_m^{(1)\prime}(x)\right]\right\}. \quad \text{(B57)}$$

Let us apply the boundary conditions for the Lagrangian streaming velocity at the bubble surface. They are written as

$$v_{Lr}^{1m} = 0 \quad \text{at} \quad r = R_0, \quad \text{(B58)}$$

$$\frac{1}{r}\frac{\partial v_{Lr}^{1m}}{\partial\theta} + \frac{\partial v_{L\theta}^{1m}}{\partial r} - \frac{v_{L\theta}^{1m}}{r} = 0 \quad \text{at} \quad r = R_0. \quad \text{(B59)}$$

Substituting Eqs. (B39), (B40), (B52), and (B53) into Eqs. (B58) and (B59), one obtains the following system of equations:

$$\begin{aligned}
&(m+1)(m+2)\bar{\gamma}_{11}\bar{x}^{-m-2} + m(m-1)\bar{\gamma}_{13}\bar{x}^{2-m} + m(m+1)\bar{\gamma}_{15}\bar{x}^{-m} - 2\bar{\gamma}_{55}\bar{x}^{-m}\\
&\quad = -(m+1)(m+2)\bar{\gamma}_{12}\bar{x}^{m+3} - m(m-1)\bar{\gamma}_{14}\bar{x}^{m-1} - m(m+1)\bar{\gamma}_{17}\bar{x}^{m+1} + 2\bar{\gamma}_{57}\bar{x}^{m+1} - \frac{S_1(\bar{x})}{6},\\
&(m+2)\bar{\gamma}_{11}\bar{x}^{-m-2} - (m-1)\bar{\gamma}_{13}\bar{x}^{2-m} + \bar{\gamma}_{15}\bar{x}^{-m} - \bar{\gamma}_{55}\bar{x}^{-m}\\
&\quad = -(m+2)\bar{\gamma}_{12}\bar{x}^{m+3} + (m-1)\bar{\gamma}_{14}\bar{x}^{m-1} - \bar{\gamma}_{17}\bar{x}^{m+1} + \bar{\gamma}_{57}\bar{x}^{m+1} - \frac{S_2(\bar{x})}{6},\\
&(m+1)(m+3)\bar{\gamma}_{11}\bar{x}^{-m-2} + m(m-2)\bar{\gamma}_{13}\bar{x}^{2-m} + (m^2+m-1)\bar{\gamma}_{15}\bar{x}^{-m} - \bar{\gamma}_{55}\bar{x}^{-m}\\
&\quad = -(m+1)(m+3)\bar{\gamma}_{12}\bar{x}^{m+3} - m(m-2)\bar{\gamma}_{14}\bar{x}^{m-1} - (m^2+m-1)\bar{\gamma}_{17}\bar{x}^{m+1} + \bar{\gamma}_{57}\bar{x}^{m+1}\\
&\qquad - \frac{1}{12}[S_1(\bar{x}) - S_3(\bar{x}) + \bar{x}S_3'(\bar{x})],\\
&(m+1)(m+2)^2\bar{\gamma}_{11}\bar{x}^{-m-2} - m(m-1)^2\bar{\gamma}_{13}\bar{x}^{2-m} + m(m+1)\bar{\gamma}_{15}\bar{x}^{-m} - (m^2+m+1)\bar{\gamma}_{55}\bar{x}^{-m}\\
&\quad = -(m+1)(m+2)^2\bar{\gamma}_{12}\bar{x}^{m+3} + m(m-1)^2\bar{\gamma}_{14}\bar{x}^{m-1} - m(m+1)\bar{\gamma}_{17}\bar{x}^{m+1} + (m^2+m+1)\bar{\gamma}_{57}\bar{x}^{m+1}\\
&\qquad - \frac{1}{12}[S_1(\bar{x}) + m(m+1)S_2(\bar{x}) + S_4(\bar{x}) - \bar{x}S_4'(\bar{x})].
\end{aligned} \quad \text{(B60)}$$

