Acoustic microstreaming produced by nonspherical oscillations of a gas bubble. II. Case of modes 1 and m

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This paper continues a study that was started in our previous paper [A. A. Doinikov et al., Phys. Rev. E 100, 033104 (2019)]. The overall aim of the study is to develop a theory for modeling the velocity field of acoustic microstreaming produced by nonspherical oscillations of an acoustically driven gas bubble. In this paper, general equations were derived that describe the velocity field of acoustic microstreaming produced by modes $n$ and $m$ of bubble oscillations. In the present paper, the above equations are solved analytically in the case that acoustic microstreaming is the result of the interaction of the translational mode (mode 1) with a mode of arbitrary order $m \geq 1$. Solutions are expressed in terms of complex mode amplitudes, which means that the mode amplitudes are assumed to be known and serve as input data for the calculation of the velocity field of acoustic microstreaming. No restrictions are imposed on the ratio of the bubble radius to the viscous penetration depth. Analytical results are illustrated by numerical examples.

I. INTRODUCTION

In Part I of our study [1], equations were derived for the velocity field of acoustic microstreaming that is produced by modes $n$ and $m$ of oscillations of a gas bubble; see Sec. II C of Part I [1]. The aim of the present paper is to apply the above general equations to the case that acoustic microstreaming is the result of the interaction of the translational mode (mode 1) with a mode of arbitrary order $m \geq 1$.

The case 1−1, where only mode 1 is involved, and the case of modes 1 and $m$ with $m > 1$ are shown in Part I [1] to be described by different equations. Therefore, the present calculation is divided into two parts. In Sec. II A, a solution for the case 1−1 is derived, while the case 1−$m$ with $m > 1$ is considered in Sec. II B.

The case 1−1 was considered previously by Davidson and Riley [2] and Longuet-Higgins [3]. We consider this case in a different formulation. In Refs. [2] and [3], it is assumed that the bubble is fixed while the liquid oscillates about it. This means that the liquid at infinity has a unidirectional velocity. Conversely, we assume that the bubble is moving while the liquid at infinity is at rest. Our results show that these assumptions lead to different solutions for the streaming. The fact that these two cases are not equivalent as far as acoustic streaming is concerned is also confirmed by results of Wu and Du [4], which are presented in more detail below. Another important distinctive feature of our solutions is that they do not impose any restrictions on the ratio of the bubble radius to the viscous penetration depth, whereas the results obtained in Refs. [2] and [3] are valid only when the bubble radius is much greater than the viscous penetration depth.

II. THEORY

We consider a gas bubble undergoing axisymmetric oscillations, which include the radial pulsation (mode 0), translation (mode 1), and shape modes of order $m \geq 2$. The liquid motion produced by the bubble oscillations is described by spherical coordinates $r$ and $\theta$ whose origin is at the equilibrium center of the bubble. The geometry of the problem is depicted by Fig. 1 of Part I [1].

A. Acoustic microstreaming produced by mode 1 alone

According to the theory developed in Part I [1], in the case 1−1, the Eulerian streaming velocity is represented by

$$\langle v_2^{11} \rangle = \nabla \times \left[ \langle \psi_2^{11}(r, \theta) \rangle e_\theta \right],$$

where $\langle \rangle$ denotes the time average, $e_\theta$ is the unit azimuth vector, and $\langle \psi_2^{11} \rangle$ is the amplitude of the vector potential of the streaming velocity that is calculated from Eq. (33) of Part I, in which $n$ is set equal to 1, giving the following result:

$$\left( \Delta_{r\theta} - \frac{1}{r^2 \sin^2 \theta} \right) \langle \psi_2^{11} \rangle = \mu \frac{\sqrt{1 - \mu^2}}{v^r} \text{Re} \left\{ k_1^2 b_1^* x_1^* h_2^{11}(x_1) - h_1^{11}(x_1) \right\}^* - k_1^2 b_1^* x_1 h_2^{11}(x_1) h_1^{11}(x_1) \right\}.$$
is the spherical Hankel function of the first kind, \( h_1^{(1)}(x_1) = \frac{d h_0^{(1)}(x_1)}{dx_1} \), and the asterisk denotes complex conjugate.

A solution to Eq. (2) (see Appendix A) is given by

\[
\psi_1^{(1)} = \mu \sqrt{1 - \mu^2} |b_1|^2 \text{Re}(F(x_1)),
\]

where the function \( F(x_1) \) is defined by Eq. (A12). Substituting Eq. (3) into Eq. (1) yields the following expressions for the radial and tangential components of the streaming velocity:

\[
\nu_r^{(1)} = \frac{|b_1|^2}{3r} \text{Re}(F(x_1)) P_2(\mu),
\]

\[
\nu_\theta^{(1)} = -\frac{|b_1|^2}{6r} \text{Re}(F(x_1)) + x_1 F'(x_1)) \mu \sqrt{1 - \mu^2},
\]

where \( P_2 \) is the Legendre polynomial of order 2 and the function \( F(x_1) \) is defined by Eq. (A21).

It should be emphasized that Eqs. (4) and (5) give the components of the Eulerian streaming velocity, the functions \( F \) and \( F' \) specifying the dependence of these components on distance. To calculate the Lagrangian streaming velocity, Eqs. (4) and (5) are added with the components of the Stokes drift velocity, which are given by Eqs. (A28) and (A29). A MATLAB code for the calculation of the Eqs. (4), (5), (A28), and (A29) is provided as Supplemental Material [5].

As said in the Introduction, we consider a case different from that considered by Davidson and Riley [2] and Longuet-Higgins [3]. We assume that the bubble is moving and the liquid at infinity is at rest, whereas the above authors assume that the bubble is fixed and the liquid at infinity is moving. The streaming velocity, as a nonlinear effect, is different in these two cases. This inference follows from our results and is corroborated by results of Wu and Du [4].

Wu and Du [4] derived approximate solutions for the streaming velocity within the thin viscous boundary layer at the outer and inner surface of a gas bubble undergoing the monopole and dipole vibrations. They assumed that the gas inside the bubble was viscous and used the non-slip boundary conditions. Therefore, their main solutions cannot be correctly compared to those of Davidson and Riley [2] and Longuet-Higgins [3], as well as our solutions. However, we can use limiting equations obtained by Wu and Du [4] in the case that the gas viscosity tends to zero, Eqs. (26') and (28') in their paper, which give the streaming velocity in the case 1–1 within the boundary layer outside the bubble. We cannot perform an exact quantitative comparison as Wu and Du [4] use a quantity \( u_0 \) called by them “the velocity amplitude of a sound source.” It is not clear how to correctly recalculate this quantity to the translational amplitude used in our theory and in the theories of Davidson and Riley [2] and Longuet-Higgins [3]. However, we can compare the sign of the components of the streaming velocity inside the viscous boundary layer.

It follows from the theory of Wu and Du [4] that \( u_0 \) can be treated as the amplitude of the liquid velocity generated by the incident acoustic wave at the center of the bubble as if the bubble were absent. Then the following relation between \( u_0 \) and the magnitude of the translational velocity of the bubble, \( v_b \), can be written: \( v_b = |s_1| = 3u_0[6] \), where \( s_1 \) is the complex amplitude of mode 1 used in our theory. Substituting this relation into Eqs. (26') and (28') of Ref. [4], we can write the components of the streaming velocity derived by Wu and Du in the following form:

\[
u_{WD_\nu} = -\frac{\omega_1^2 |s_1|^2 \rho_\nu}{2\eta r^2} \left( 1 - \frac{r}{R_0} \right) P_2(\mu),
\]

\[
u_{WD_\theta} = -\frac{\omega_1^2 |s_1|^2 \rho_\nu}{4\eta r} \mu \sqrt{1 - \mu^2},
\]

where \( \eta \) is the dynamic liquid viscosity and \( \rho_\nu \) is the equilibrium gas density.

According to the theory of Longuet-Higgins [3], in case 1–1, the components of the Lagrangian streaming velocity inside the viscous boundary layer are calculated by

\[
u_{L_\nu} = \frac{180 \omega_1 |s_1|^2 \delta_1^2}{R_0 r^2} \left( e^{-\xi} \cos \xi - 1 + \frac{3}{2} \xi e^{\xi} \sin \xi \right) P_2(\mu),
\]

\[
u_{L_\theta} = \frac{9 \omega_1 |s_1|^2 \delta_1}{4R_0 r} \left[ 2 e^{-\xi} (\cos \xi + 2 \sin \xi) + 2 \xi e^{-\xi} (\cos \xi \sin \xi \sin \xi + \frac{3}{5}) \mu \sqrt{1 - \mu^2} \right],
\]

where \( \xi = (r - R_0)/\delta_1 \). These equations follow from Eq. (6.5) of Ref. [3].

In Fig. 1, we compare the dependence on \( r \) given by Eqs. (6)–(9) to the results of our theory. Since Longuet-Higgins [3] states that his results for the case 1–1 are identical to those of Davidson and Riley [2], we only provide the results of Longuet-Higgins [3] in Fig. 1. The simulations were made at the following values of the physical parameters: \( R_0 = 50 \mu \text{m} \), \( \omega_1 = 2\pi / 50 \text{ kHz} \), the liquid density \( \rho = 1000 \text{ kg/m}^3 \), \( \eta = 0.001 \text{ Pa s} \), and \( \rho_\nu = 1.2 \text{ kg/m}^3 \). The components of the streaming velocity were normalized by

![Graph showing the comparison of streaming velocity components given by different theories inside the viscous boundary layer.](image-url)
the factor \( \omega_1 |s_1|^2/R_0 \). The comparison is carried out within the viscous boundary layer, from \( r/R_0 = 1 \) up to \( r/R_0 = 1 + \delta_1/R_0 \), where \( \delta_1/R_0 = 0.05 \) for the above mentioned parameters. As one can see, the velocity components of Wu and Du [4] are of the same sign as those predicted by our theory inside the viscous boundary layer, whereas the velocity components of Longuet-Higgins [3] are of opposite sign.

Outside the viscous boundary layer, the theory of Longuet-Higgins [3] gives

\[
v_{1 H r} = \frac{27 \omega_1 |s_1|^2 \delta_1}{20 R_0^2} \left( \frac{R_0}{r} \right)^2 P_2(\mu),
\]

\[
v_{1 H t} = \frac{27 \omega_1 |s_1|^2 \delta_1}{20 R_0^2} \mu \sqrt{1 - \mu^2}.
\]

These equations follow from Eq. (6.7) of Ref. [3]. Figure 2 compares the dependence on \( r \) given by Eqs. (10) and (11) to the results of our theory. The parameters are the same as in Fig. 1. The velocity components are normalized by the factor \( \omega_1 |s_1|^2/R_0 \). As one can see, our theory predicts a greater velocity magnitude. However, it should be emphasized once again that Fig. 2 compares two different physical cases.

B. Acoustic microstreaming produced by modes 1 and \( m \) with \( m > 1 \)

In the case \( 1 - m \), the Eulerian streaming velocity is represented by

\[
\langle v_{2}^{1m} \rangle = \mathbf{v} \times \left[ \langle \psi_{2}^{1m}(r, \theta) \rangle \mathbf{e}_{r} \right],
\]

where \( \langle \psi_{2}^{1m} \rangle \) is calculated from Eq. (32) of Part I [1], in which \( n \) is set equal to 1, leading to

\[
\left( \frac{\Delta_{\phi} - \frac{1}{r^2 \sin^2 \theta}}{r^2} \right)^2 \langle \psi_{2}^{1m} \rangle = \frac{1}{v_F} \mu P_m^1(\mu) \text{Re} \left\{ k_1^2 a_1 b_m^* \left( \frac{R_0}{r} \right) \left[ 2 h_{m}^{(1)}(x_1) - x_1 h_{m}^{(1)}(x_1) \right] + k_1^2 b_1 a_m \left( \frac{R_0}{r} \right) \left[ \left( m + 1 \right) h_{m}^{(1)}(x_1) - x_1 h_{m}^{(1)}(x_1) \right] \right\}
\]

\[
- \frac{m + 1}{2 v_F^2} \left( \frac{R_0}{r} \right)^{m+1} \left[ k_1^2 a_1 b_m^* \left[ x_1 h_{m}^{(1)}(x_1) h_{m}^{(1)}(x_1) + x_1 h_{m}^{(1)}(x_1) h_{m}^{(1)}(x_1) \right] \right]
\]

\[
+ \frac{\sqrt{1 - \mu^2}}{2 v_F^2} \left( \frac{R_0}{r} \right)^{m+1} \left[ k_1^2 a_1 b_m^* \left[ x_1 h_{m}^{(1)}(x_1) + x_1 h_{m}^{(1)}(x_1) \right] \right]
\]

\[
= \frac{1}{v_F} \mu P_m^1(\mu) \text{Re} \left\{ b_1^* b_m \left[ \mu P_m^1(\mu) F_1(x_1) + \sqrt{1 - \mu^2} P_m(\mu) F_2(x_1) \right] \right\},
\]

where the functions \( F_1(x_1) \) and \( F_2(x_1) \) are defined by Eqs. (B35) and (B36). Substituting Eq. (14) into Eq. (12) yields the following expressions for the radial and tangential components of the streaming velocity:

\[
\langle v_{2}^{1m} \rangle = - \frac{1}{v_F} \text{Re} \left\{ b_1^* b_m \left[ \mu P_m^1(\mu) m(m + 1) F_1(x_1) - 2 F_2(x_1) \right] \right\} + \sqrt{1 - \mu^2} P_m^1(\mu) \left[ F_1(x_1) - F_2(x_1) \right],
\]

where \( a_m \) and \( b_m \) are defined by Eqs. (B1) and (B2), \( P_m \) is the Legendre polynomial of order \( m \), and \( P_m^1(\mu) \) is the associated Legendre polynomial of the first order and of degree \( m \).

A solution to Eq. (13) (see Appendix B) is given by

\[
\langle v_{2}^{1m} \rangle = \frac{1}{v_F} \text{Re} \left\{ b_1^* b_m \left[ \mu P_m^1(\mu) F_1(x_1) + \sqrt{1 - \mu^2} P_m(\mu) F_2(x_1) \right] \right\},
\]
\[
\{v_{lm}\} = -\frac{1}{\nu P} \text{Re}\left[ b_l b_m \left[ \mu P_m(\mu) [F_1(x_1) + x_1 F'_1(x_1)]
+ \sqrt{1 - \mu^2} P_m(\mu) [F_2(x_1) + x_1 F'_2(x_1)] \right]\right], \tag{16}
\]
where the functions \(F'_1(x_1)\) and \(F'_2(x_1)\) are defined by Eqs. (B41) and (B42).

It should be emphasized that Eqs. (15) and (16) give the components of the Eulerian streaming velocity. As one can see, the dependence of these components on distance is determined by the functions \(F_1, F_2\) and their derivatives. Note that the above functions are independent of the phase shift between the modes. The phases of the modes are included in the coefficients \(b_1\) and \(b_m\), which, as Eqs. (A3) and (B2) show, are proportional to the complex amplitudes of the modes, \(s_1\) and \(s_m\). These amplitudes are defined as \(s_m = |s_m| \exp(i\phi_m)\), where \(|s_m|\) and \(\phi_m\) are the magnitude and the phase of mode \(m\), respectively.

To calculate the Lagrangian streaming velocity, Eqs. (15) and (16) are added with the components of the Stokes drift velocity, which are given by Eqs. (B52) and (B53). A MATLAB code for the calculation of Eqs. (15), (16), (B52), and (B53) is provided as Supplemental Material [5].

The case \(1 - m\) with \(m > 1\) was considered previously by Spelman and Lauga [7]. Just as in the case of the microstreaming produced by modes 0 and \(m\) discussed in Part I [1], the difference between their theoretical model and ours is that they assume that the bubble is fixed while the liquid oscillates around it, whereas we assume that the bubble is moving while the liquid at infinity is at rest. This means that at infinity the first-order liquid velocity tends to a nonzero value in their case. Hence our model and the one of Spelman and Lauga describe two different physical situations. Figure 3 compares the radial and tangential components of the Lagrangian velocity given by the theory of Spelman and Lauga [7] and the present model for modes 1 and 4. The velocity components are plotted for three angles: (a), (b) \(\theta = 0\); (c), (d) \(\theta = \pi/4\); (e), (f) \(\theta = \pi/2\).

FIG. 3. Evolution of the radial (left column) and tangential (right column) components of the Lagrangian streaming velocity given by the theory of Spelman and Lauga [7] and the present model for modes 1 and 4. The velocity components are plotted for three angles: (a), (b) \(\theta = 0\); (c), (d) \(\theta = \pi/4\); (e), (f) \(\theta = \pi/2\).

presented in Fig. 4 show that for modes 1 and \(m\) with \(m > 1\), the number of lobes is equal to \(2(m - 1)\). It is interesting to note that the streamline patterns in the cases 1–1 and 1–3 look identical.

Figure 5 shows the dependence of the normalized magnitude of the Eulerian streaming velocity on the distance from the bubble surface at various values of the phase shift between modes. The case of modes 1 and 3 is presented. The variation of the streaming velocity along three directions is shown: \(\theta = 0, \theta = \pi/4\), and \(\theta = \pi/2\). As one can see, a change in the phase shift leads to a considerable change in the magnitude of the streaming velocity. As the phase shift increases, the magnitude of the streaming velocity first decreases but then again increases. The bend of the \(\phi = \pi/6\) curve in Fig. 5(c) results from the fact that the sign of the velocity changes at this spatial point.

III. NUMERICAL RESULTS

Numerical simulations were made at the following values of physical parameters: \(\rho = 1000\text{ kg/m}^3\), \(\eta = 0.001\text{ Pa s}\), \(f = 50\text{ kHz}\), and \(R_0 = 100\mu\text{m}\). The streaming velocity was normalized by the factor \(\omega_1|s_1||s_m|/R_0\).

Figure 4 exemplifies Lagrangian streamline patterns produced by modes 1–1, 1–2, 1–3, and 1–4. The phase shift between the modes was set zero. As one can see, the main vortices have a form of lobes. The numerical examples presented in Fig. 4 show that for modes 1 and \(m\) with \(m > 1\), the number of lobes is equal to \(2(m - 1)\). It is interesting to note that the streamline patterns in the cases 1–1 and 1–3 look identical.