Solving these equations for four remaining constants, one gets finally

$$\bar{\gamma}_{11} = -\bar{\gamma}_{12}\bar{x}^{2m+5} + \frac{\bar{x}^{m+2}}{12(2m+1)(2m+3)}\left\{\frac{m[(m^2-3)S_1(\bar{x}) + (m^3-m+4)S_2(\bar{x})]}{(m-1)(m+2)} + m[S_3(\bar{x}) - \bar{x}S_3'(\bar{x})] - S_4(\bar{x}) + \bar{x}S_4'(\bar{x})\right\}, \quad \text{(B61)}$$

$$\bar{\gamma}_{13} = -\bar{\gamma}_{14}\bar{x}^{2m-3} - \frac{\bar{x}^{m-2}}{12(2m-1)(2m+1)}\left\{\frac{(m+1)[(m^2+2m-2)S_1(\bar{x}) - (m^3+3m^2+2m-4)S_2(\bar{x})]}{(m-1)(m+2)} + (m+1)[S_3(\bar{x}) - \bar{x}S_3'(\bar{x})] + S_4(\bar{x}) - \bar{x}S_4'(\bar{x})\right\}, \quad \text{(B62)}$$

$$\bar{\gamma}_{15} = -\bar{\gamma}_{17}\bar{x}^{2m+1} - \frac{\bar{x}^{m}}{12(2m-1)(2m+3)}\left\{\frac{m(m+1)[3S_1(\bar{x}) + 2(m^2+m-5)S_2(\bar{x})]}{(m-1)(m+2)} - 3[S_3(\bar{x}) - \bar{x}S_3'(\bar{x})] - 2[S_4(\bar{x}) - \bar{x}S_4'(\bar{x})]\right\}, \quad \text{(B63)}$$

$$\bar{\gamma}_{55} = -\bar{\gamma}_{57}\bar{x}^{2m+1} + \frac{\bar{x}^{m}}{12(2m-1)(2m+3)}\left\{\frac{(2m^4+4m^3-9m^2-11m+6)S_1(\bar{x}) + m(m+1)(3m^2+3m+2)S_2(\bar{x})}{(m-1)(m+2)} + 2m(m+1)[S_3(\bar{x}) - \bar{x}S_3'(\bar{x})] + S_4(\bar{x}) - \bar{x}S_4'(\bar{x})\right\}. \quad \text{(B64)}$$

Expressions for $S_3'(\bar{x})$ and $S_4'(\bar{x})$ are provided in Appendix D.

APPENDIX C: EQUATIONS USED FOR THE CALCULATION OF EQ. (B9)

This Appendix provides equations that were used in the course of the derivation of Eq. (B9).

$$\begin{aligned} D\big[\mu P_m^1(\mu)F_1(x)\big] &= \frac{k_1^2}{x^2}\left\{\mu P_m^1(x^2F_1')' + F_1\left[(1-\mu^2)\big(\mu P_m^1\big)'' - 2\mu\big(\mu P_m^1\big)' - \frac{\mu P_m^1}{1-\mu^2}\right]\right\} \\ &= \frac{k_1^2}{x^2}\left\{\mu P_m^1(x^2F_1')' + F_1\left[\mu(1-\mu^2)P_m^{1\prime\prime} - 2\mu^2 P_m^{1\prime} + 2(1-\mu^2)P_m^{1\prime} - 2\mu P_m^1 - \frac{\mu P_m^1}{1-\mu^2}\right]\right\} \\ &= k_1^2\left\{\mu P_m^1\left[F_1'' + \frac{2}{x}F_1' - \frac{m(m+1)}{x^2}F_1\right] + 2m(m+1)\sqrt{1-\mu^2}P_m\frac{F_1}{x^2}\right\}. \end{aligned} \tag{C1}$$

Here, we have used the following equations:

$$(1-\mu^2)P_m^{1\prime\prime} - 2\mu P_m^{1\prime} = \frac{P_m^1}{1-\mu^2} - m(m+1)P_m^1, \tag{C2}$$

$$(1-\mu^2)P_m^{1\prime}(\mu) = \mu P_m^1(\mu) + m(m+1)\sqrt{1-\mu^2}P_m(\mu), \tag{C3}$$

$$\begin{aligned} D[\sqrt{1-\mu^2}P_m(\mu)F_2(x)] &= \frac{k_1^2}{x^2}\left\{\sqrt{1-\mu^2}P_m(x^2F_2')' + F_2\left[(1-\mu^2)(\sqrt{1-\mu^2}P_m)'' - 2\mu(\sqrt{1-\mu^2}P_m)' - \frac{P_m}{\sqrt{1-\mu^2}}\right]\right\} \\ &= \frac{k_1^2}{x^2}\{\sqrt{1-\mu^2}P_m(x^2F_2')' + F_2[\sqrt{1-\mu^2}((1-\mu^2)P_m'' - 2\mu P_m') - 2\sqrt{1-\mu^2}P_m - 2\mu\sqrt{1-\mu^2}P_m']\} \\ &= k_1^2\left[\sqrt{1-\mu^2}P_m\left(F_2'' + \frac{2}{x}F_2' - \frac{m^2+m+2}{x^2}F_2\right) + 2\mu P_m^1\frac{F_2}{x^2}\right]. \end{aligned} \tag{C4}$$