IV. CONCLUSIONS

In the present paper, a general theory developed in our previous paper [1] has been applied to the case that acoustic microstreaming is produced by the interaction between the bubble translation (mode 1) and a mode of arbitrary order \(m \geq 1\). Since the case 1–1, where only mode 1 is involved, and the case of modes 1 and \(m\) with \(m > 1\) are described by different equations [1], solutions were obtained separately for these cases. Analytical results were then used to carry out numerical simulations. The simulations have shown that in the
case $1 - m$ with $m > 1$ streamlines form lobes whose number is equal to $2(m - 1)$.

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**APPENDIX A: SOLUTION OF EQ. (2)**

Let us first define the operator $\Delta_{r\theta}$ and the constants $a_1$ and $b_1$ that appear in Eq. (2).

According to Eq. (A9) of Part I [1], $\Delta_{r\theta}$ is given by

$$\Delta_{r\theta} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right).$$  \hfill (A1)

The constants $a_1$ and $b_1$ are known as the linear scattering coefficients of, respectively, the potential and the vortical parts of the scattered wave from the bubble. According to Eqs. (18) and (19) of Part I [1], they are calculated by

$$a_1 = \frac{i R_0 \omega_1 s_1 k_1^2 h_1^{(1)(\nu)}(\bar{x}_1)}{2 [ \bar{x}_1^2 h_1^{(1)(\nu)}(\bar{x}_1) + 6 h_1^{(1)}(\bar{x}_1) ]},$$ \hfill (A2)

$$b_1 = \frac{3i R_0 \omega_1 s_1}{\bar{x}_1^2 h_1^{(1)(\nu)}(\bar{x}_1) + 6 h_1^{(1)}(\bar{x}_1)},$$ \hfill (A3)

where $s_1$ is the complex amplitude of mode 1 and $\bar{x}_1 = k_1 R_0$.

Making use of Eqs. (A1)–(A3) to express $\Delta_{r\theta}$ in terms of $x_1$ and $\mu$ and $a_1$ in terms of $b_1$, Eq. (A2) is transformed to

$$D^2[\psi_{11}^{(1)}] = \mu \sqrt{1 - \mu} \frac{k_1^4 |b_1|^2}{6 \bar{x}_1^2}$$

$$\times \text{Re} \left\{ \bar{x}_1^4 h_1^{(1)(\nu)}(\bar{x}_1) \left[ x_1 h_1^{(1)}(x_1) - h_1^{(1)}(x_1) \right]^* - 6 x_1^3 h_1^{(1)}(x_1) h_1^{(1)(\nu)}(x_1) \right\},$$ \hfill (A4)
where the operator $D$ is given by

$$D = \frac{k^2}{x^3} \left[ \frac{\partial^2}{\partial x_1} \left( x^2 \frac{\partial}{\partial x_1} \right) + (1 - \mu^2) \frac{\partial^2}{\partial \mu^2} - \frac{1}{1 - \mu^2} \right].$$

(A5)

Taking $\langle \psi_1^{(1)} \rangle$ in the form

$$\langle \psi_1^{(1)} \rangle = \mu \sqrt{1 - \mu^2} \frac{|b_1|}{6v} \text{Re}[F(x_1)],$$

(A6)

and substituting into Eq. (A4), one obtains the following equation for $F(x_1)$:

$$\frac{d^4F}{dx_1^4} + \frac{4}{x_1} \frac{d^3F}{dx_1^3} - \frac{12}{x^2_1} \frac{d^2F}{dx_1^2} + \frac{24F}{x^3_1} = G(x_1),$$

(A7)

where $G(x_1)$ is given by

$$G(x_1) = \frac{1}{x_1^4} \left[ x^3 \frac{d}{dx_1} \left( x^4 \frac{d}{dx_1} \right) \right] = 6x^3 \frac{d}{dx_1} \left( x^4 \frac{d}{dx_1} \right).$$

Equation (A7) is solved by the method of variation of parameters [8], which means that we first solve a homogeneous equation that corresponds to Eq. (A7),

$$\frac{d^4F}{dx_1^4} + \frac{4}{x_1} \frac{d^3F}{dx_1^3} - \frac{12}{x^2_1} \frac{d^2F}{dx_1^2} + \frac{24F}{x^3_1} = 0.$$  

(A9)

Assuming that partial solutions to Eq. (A9) are given by $x^\lambda$ and substituting them into Eq. (A9), one obtains a polynomial for $\lambda$,

$$\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + 4\lambda(\lambda - 1)(\lambda - 2) - 12\lambda(\lambda - 1) + 24 = 0.$$  

(A10)

The roots of Eq. (A10) are $-3$, $-1$, $2$, and $4$, which means that the general solution of Eq. (A9) is given by

$$F(x_1) = \frac{C_1}{x_1^4} + \frac{C_2}{x_1^3} + C_3 x^2_1 + C_4 x^4_1,$$  

(A11)

and hence the solution of Eq. (A7) can be written as

$$F(x_1) = \frac{C_1(x_1)}{x_1^3} + \frac{C_2(x_1)}{x_1^2} + C_3(x_1) x^2_1 + C_4(x_1) x^4_1,$$  

(A12)

where $C_n(x_1)$ should obey the following system of equations:

$$C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4 = 0,$$

$$C_1 y_1' + C_2 y_2' + C_3 y_3' + C_4 y_4' = 0,$$

$$C_1 y_1'' + C_2 y_2'' + C_3 y_3'' + C_4 y_4'' = 0,$$

$$C_1 y_1''' + C_2 y_2''' + C_3 y_3''' + C_4 y_4''' = G(x_1).$$

(A13)

Here, the prime denotes the derivative with respect to $x_1$ and the functions $y_n$ are given by

$$y_1 = x_1^{-3}, \quad y_2 = x_1^{-1}, \quad y_3 = x_1^2, \quad y_4 = x_1^4.$$  

(A14)

Solving system (A13) for $C_n$ and integrating the solutions, one obtains

$$C_1(x_1) = C_{10} - \frac{1}{70} \int_{x_i}^{x_f} s^2 G(s) ds,$$  

(A15)

$$C_2(x_1) = C_{20} + \frac{1}{30} \int_{x_i}^{x_f} s G(s) ds,$$  

(A16)

$$C_3(x_1) = C_{30} - \frac{1}{30} \int_{x_i}^{x_f} s G(s) ds,$$  

(A17)

$$C_4(x_1) = C_{40} + \frac{1}{70} \int_{x_i}^{x_f} G(s) ds,$$  

(A18)

where $C_{m0}$ are constants to be determined by boundary conditions.

To apply the boundary conditions, we first calculate the components of the Eulerian streaming velocity, using...
where, as follows from Eqs. (A12) and (A13),

\[ v_{2o}^{(1)} = -\frac{1}{r} \frac{\partial}{\partial \theta} (x_1 |\psi_1^{(1)}|) \]

\[ = \frac{|b_1|^2}{6 \nu r} (3 \mu^2 - 1) \text{Re} \{ F(x_1) \}, \]  
(A19)

\[ v_{2o}^{(1)} = -\frac{1}{r} \frac{\partial}{\partial x_1} (x_1 |\psi_1^{(1)}|) \]

\[ = -\frac{|b_1|^2}{6 \nu r} \mu \sqrt{1 - \mu^2} \text{Re} \{ F(x_1) + x_1 F'(x_1) \}, \]  
(A20)

where, as follows from Eqs. (A12) and (A13),

\[ F'(x_1) = -\frac{3C_1(x_1)}{x_1^2} - \frac{C_2(x_1)}{x_1^2} + 2C_3(x_1)x_1 + 4C_4(x_1)x_1^3. \]  

(A21)

From the condition \( \langle v_2^{(1)} \rangle \to 0 \) for \( r \to \infty \), it follows that

\[ C_{30} = \frac{1}{30} \int_{x_1}^{\infty} sG(s)ds, \]  
(A22)

\[ C_{40} = \frac{1}{70} \int_{x_1}^{\infty} \frac{G(s)}{s}ds. \]  
(A23)

To apply boundary conditions at the bubble surface, we need the Lagrangian streaming velocity, which is defined by [3]

\[ u_{L}^{(1)} = \langle v_2^{(1)} \rangle + v_{s}^{(1)}, \]  
(A24)

where \( v_{s}^{(1)} \), called the Stokes drift velocity, is calculated by [3]

\[ v_{s}^{(1)} = \left( \int \left( \int v^{(1)} dt \cdot \nabla \right) v^{(1)} \right) \] 
\[ = \frac{1}{20 \nu} \text{Re} \{ i \langle v^{(1)} \cdot \nabla \rangle v^{(1)} \}. \]  
(A25)

\( v^{(1)} \) being the linear liquid velocity produced by mode 1.

From Eqs. (11) and (12) of Part I [1], it follows that

\[ v_{s}^{(1)} = -\frac{1}{x_1} v_{s}^{(1)} \frac{\mu}{3} \left[ x_1^4 h_1^{(1)\nu}(x_1) + x_1^2 h_1^{(1)}(x_1) \right], \]

(A26)

\[ v_{s}^{(1)} = -\frac{1}{x_1^2} \frac{\mu}{6} \left[ x_1^2 h_1^{(1)\nu}(x_1) - 6x_1^2 h_1^{(1)}(x_1) \right]. \]

(A27)

Substituting Eqs. (A26) and (A27) into Eq. (A25) yields

\[ v_{s}^{(1)} = \frac{|b_1|^2}{6 \nu x_1^3} (1 - 3 \mu^2) \text{Re} \left\{ 6x_1 h_1^{(1)\nu}(x_1) h_1^{(1)\nu}(x_1) \right\} \]

\[ - \frac{i}{x_1^2} \left[ x_1^4 h_1^{(1)\nu}(x_1) \right] \left[ 2h_1^{(1)}(x_1) + x_1 h_1^{(1)\nu}(x_1) \right]. \]  
(A28)

\[ v_{s}^{(1)} = \frac{|b_1|^2}{6 \nu x_1^2} \sqrt{1 - \mu^2} \text{Re} \left\{ 6x_1^2 h_1^{(1)}(x_1) h_1^{(1)\nu}(x_1) \right\} \]

\[ + \frac{i}{x_1^2} \left[ x_1^4 h_1^{(1)\nu}(x_1) \right] \left[ 6h_1^{(1)}(x_1) - x_1^2 h_1^{(1)\nu}(x_1) \right]. \]  
(A29)

The boundary conditions at the bubble surface are written as (see Part I [1] for more detail)

\[ v_{L}^{(1)} = 0 \quad \text{at} \quad r = R_0, \]  
(A30)

\[ \frac{1}{r} \frac{\partial v_{L}^{(1)}}{\partial \theta} + \frac{\partial v_{L}^{(1)}}{\partial r} - \frac{v_{L}^{(1)}}{r} = 0 \quad \text{at} \quad r = R_0. \]  
(A31)

Substituting Eqs. (A19), (A20), (A28), and (A29) into Eqs. (A30) and (A31) yields

\[ C_{10} + x_1^2 C_{20} = A, \]  
(A32)

\[ 16C_{10} + 6x_1^2 C_{20} = B, \]  
(A33)

where

\[ A = -C_{30} x_1^5 + C_{40} x_1^7 + \bar{\chi}_1 h_1^{(1)\nu}(\bar{x}_1) \left[ 2h_1^{(1)}(\bar{x}_1) + \bar{\chi}_1 h_1^{(1)\nu}(\bar{x}_1) \right] \]

\[ - 6x_1^2 h_1^{(1)\nu}(\bar{x}_1) h_1^{(1)\nu}(\bar{x}_1), \]  
(A34)

\[ B = -6C_{30} x_1^5 - 6C_{40} x_1^7 + \bar{\chi}_1 h_1^{(1)\nu}(\bar{x}_1) \left[ x_1^2 h_1^{(1)\nu}(\bar{x}_1) - 6h_1^{(1)\nu}(\bar{x}_1) \right] \]

\[ - 2\bar{\chi}_1 h_1^{(1)\nu}(\bar{x}_1) \left[ 2x_1^2 h_1^{(1)\nu}(\bar{x}_1) + 3x_1^2 h_1^{(1)\nu}(\bar{x}_1) - 6h_1^{(1)\nu}(\bar{x}_1) \right] + 48x_1^3 h_1^{(1)\nu}(\bar{x}_1) h_1^{(1)}(\bar{x}_1) - 36x_1^2 h_1^{(1)}(\bar{x}_1) h_1^{(1)}(\bar{x}_1). \]  
(A35)

It follows from Eqs. (A32) and (A33) that

\[ C_{10} = \frac{B - 6A}{10}, \]  
(A36)

\[ C_{20} = \frac{16A - B}{10x_1^2}. \]  
(A37)

**APPENDIX B: SOLUTION OF EQ. (13)**

From Eqs. (18) and (19) of Part I [1], it follows that

\[ a_m = \frac{\bar{x}_1^2 h_1^{(1)\nu}(\bar{x}_m) - (m - 1)(m + 2) h_1^{(1)}(\bar{x}_m)}{2(m + 2)} b_m \quad \text{for} \quad m \geq 1, \]  
(B1)

\[ b_m = \frac{2i R_0 (m + 2) \omega_m s_m}{\omega_m (m + 1) \left[ \bar{x}_m h_1^{(1)\nu}(\bar{x}_m) + (m^2 + 3m + 2) h_1^{(1)}(\bar{x}_m) \right]}, \]  
(B2)

where \( \bar{x}_m = k_m R_0, k_m = (1 + i)/\delta_m, \delta_m = \sqrt{\nu/\omega_m}, \omega_m \) is the frequency of the mth mode, and \( s_m \) is the complex amplitude of the mth mode. In the case under consideration, it is assumed that modes 1 and \( m \) oscillate at the same frequency \( \omega_0 \), so \( \omega_m = \omega_0, k_m = k_1, \) and \( \bar{x}_m = \bar{x}_1 \).

By using Eqs. (B1) and (B2) and the transformation

\[ \sqrt{1 - \mu^2} \left[ \sqrt{1 - \mu^2} P_m(\mu) \right] = m(m + 1) \sqrt{1 - \mu^2} P_m(\mu), \]  
(B3)

Eq. (13) is rearranged to

\[ D^2 \langle \psi_2^{1m} \rangle = \frac{k_1^2}{\nu} \text{Re} \left\{ c_{1}^{\nu} b_m \left[ \mu P_m(\mu) G_1(x) \right. \right\} \]

\[ + \sqrt{1 - \mu^2} P_m(\mu) G_2(x) \}, \]  
(B4)

where, for convenience, we denote \( x_1 = x \) and \( \bar{x}_1 = \bar{x} \), the operator \( D \) is given by Eq. (A5), and \( G_1(x) \) and \( G_2(x) \) are
calculated by
\[ G_1(x) = \frac{\bar{h}_1^{(1)v}(\bar{x})}{6x} \left[ 2h_m^{(1)}(x) - x h_m^{(1)v}(x) \right] + \frac{1}{x} h_1^{(1)v}(x) h_m^{(1)}(x) - \frac{1}{x} h_1^{(1)v}(x) h_m^{(1)}(x), \]  
\[ G_2(x) = (m + 1) h_m^{(1)}(x) \left[ \frac{\bar{h}_1^{(1)v}(\bar{x})}{12x^3} - \frac{1}{x} h_1^{(1)v}(x) \right] \]  
\[ + \frac{(m + 1)x^{m+1}}{4(m + 2)x^{m+3}} \left[ \bar{h}_m^{(1)v}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x}) \right] \left[ x^2 h_1^{(1)}(x) - x^2 h_1^{(1)v}(x) \right]. \]  
\[ (B5) \]
\[ (B6) \]

A solution to Eq. (B4) is sought in the following form:
\[ \langle \psi_2^m \rangle = \frac{1}{\sqrt{v}} \text{Re} \left[ h_1^m \left[ \mu P^1_m(\mu) F_1(x) + \sqrt{1 - \mu^2 P^1_m(\mu)} F_2(x) \right] \right]. \]  
\[ (B7) \]

Substitution of Eq. (B7) into Eq. (B4) yields
\[ D^2 \left[ \mu P^1_m(\mu) F_1(x) \right] + D^2 \left[ \sqrt{1 - \mu^2 P^1_m(\mu)} F_2(x) \right] = k_1^2 \mu P^1_m(\mu) G_1(x) + k_1^2 \left[ \sqrt{1 - \mu^2 P^1_m(\mu)} G_2(x) \right]. \]  
\[ (B8) \]

By using Eqs. (C10) and (C14) from Appendix C, one obtains
\[ \mu P^1_m \left[ \frac{F_1^{IV}}{x} + \frac{4}{x} F_1^{IV} - \frac{2m(m + 1)}{x^2} F_1^{IV} + \frac{m(m + 1)(m^2 + m + 2)}{x^4} F_1 + \frac{4}{x} F_2^{IV} - \frac{4m(m + 1)}{x^2} F_2 \right] \]  
\[ + \sqrt{1 - \mu^2 P_m} \left[ \frac{F_1^{IV}}{x} + \frac{4}{x} F_1^{IV} - \frac{2m(m + 1)}{x^2} F_1^{IV} + \frac{m(m + 1)(m^2 + m + 6)}{x^4} F_1 + \frac{4}{x} F_2^{IV} - \frac{4m(m + 1)}{x^2} F_2 \right] \]  
\[ = \mu P^1_m G_1 + \sqrt{1 - \mu^2 P_m} G_2. \]  
\[ (B9) \]

Equation (B9) is divided into the following two equations:
\[ F_1^{IV} + \frac{4}{x} F_1^{IV} - \frac{2m(m + 1)}{x^2} F_1^{IV} + \frac{m(m + 1)(m^2 + m + 2)}{x^4} F_1 + \frac{4}{x} F_2^{IV} - \frac{4m(m + 1)}{x^2} F_2 = G_1, \]  
\[ (B10) \]
\[ F_2^{IV} + \frac{4}{x} F_2^{IV} - \frac{2m(m + 1)}{x^2} F_2^{IV} + \frac{m(m + 1)(m^2 + m + 6)}{x^4} F_1 + \frac{4}{x} F_2^{IV} - \frac{4m(m + 1)}{x^2} F_2 = G_2. \]  
\[ (B11) \]