Here, we have used the following equations:

$$(1-\mu^2)P_m''(\mu) - 2\mu P_m'(\mu) = -m(m+1)P_m(\mu), \tag{C5}$$

$$\sqrt{1-\mu^2}P_m'(\mu) = -P_m^1(\mu). \tag{C6}$$

Applying the operator D to Eq. (C1), one has

$$D^2\big[\mu P_m^1(\mu)F_1(x)\big] = k_1^2 D\big[\mu P_m^1 H_1\big] + 2m(m+1)k_1^2 D[\sqrt{1-\mu^2}P_m H_2], \tag{C7}$$

where

$$H_1 = F_1'' + \frac{2}{x}F_1' - \frac{m(m+1)}{x^2}F_1, \quad H_2 = \frac{F_1}{x^2}. \tag{C8}$$

By using Eqs. (C1) and (C4), one obtains

$$\begin{aligned} D^2\big[\mu P_m^1(\mu)F_1(x)\big] &= k_1^4\left\{\mu P_m^1\left[H_1'' + \frac{2}{x}H_1' - \frac{m(m+1)}{x^2}H_1\right] + 2m(m+1)\sqrt{1-\mu^2}P_m\frac{H_1}{x^2}\right\} \\ &\quad + 2m(m+1)k_1^4\left[\sqrt{1-\mu^2}P_m\left(H_2'' + \frac{2}{x}H_2' - \frac{m^2+m+2}{x^2}H_2\right) + 2\mu P_m^1\frac{H_2}{x^2}\right]. \end{aligned} \tag{C9}$$

Substitution of Eq. (C8) into Eq. (C9) yields

$$\begin{aligned} D^2\big[\mu P_m^1(\mu)F_1(x)\big] &= 4m(m+1)k_1^4\sqrt{1-\mu^2}P_m\left[\frac{F_1''}{x^2} - \frac{m(m+1)}{x^4}F_1\right] \\ &\quad + k_1^4\mu P_m^1\left[F_1^{IV} + \frac{4}{x}F_1''' - \frac{2m(m+1)}{x^2}F_1'' + \frac{m(m+1)(m^2+m+2)}{x^4}F_1\right]. \end{aligned} \tag{C10}$$

Applying the operator D to Eq. (C4), one has

$$D^2[\sqrt{1-\mu^2}P_m(\mu)F_2(x)] = k_1^2 D[\sqrt{1-\mu^2}P_m J_1] + 2k_1^2 D\big[\mu P_m^1 J_2\big], \tag{C11}$$

where

$$J_1 = F_2'' + \frac{2}{x}F_2' - \frac{m^2+m+2}{x^2}F_2, \quad J_2 = \frac{F_2}{x^2}. \tag{C12}$$

By using Eqs. (C1) and (C4), one obtains

$$D^2\left[\sqrt{1-\mu^2}P_m(\mu)F_2(x)\right] = 2k_1^4\mu P_m^1\left[J_2'' + \frac{2}{x}J_2' - \frac{m(m+1)}{x^2}J_2 + \frac{1}{x^2}J_1\right] + k_1^4\sqrt{1-\mu^2}P_m\left[J_1'' + \frac{2}{x}J_1' - \frac{m^2+m+2}{x^2}J_1 + \frac{4m(m+1)}{x^2}J_2\right]. \quad \text{(C13)}$$

Substitution of Eq. (C12) into Eq. (C13) results in

$$D^2[\sqrt{1-\mu^2}P_m(\mu)F_2(x)] = 4k_1^4\mu P_m^1\left[\frac{1}{x^2}F_2'' - \frac{m(m+1)}{x^4}F_2\right] + k_1^4\sqrt{1-\mu^2}P_m\left[F_2^{IV} + \frac{4}{x}F_2''' - \frac{2(m^2+m+2)}{x^2}F_2'' + \frac{m(m+1)(m^2+m+6)}{x^4}F_2\right]. \quad \text{(C14)}$$

APPENDIX D: EXPRESSIONS FOR $S_3'(\bar{x})$ AND $S_4'(\bar{x})$

This Appendix provides expressions for $S_3'(\bar{x})$ and $S_4'(\bar{x})$, which appear in Eqs. (B61)–(B64). They are calculated by differentiating Eqs. (B56) and (B57).