To solve Eqs. (B10) and (B11), we need to transform them into a system of ordinary differential equations of first order [8]. To this end, we introduce the following functions:
\[ Y_1 = F_1, \quad Y_2 = xY_1', \quad Y_3 = xY_2', \quad Y_4 = xY_3', \quad Y_5 = F_2, \quad Y_6 = xY_5', \quad Y_7 = xY_6', \quad Y_8 = xY_7'. \]  
\[ (B12) \]
\[ (B13) \]

Equation (B12) gives the equations
\[ Y_2 = xF_1', \quad Y_3 = x^2 F_1'' + xF_1', \quad Y_4 = x^3 F_1''' + x^2 F_1'' + xF_1', \quad Y_5 = x^3 F_1'' + 6x^2 F_1''' + 7x F_1'' + F_1', \]  
\[ (B14) \]
from which we obtain
\[ F_1' = \frac{1}{x} Y_2, \quad F_1'' = \frac{1}{x^2} (Y_3 - Y_2), \quad F_1''' = \frac{1}{x^3} (Y_4 - 3Y_3 + 2Y_2), \quad F_1^{IV} = \frac{1}{x^4} Y_4' - \frac{1}{x^5} (6Y_4 - 11Y_3 + 6Y_2). \]  
\[ (B15) \]

In a similar way, from Eq. (B13) we find
\[ F_2' = \frac{1}{x} Y_6, \quad F_2'' = \frac{1}{x^2} (Y_7 - Y_6), \quad F_2''' = \frac{1}{x^3} (Y_8 - 3Y_7 + 2Y_6), \quad F_2^{IV} = \frac{1}{x^4} Y_8' - \frac{1}{x^5} (6Y_8 - 11Y_7 + 6Y_6). \]  
\[ (B16) \]

Substitution of Eqs. (B15) and (B16) into Eqs. (B10) and (B11) yields
\[ Y_4' = -\frac{m(m + 1)(m^2 + m + 2)}{x} Y_1 - \frac{2m(m + 1)}{x} Y_2 + \frac{2m(m + 1) + 1}{x} Y_3 + \frac{2}{x} Y_4 + \frac{4m(m + 1)}{x} Y_5 - \frac{4}{x} Y_6 - \frac{4}{x} Y_7 + x^3 G_1, \]  
\[ (B17) \]
\[
Y_8' = \frac{4m(m+1)}{x} [m(m+1)Y_1 + Y_2 - Y_3] - \frac{m(m+1)(m^2 + m + 6)}{x} Y_5 - \frac{2(m^2 + m + 2) + 2}{x} Y_6 \\
+ \frac{2(m^2 + m + 2) + 1}{x} Y_7 + \frac{2}{x} Y_8 + x^3 G_2.
\]  
(B18)

Equations (B12), (B13), (B17), and (B18) are combined into the following system:

\[
Y_1' = \frac{1}{x} Y_2, \quad Y_2' = \frac{1}{x} Y_3, \quad Y_3' = \frac{1}{x} Y_4, \\
Y_4' = -\frac{m(m+1)(m^2 + m + 2)}{x} Y_1 - \frac{2(m^2 + m + 1)}{x} Y_2 + \frac{2m^2 + 2m + 1}{x} Y_3 + \frac{2}{x} Y_4 + \frac{4m(m+1)}{x} Y_5 + \frac{4}{x} Y_6 - \frac{4}{x} Y_7 + x^3 G_1, \\
Y_5' = \frac{1}{x} Y_6, \quad Y_6' = \frac{1}{x} Y_7, \quad Y_7' = \frac{1}{x} Y_8, \\
Y_8' = \frac{4m(m+1)}{x} [m(m+1)Y_1 + Y_2 - Y_3] - \frac{m(m+1)(m^2 + m + 6)}{x} Y_5 - \frac{2(m^2 + m + 3)}{x} Y_6 \\
+ \frac{2m^2 + 2m + 5}{x} Y_7 + \frac{2}{x} Y_8 + x^3 G_2.
\]  
(B19)

The system of Eqs. (B19) is solved by the method of variation of parameters [8]. We first seek solutions to homogeneous equations corresponding to system Eqs. (B19), setting \( G_1 = G_2 = 0 \). Partial solutions are sought as \( Y_n = \gamma_n x^k \) \((n = 1, 2, \ldots, 8)\) with \( \gamma_n \) being a constant. Substituting this expression into system Eqs. (B19) with \( G_1 = G_2 = 0 \), one obtains a system of equations for \( \gamma_n \),

\[
\lambda \gamma_1 - \gamma_2 = 0, \quad \lambda \gamma_2 - \gamma_3 = 0, \quad \lambda \gamma_3 - \gamma_4 = 0, \\
m(m+1)(m^2 + m + 2)\gamma_1 + 2(m^2 + m + 1)\gamma_2 - (2m^2 + 2m + 1)\gamma_3 + (\lambda - 2)\gamma_4 - 4m(m+1)\gamma_5 - 4\gamma_6 + 4\gamma_7 = 0, \\
\lambda \gamma_5 - \gamma_6 = 0, \quad \lambda \gamma_6 - \gamma_7 = 0, \quad \lambda \gamma_7 - \gamma_8 = 0, \\
4m(m+1)(m^2 + m + 1)\gamma_1 + \gamma_2 - \gamma_3 - m(m+1)(m^2 + m + 6)\gamma_5 - 2(m^2 + m + 3)\gamma_6 + 2(m^2 + 2m + 5)\gamma_7 + (2 - \lambda)\gamma_8 = 0.
\]  
(B20)

This system has nonzero solutions for \( \gamma_n \) only if its determinant is equal to zero. This condition gives an equation for \( \lambda \),

\[
(\lambda^2 - \lambda - m - m^2)^2[(\lambda^2 - \lambda)^2 - 2(4 + m + m^2)(\lambda^2 - \lambda) + 12 - 8m - 7m^2 + 2m^3 + m^4] = 0.
\]  
(B21)

This equation is transformed to

\[
(\lambda^2 - \lambda - m - m^2)^2(\lambda^2 - \lambda - m - 5m - 6)(\lambda^2 - \lambda - m - 2m - 3) = 0.
\]  
(B22)

It is easy to check that the roots of Eq. (59) are given by

\[
\lambda_1 = -m - 2, \quad \lambda_2 = m + 3, \quad \lambda_3 = 2 - m, \quad \lambda_4 = m - 1, \quad \lambda_{5,6} = -m, \quad \lambda_{7,8} = m + 1.
\]  
(B23)

When we substitute \( \lambda_1 \) into system Eqs. (B20), we get equations that give the values of \( \gamma_n \) corresponding to the root \( \lambda_1 \). Let us denote these values as \( \gamma_{n1} \) \((n = 1, 2, \ldots, 8)\). Since the determinant of system Eqs. (B20) is equal to zero, only seven equations are independent. This means that one of the unknowns \( \gamma_{n1} \) should be taken as an arbitrary constant and then the other unknowns \( \gamma_{n1} \) can be expressed in terms of this constant. Let us set \( \gamma_{11} \) as an arbitrary constant. Then, substituting \( \lambda_1 \) into system Eqs. (B20) and solving this latter for \( \gamma_{n1} \) with \( n = 2, 3, \ldots, 8 \), we obtain

\[
\gamma_{21} = -(m + 2)\gamma_{11}, \quad \gamma_{31} = (m + 2)^2 \gamma_{11}, \quad \gamma_{41} = -(m + 2)^3 \gamma_{11}, \quad \gamma_{51} = -(m + 1)\gamma_{11}, \\
\gamma_{61} = (m + 1)(m + 2)\gamma_{11}, \quad \gamma_{71} = -(m + 1)(m + 2)^2 \gamma_{11}, \quad \gamma_{81} = (m + 1)(m + 2)^3 \gamma_{11}.
\]  
(B24)

In an analogous way, for the roots \( \lambda_2, \lambda_3, \text{ and } \lambda_4 \), one finds

\[
\gamma_{22} = (m + 3)\gamma_{12}, \quad \gamma_{32} = (m + 3)^2 \gamma_{12}, \quad \gamma_{42} = (m + 3)^3 \gamma_{12}, \quad \gamma_{52} = -(m + 1)\gamma_{12}, \\
\gamma_{62} = -(m + 1)(m + 3)\gamma_{12}, \quad \gamma_{72} = -(m + 1)(m + 3)^2 \gamma_{12}, \quad \gamma_{82} = -(m + 1)(m + 3)^3 \gamma_{12}, \\
\gamma_{23} = (2 - m)\gamma_{13}, \quad \gamma_{33} = (2 - m)^2 \gamma_{13}, \quad \gamma_{43} = (2 - m)^3 \gamma_{13}, \quad \gamma_{53} = m\gamma_{13}, \\
\gamma_{63} = m(2 - m)\gamma_{13}, \quad \gamma_{73} = m(2 - m)^2 \gamma_{13}, \quad \gamma_{83} = m(2 - m)^3 \gamma_{13}, \\
\gamma_{24} = -(m - 1)\gamma_{14}, \quad \gamma_{34} = -(m - 1)^2 \gamma_{14}, \quad \gamma_{44} = -(m - 1)^3 \gamma_{14}, \quad \gamma_{54} = m\gamma_{14}, \\
\gamma_{64} = m(m - 1)\gamma_{14}, \quad \gamma_{74} = m(m - 1)^2 \gamma_{14}, \quad \gamma_{84} = m(m - 1)^3 \gamma_{14},
\]  
(B25)

where \( \gamma_{12}, \gamma_{13}, \text{ and } \gamma_{14} \) are arbitrary constants.
We have two pairs of repeated roots: \( \lambda_5 = \lambda_6 \) and \( \lambda_7 = \lambda_8 \). Partial solutions corresponding to these roots are written as [8]

\[
Y_n = (\gamma_{65} + \gamma_{66} \ln x) x^{\lambda_5}, \quad Y_n = (\gamma_{57} + \gamma_{68} \ln x) x^{\lambda_7}, \quad n = 1, 2, \ldots, 8.
\]  

(B28)

To find \( \gamma_{65} \) and \( \gamma_{66} \), we substitute the partial solutions given by the first expression of Eq. (B28) into system Eqs. (B19) with \( G_1 = G_2 = 0 \). As a result, we obtain

\[
m\gamma_{15} - \gamma_{25} + (m\gamma_{16} + \gamma_{26}) \ln x = 0,
\]

\[
m\gamma_{25} - \gamma_{26} + \gamma_{35} + (m\gamma_{26} + \gamma_{36}) \ln x = 0,
\]

\[
m\gamma_{35} - \gamma_{25} + (m\gamma_{36} + \gamma_{46}) \ln x = 0,
\]

\[
m(m + 1)(m^2 + m + 2)(\gamma_{15} + \gamma_{16} \ln x) + 2(m^2 + m + 1)(\gamma_{25} + \gamma_{26} \ln x) - (2m^2 + 2m + 1)(\gamma_{35} + \gamma_{36} \ln x)
\]

\[
- (m + 2)(\gamma_{45} + \gamma_{46} \ln x) + \gamma_{46} - 4m(m + 1)(\gamma_{55} + \gamma_{56} \ln x) - 4(\gamma_{65} + \gamma_{66} \ln x) + 4(\gamma_{75} + \gamma_{76} \ln x) = 0,
\]

\[
m\gamma_{55} - \gamma_{56} + \gamma_{65} + (m\gamma_{56} + \gamma_{66}) \ln x = 0,
\]

\[
m\gamma_{65} - \gamma_{56} + \gamma_{75} + (m\gamma_{66} + \gamma_{76}) \ln x = 0,
\]

\[
m\gamma_{75} - \gamma_{76} + \gamma_{55} + (m\gamma_{76} + \gamma_{66}) \ln x = 0,
\]

\[
4m(m + 1)(m(m + 1)(\gamma_{15} + \gamma_{16} \ln x) + \gamma_{25} - \gamma_{35} + (\gamma_{26} - \gamma_{36}) \ln x) - m(m + 1)(m^2 + m + 6)(\gamma_{55} + \gamma_{56} \ln x)
\]

\[
- (m + 2)(m^2 + m + 3)(\gamma_{65} + \gamma_{66} \ln x) + (m^2 + m + 5)(\gamma_{75} + \gamma_{76} \ln x) + (m + 2)(\gamma_{85} + \gamma_{86} \ln x) - \gamma_{76} = 0.
\]  

(B29)

From these equations, it follows that all the constants \( \gamma_{66} = 0 \) and the constants \( \gamma_{65} \) are defined by

\[
\gamma_{25} = -m\gamma_{15}, \quad \gamma_{35} = m^2\gamma_{15}, \quad \gamma_{45} = -m\gamma_{55}, \quad \gamma_{55} = m^2\gamma_{55}, \quad \gamma_{65} = -m^3\gamma_{55},
\]  

(B30)

where \( \gamma_{15} \) and \( \gamma_{55} \) are arbitrary constants.

In a similar way, one obtains for the roots \( \lambda_7 = \lambda_8 \) that \( \gamma_{68} = 0 \) and the constants \( \gamma_{67} \) are defined by

\[
\gamma_{27} = (m + 1)\gamma_{17}, \quad \gamma_{37} = (m + 1)^2\gamma_{17}, \quad \gamma_{47} = (m + 1)^3\gamma_{17},
\]

\[
\gamma_{57} = (m + 1)^4\gamma_{57}, \quad \gamma_{67} = (m + 1)^5\gamma_{57}, \quad \gamma_{77} = (m + 1)^6\gamma_{57},
\]  

(B31)

where \( \gamma_{17} \) and \( \gamma_{57} \) are arbitrary constants.

The general homogeneous solutions are written as

\[
Y_n = \gamma_{41}x^{-m-2} + \gamma_{42}x^{m+3} + \gamma_{43}x^{2-m} + \gamma_{44}x^{m-1} + \gamma_{45}x^{-m} + \gamma_{46}x^{m+1}, \quad n = 1, 2, \ldots, 8.
\]  

(B32)

Recall that eight of the constants \( \gamma_{4n} \) are arbitrary.

To find solutions to the inhomogeneous system Eqs. (B19), we assume that eight arbitrary constants \( \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}, \gamma_{17}, \gamma_{35}, \gamma_{57} \) are functions of \( x \). With this assumption, the substitution of the general solutions given by Eq. (B32) into system Eqs. (B19) yields

\[
\gamma_{11}x^{-m-2} + \gamma_{12}x^{m+3} + \gamma_{13}x^{2-m} + \gamma_{14}x^{m-1} + \gamma_{15}x^{-m} + \gamma_{17}x^{m+1} = 0,
\]

\[
(m + 2)\gamma_{11}x^{-m-2} - (m + 3)\gamma_{12}x^{m+3} + (m - 2)\gamma_{13}x^{2-m} - (m - 1)\gamma_{14}x^{m-1} + m\gamma_{15}x^{-m} - (m + 1)\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 2)^2\gamma_{11}x^{-m-2} + (m + 3)^2\gamma_{12}x^{m+3} + (m - 2)^2\gamma_{13}x^{2-m} + (m - 1)^2\gamma_{14}x^{m-1} + m^2\gamma_{15}x^{-m} + (m + 1)^2\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 2)^3\gamma_{11}x^{-m-2} - (m + 3)^3\gamma_{12}x^{m+3} + (m - 2)^3\gamma_{13}x^{2-m} - (m - 1)^3\gamma_{14}x^{m-1} + m^3\gamma_{15}x^{-m} + (m + 1)^3\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 1)\gamma_{11}x^{-m-2} + (m + 1)^2\gamma_{12}x^{m+3} - m\gamma_{13}x^{2-m} - m\gamma_{14}x^{m-1} - \gamma_{15}x^{-m} - m\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 1)(m + 2)\gamma_{11}x^{-m-2} - (m + 1)(m + 3)\gamma_{12}x^{m+3} - m(m - 2)\gamma_{13}x^{2-m} + m(m - 1)\gamma_{14}x^{m-1} - m\gamma_{15}x^{-m} + (m + 1)\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 1)(m + 2)^2\gamma_{11}x^{-m-2} + (m + 1)(m + 3)^2\gamma_{12}x^{m+3} - m(m - 2)^2\gamma_{13}x^{2-m} - m(m - 1)^2\gamma_{14}x^{m-1} - m^2\gamma_{15}x^{-m} + m^2\gamma_{17}x^{m+1} = 0,
\]

\[
(m + 1)(m + 2)^3\gamma_{11}x^{-m-2} - (m + 1)(m + 3)^3\gamma_{12}x^{m+3} + m(m - 2)^3\gamma_{13}x^{2-m} + m(m - 1)^3\gamma_{14}x^{m-1} - m^3\gamma_{15}x^{-m} + (m + 1)^3\gamma_{17}x^{m+1} = x^3G_2.
\]  

(B33)
From these equations, one obtains that
\[
\begin{align*}
\gamma_{11}(x) &= \tilde{\gamma}_{11} + \frac{1}{2(2m + 1)(2m + 3)(2m + 5)} \int_{x}^{\infty} [G_2(s) - mG_1(s)]s^{m+5} ds, \\
\gamma_{12}(x) &= \tilde{\gamma}_{12} + \frac{1}{2(2m + 1)(2m + 3)(2m + 5)} \int_{x}^{\infty} [mG_1(s) - G_2(s)]s^{-m} ds, \\
\gamma_{13}(x) &= \tilde{\gamma}_{13} + \frac{1}{2(2m + 1)(2m - 1)(2m - 3)} \int_{x}^{\infty} [(m + 1)G_1(s) + G_2(s)]s^{m+1} ds, \\
\gamma_{14}(x) &= \tilde{\gamma}_{14} - \frac{1}{2(2m + 1)(2m - 1)(2m - 3)} \int_{x}^{\infty} [(m + 1)G_1(s) + G_2(s)]s^{-m} ds, \\
\gamma_{15}(x) &= \tilde{\gamma}_{15} - \frac{1}{2(2m + 1)(2m + 1)(2m + 3)} \int_{x}^{\infty} [3G_1(s) + 2G_2(s)]s^{m+3} ds, \\
\gamma_{17}(x) &= \tilde{\gamma}_{17} + \frac{1}{2(2m + 1)(2m + 1)(2m + 3)} \int_{x}^{\infty} [3G_1(s) + 2G_2(s)]s^{-m} ds, \\
\gamma_{55}(x) &= \tilde{\gamma}_{55} - \frac{1}{2(2m + 1)(2m + 1)(2m + 3)} \int_{x}^{\infty} [2(m + 1)G_1(s) + G_2(s)]s^{m+3} ds, \\
\gamma_{57}(x) &= \tilde{\gamma}_{57} + \frac{1}{2(2m + 1)(2m + 1)(2m + 3)} \int_{x}^{\infty} [2(m + 1)G_1(s) + G_2(s)]s^{-m} ds,
\end{align*}
\]
where \(\tilde{\gamma}_{11}, \tilde{\gamma}_{12},\) etc. are constants to be determined by boundary conditions.