$$\begin{aligned} S_3'(\bar{x}) = {} & \left[\frac{6}{\bar{x}^2}h_1^{(1)\prime}(\bar{x}) - \frac{14}{\bar{x}}h_1^{(1)\prime\prime}(\bar{x})\right]^* \left[\frac{\bar{x}^2h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2)h_m^{(1)}(\bar{x})}{2(m+2)} - \bar{x}h_m^{(1)\prime}(\bar{x}) - h_m^{(1)}(\bar{x})\right] \\ & + \left[2h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{\bar{x}}h_1^{(1)\prime}(\bar{x})\right]^* \left[\frac{m(m+1)}{2\bar{x}}h_m^{(1)}(\bar{x}) - h_m^{(1)\prime}(\bar{x}) - \frac{3}{2}\bar{x}h_m^{(1)\prime\prime}(\bar{x})\right] \\ & + \left[\frac{4}{\bar{x}}h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{\bar{x}^2}h_1^{(1)\prime}(\bar{x}) + \frac{12}{\bar{x}^3}h_1^{(1)}(\bar{x})\right]^*\left[\frac{3}{2}\bar{x}^2h_m^{(1)\prime\prime}(\bar{x}) + \bar{x}h_m^{(1)\prime}(\bar{x}) - \frac{m(m+1)}{2}h_m^{(1)}(\bar{x})\right] \\ & - \left[h_1^{(1)\prime\prime}(\bar{x}) + \frac{6}{\bar{x}^2}h_1^{(1)}(\bar{x})\right]^* \left[\bar{x}^2h_m^{(1)\prime\prime\prime}(\bar{x}) - \frac{m-2}{2}\bar{x}h_m^{(1)\prime\prime}(\bar{x}) - h_m^{(1)\prime}(\bar{x}) + \frac{(m+2)(m^2+m-2)+2}{2\bar{x}}h_m^{(1)}(\bar{x})\right], \end{aligned} \quad \text{(D1)}$$

$$\begin{aligned} S_4'(x) = \frac{m+1}{2}\Bigg\{ & \left[\frac{6}{\bar{x}^3}h_1^{(1)}(\bar{x}) - \frac{6}{\bar{x}^2}h_1^{(1)\prime}(\bar{x}) - \frac{3}{\bar{x}}h_1^{(1)\prime\prime}(\bar{x}) + 3h_1^{(1)\prime\prime\prime}(\bar{x})\right]^* \left[\frac{\bar{x}^2h_m^{(1)\prime\prime}(\bar{x})}{m+2} + (m+1)h_m^{(1)}(\bar{x})\right] \\ & + \left[9h_1^{(1)\prime\prime}(\bar{x}) + \frac{6}{\bar{x}}h_1^{(1)\prime}(\bar{x}) - \frac{6}{\bar{x}^2}h_1^{(1)}(\bar{x})\right]^* \left[\frac{(m-2)(m+1)}{2\bar{x}}h_m^{(1)}(\bar{x}) + mh_m^{(1)\prime}(\bar{x}) - \frac{\bar{x}}{2}h_m^{(1)\prime\prime}(\bar{x})\right] \\ & + \left[\frac{6}{\bar{x}^3}h_1^{(1)}(\bar{x}) - \frac{5}{\bar{x}}h_1^{(1)\prime\prime}(\bar{x})\right]^* \left[\frac{(m+1)[\bar{x}^2h_m^{(1)\prime\prime}(\bar{x}) - (m^2+m-2)h_m^{(1)}(\bar{x})]}{m+2} - m\bar{x}h_m^{(1)\prime}(\bar{x})\right] \\ & + \left[h_1^{(1)\prime\prime}(\bar{x}) - \frac{6}{\bar{x}}h_1^{(1)\prime}(\bar{x}) - \frac{6}{\bar{x}^2}h_1^{(1)}(\bar{x})\right]^* \left[\frac{(m+1)(m^2+m-2)h_m^{(1)}(\bar{x})}{2\bar{x}} - \frac{3m+1}{2}\bar{x}h_m^{(1)\prime\prime}(\bar{x})\right]\Bigg\}. \end{aligned} \quad \text{(D2)}$$

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[5] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.100.033105 for a MATLAB code that implements the calculation of the Lagrangian streaming velocity for the 1 − 1 case (MainProgramm_Case_11.m) and for the 1 − m case (MainProgram_Case_1m.m).

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