It will be recalled that \(Y_1 = F_1\) and \(Y_2 = F_2\) so we have
\[
\begin{align*}
F_1(x) &= \gamma_{11}(x)x^{-m-2} + \gamma_{12}(x)x^{m+3} + \gamma_{13}(x)x^{2-m} + \gamma_{14}(x)x^{m-1} + \gamma_{15}(x)x^{-m} + \gamma_{17}(x)x^{m+1}, \\
F_2(x) &= -(m + 1)\gamma_{11}(x)x^{-m-2} - (m + 1)\gamma_{12}(x)x^{m+3} + m\gamma_{13}(x)x^{2-m} + m\gamma_{14}(x)x^{m-1} + \gamma_{55}(x)x^{-m} + \gamma_{57}(x)x^{m+1}.
\end{align*}
\]
The components of the Eulerian streaming velocity are calculated by
\[
\begin{align*}
\langle v_{2r}^{1m} \rangle &= -\frac{1}{v_f} \nabla \cdot \frac{\partial}{\partial x} \frac{1}{\varrho} \left[ \langle \psi_{2r}^{1m} \rangle \sqrt{1 - \mu^2} \right], \\
\langle v_{20}^{1m} \rangle &= -\frac{1}{v_f} \nabla \cdot \frac{\partial}{\partial x} \left[ \langle \psi_{20}^{1m} \rangle \right].
\end{align*}
\]
Substitution of Eq. (B7) into Eqs. (B37) and (B38) yields
\[
\begin{align*}
\langle v_{2r}^{1m} \rangle &= -\frac{1}{v_f} \nabla \cdot \frac{\partial}{\partial x} \left[ \frac{1}{\varrho} \left[ \frac{\partial}{\partial \mu} \mu^2 P_1(\mu) (m+1)(2m+5) \right] F_1(x) \right], \\
\langle v_{20}^{1m} \rangle &= -\frac{1}{v_f} \nabla \cdot \frac{\partial}{\partial x} \left[ \frac{1}{\varrho} \left[ \frac{\partial}{\partial \mu} \mu^2 P_1(\mu) (m+1)(2m+3) \right] F_2(x) \right],
\end{align*}
\]
where
\[
\begin{align*}
F_1(x) &= -(m + 2)\gamma_{11}(x)x^{-m-3} + (m + 3)\gamma_{12}(x)x^{m+3} + (2 - m)\gamma_{13}(x)x^{2-m} + (m + 1)\gamma_{17}(x)x^{m+1} \\
&\quad - m\gamma_{15}(x)x^{-m} + (m + 1)\gamma_{14}(x)x^{m-1} + (m - 1)\gamma_{14}(x)x^{m-2}, \\
F_2(x) &= -(m + 1)(m + 2)\gamma_{11}(x)x^{-m-3} - (m + 1)(m + 3)\gamma_{12}(x)x^{m+3} + m(2 - m)\gamma_{13}(x)x^{2-m} \\
&\quad + m(m + 1)\gamma_{14}(x)x^{m-2} - m\gamma_{55}(x)x^{-m} - (m + 1)\gamma_{57}(x)x^{m} + (m - 1)\gamma_{14}(x)x^{m-1}.
\end{align*}
\]
From the condition \(\langle v_{2r}^{1m} \rangle \to 0\) for \(r \to \infty\), it follows that
\[
\begin{align*}
\tilde{\gamma}_{12} &= -\frac{1}{2(2m + 1)(2m + 3)(2m + 5)} \int_{x}^{\infty} [mG_1(s) - G_2(s)]s^{-m} ds, \\
\tilde{\gamma}_{14} &= \frac{1}{2(2m + 1)(2m - 1)(2m - 3)} \int_{x}^{\infty} [(m + 1)G_1(s) + G_2(s)]s^{4-m} ds, \\
\tilde{\gamma}_{17} &= -\frac{1}{2(2m + 1)(2m - 1)(2m + 3)} \int_{x}^{\infty} [3G_1(s) + 2G_2(s)]s^{2-m} ds, \\
\tilde{\gamma}_{57} &= -\frac{1}{2(2m - 1)(2m + 1)(2m + 3)} \int_{x}^{\infty} [2(m + 1)G_1(s) + G_2(s)]s^{2-m} ds.
\end{align*}
\]
To find the other constants, we need to calculate the Lagrangian streaming velocity, \( \mathbf{v}_L^{1m} = (\mathbf{v}_x^{1m}) + \mathbf{v}_z^{1m} \), where \( \mathbf{v}_z^{1m} \) denotes the Stokes drift velocity, which is calculated by [3]

\[
\mathbf{v}_z^{1m} = \left( \int \mathbf{v}_1 dt \cdot \nabla \right) \mathbf{v}_1 \Bigg|_{1m} = \frac{1}{2\omega_1} \text{Re}[i(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_z^1]_{1m}. \tag{B47}
\]

In Eq. (B47), \( \mathbf{v}_1 \) is the linear liquid velocity and the subscript \( 1m \) means that cross terms produced by modes 1 and \( m \) should be only kept. Equation (B47) gives

\[
v_{1r}^{1m} = \frac{1}{2\omega_1} \text{Re} \left\{ iv_{1r}^0 \frac{\partial v_x^{1m}}{\partial r} - iv_{1r}^0 \frac{\partial v_y^{1m}}{\partial \theta} + \frac{iv_{1r}^m}{r} \frac{\partial v_x^{1m}}{\partial \theta} - \frac{iv_{1r}^m}{r} \frac{\partial v_y^{1m}}{\partial r} \right\}, \tag{B48}
\]

\[
v_{1\theta}^{1m} = \frac{1}{2\omega_1} \text{Re} \left\{ iv_{1\theta}^0 \frac{\partial v_y^{1m}}{\partial r} - iv_{1\theta}^0 \frac{\partial v_x^{1m}}{\partial \theta} + \frac{iv_{1\theta}^m}{r} \frac{\partial v_y^{1m}}{\partial \theta} - \frac{iv_{1\theta}^m}{r} \frac{\partial v_x^{1m}}{\partial r} \right\}. \tag{B49}
\]

where \( v_{1r}^{1m} \) and \( v_{1\theta}^{1m} \) are the radial and tangential components of the linear liquid velocity \( \mathbf{v}_1^{1m} \) produced by mode \( m \).

From Eqs. (11) and (12) of Part I [1] and Eq. (B1), one obtains

\[
v_{1r}^{1m} = -e^{-i\omega t} b_m P_m(\mu) m + 1 \frac{1}{R_0} \left[ \frac{\bar{v}_m^{1m}(x)}{x} - (m^2 + m - 2) \frac{h^{1m}(\bar{x})}{x} \right] \left( \frac{R_0}{r} \right)^{m+1} + \frac{m\bar{x}}{x} \frac{h^{1m}(x)}{x}, \tag{B50}
\]

\[
v_{1\theta}^{1m} = -e^{-i\omega t} b_m P_m(\mu) m + 1 \frac{1}{R_0} \left[ \frac{\bar{v}_m^{1m}(x)}{x} - (m^2 + m - 2) \frac{h^{1m}(\bar{x})}{x} \right] \left( \frac{R_0}{r} \right)^{m+1} - \frac{\bar{x}}{x} \frac{h^{1m}(\bar{x})}{x} - \frac{\bar{x}}{x} \frac{h^{1m}(x)}{x}. \tag{B51}
\]

Substitution of Eqs. (B50) and (B51) into Eqs. (B48) and (B49) yields

\[
v_{1r}^{1m} = -\frac{1}{6\nu R_0} \text{Re} \left\{ b_m^* b_m \left[ \mu P_m(\mu) S_1(x) + \sqrt{1 - \mu^2} P_m(\mu) S_2(x) \right] \right\}, \tag{B52}
\]

\[
v_{1\theta}^{1m} = -\frac{1}{6\nu R_0} \text{Re} \left\{ b_m^* b_m \left[ \mu P_m(\mu) S_3(x) + \sqrt{1 - \mu^2} P_m(\mu) S_4(x) \right] \right\}, \tag{B53}
\]

where

\[
S_1(x) = (m + 1) \left[ \frac{3\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) - \frac{6}{x} h_1^{(1\nu)}(x) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \times \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right] - \left[ \frac{\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) + \frac{6}{x} h_1^{(1\nu)}(x) \right]^* \times \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} - m\bar{x} h_m^{(1\nu)}(x) + \frac{m\bar{x}}{x} h_m^{(1\nu)}(x) \right] \right\}, \tag{B54}
\]

\[
S_2(x) = \frac{m + 1}{2} \left[ \frac{\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) - \frac{6}{x} h_1^{(1\nu)}(x) - \frac{6}{x^2} h_1^{(1)}(x) \right]^* \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} + \frac{m\bar{x}}{x} h_m^{(1)}(x) \right] - \left[ \frac{\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) + \frac{6}{x} h_1^{(1\nu)}(x) \right]^* \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} - m\bar{x} h_m^{(1\nu)}(x) - \frac{m\bar{x}}{x} h_m^{(1\nu)}(x) \right] \right\}, \tag{B55}
\]

\[
S_3(x) = \left[ \frac{2\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) - \frac{6}{x} h_1^{(1\nu)}(x) \right]^* \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} - \bar{x} h_m^{(1\nu)}(x) - \frac{\bar{x}}{x} h_m^{(1\nu)}(x) \right] - \left[ \frac{\bar{x}^4}{x^4} h_1^{(1\nu)}(\bar{x}) + \frac{6}{x^2} h_1^{(1)}(x) \right]^* \left[ \frac{\bar{x}^2 h_m^{(1\nu)}(\bar{x}) - (m^2 + m - 2) h_m^{(1)}(\bar{x})}{2(m + 2)} \left( \frac{\bar{x}}{x} \right)^{m+2} + \bar{x} h_m^{(1\nu)}(x) + \bar{x} h_m^{(1\nu)}(x) - \frac{\bar{x}}{x} h_m^{(1)}(x) \right] \right\}, \tag{B56}
\]
Let us apply the boundary conditions for the Lagrangian streaming velocity at the bubble surface. They are written as

\[ \frac{1}{r} \partial v_{l,m}^{1 \mu} \left\{ \begin{array}{l} \partial v_{l,0}^{1 \mu} - \frac{1}{r} \frac{\partial v_{l,0}^{1 \mu}}{\partial \theta} - \frac{1}{m} \frac{\partial v_{l,0}^{1 \mu}}{\partial r} = 0 \quad \text{at} \quad r = R_0. \end{array} \right. \]

Substituting Eqs. (B39), (B40), (B52), and (B53) into Eqs. (B58) and (B59), one obtains the following system of equations:

\[ \begin{align*}
(m+1)(m+2)\bar{\gamma}_{11}x^{m-2} + m(m-1)\bar{\gamma}_{11}x^{2-m} + m(m+1)\bar{\gamma}_{11}x^{-m} - 2\bar{\gamma}_{55}x^{-m} & = -(m+1)(m+2)\bar{\gamma}_{12}x^{m+3} - m(m-1)\bar{\gamma}_{14}x^{m-1} - m(m+1)\bar{\gamma}_{17}x^{m+1} + 2\bar{\gamma}_{57}x^{-m} - \frac{S_1(x)}{6}, \\
(m+2)\bar{\gamma}_{11}x^{m-2} - m(m-1)\bar{\gamma}_{13}x^{2-m} + \bar{\gamma}_{15}x^{-m} - \bar{\gamma}_{55}x^{-m} & = -(m+2)\bar{\gamma}_{12}x^{m+3} - m(m+1)\bar{\gamma}_{14}x^{m-1} - \bar{\gamma}_{17}x^{m+1} + \frac{S_2(x)}{6}, \\
(m+1)(m+3)\bar{\gamma}_{11}x^{m-2} + m(m-2)\bar{\gamma}_{13}x^{2-m} + (m^2 + m + 1)\bar{\gamma}_{15}x^{-m} - \bar{\gamma}_{55}x^{-m} & = -(m+1)(m+3)\bar{\gamma}_{12}x^{m+3} - m(m-2)\bar{\gamma}_{14}x^{m-1} - (m^2 + m + 1)\bar{\gamma}_{17}x^{m+1} + \bar{\gamma}_{57}x^{-m} - \frac{1}{12}[S_1(x) - S_2(x) + \bar{\delta}S_4(x)], \\
(m+1)(m+2)^2\bar{\gamma}_{11}x^{m-2} - m(m-1)^2\bar{\gamma}_{13}x^{2-m} + m(m+1)\bar{\gamma}_{15}x^{-m} - (m^2 + m + 1)\bar{\gamma}_{55}x^{-m} & = -(m+1)(m+2)^2\bar{\gamma}_{12}x^{m+3} + m(m-1)^2\bar{\gamma}_{14}x^{m-1} - m(m+1)\bar{\gamma}_{17}x^{m+1} + (m^2 + m + 1)\bar{\gamma}_{57}x^{-m} - \frac{1}{12}[S_1(x) + m(m+1)S_2(x) + S_4(x) - \bar{\delta}S_4(x)].
\end{align*} \]

Solving these equations for four remaining constants, one gets finally

\[ \begin{align*}
\bar{\gamma}_{11} & = -\bar{\gamma}_{12}x^{2m+5} + \frac{x^{m+2}}{12(m+1)(2m+3)} \left\{ \frac{m[(m^2 - 3)S_1(x) + (m^3 - m + 4)S_2(x)]}{m-1(m+2)} + m[S_3(x) - \bar{\delta}S_4(x)] - S_4(x) + \bar{\delta}S_4(x) \right\}, \\
\bar{\gamma}_{13} & = -\bar{\gamma}_{14}x^{2m-3} - \frac{x^{m-2}}{12(2m-1)(2m+1)} \left\{ (m+1)[(m^2 + 2m - 2)S_1(x) - (m^3 + 3m^2 + 2m - 4)S_2(x)] + (m+1)[S_3(x) - \bar{\delta}S_4(x)] + S_4(x) - \bar{\delta}S_4(x) \right\}, \\
\bar{\gamma}_{15} & = -\bar{\gamma}_{17}x^{2m+1} - \frac{x^m}{12(2m-1)(2m+3)} \left\{ \frac{m[(m+1)[3S_1(x) + 2(m^2 + m - 5)S_2(x)]}{m-1(m+2)} - 3[S_3(x) - \bar{\delta}S_4(x)] - 2[S_4(x) - \bar{\delta}S_4(x)] \right\}, \\
\bar{\gamma}_{55} & = -\bar{\gamma}_{57}x^{2m+1} + \frac{x^m}{12(2m-1)(2m+3)} \left\{ \frac{2(m^4 + 4m^3 - 9m^2 - 11m + 6)S_1(x) + m(m+1)(3m^2 + 3m + 2)S_2(x)}{m-1(m+2)} + 2m(m+1)[S_3(x) - \bar{\delta}S_4(x)] + S_4(x) - \bar{\delta}S_4(x) \right\}.
\end{align*} \]

Expressions for \( S'_4(x) \) and \( S''_4(x) \) are provided in Appendix D.
APPENDIX C: EQUATIONS USED FOR THE CALCULATION OF Eq. (B9)

This Appendix provides equations that were used in the course of the derivation of Eq. (B9).

\[
D[\mu P_m^1(\mu)F_1(x)] = \frac{k_1^2}{x^2} \left\{ \mu P_m^1(x^2 F_1'') + F_1 \left[ (1 - \mu^2)(\mu^2 P_m'') - 2\mu(\mu^2 P_m') - \frac{\mu P_m^1}{1 - \mu^2} \right] \right\}
\]

\[
= \frac{k_1^2}{x^2} \left\{ \mu P_m^1(x^2 F_1'') + F_1 \left[ (1 + \mu^2)(\mu^2 P_m'') - 2\mu(\mu^2 P_m') + 2(1 - \mu^2) P_m'' - 2\mu P_m^1 - \frac{\mu P_m^1}{1 - \mu^2} \right] \right\}
\]

\[
= \frac{k_1^2}{x^2} \left\{ \mu P_m^1 \left[ \frac{F_1'}{x} + \frac{2}{x} F_1' - \frac{m(m + 1)}{x^2} F_1 \right] + 2m(m + 1) \sqrt{1 - \mu^2} P_m F_1 \right\}.
\]  
(C1)

Here, we have used the following equations:

\[
(1 - \mu^2)P_m'' - 2\mu P_m' = \frac{P_m^1}{1 - \mu^2} - m(m + 1)P_m^1,
\]  
(C2)

\[
(1 - \mu^2)P_m''(\mu) = \mu P_m^1(\mu) + m(m + 1)\sqrt{1 - \mu^2} P_m(\mu),
\]  
(C3)

\[
D[\sqrt{1 - \mu^2} P_m(\mu)F_2(x)] = \frac{k_1^2}{x^2} \left\{ \sqrt{1 - \mu^2} P_m(x^2 F_2'') + F_2 \left[ (1 - \mu^2)(\sqrt{1 - \mu^2} P_m)'' - 2\mu(\sqrt{1 - \mu^2} P_m) - \frac{P_m}{1 - \mu^2} \right] \right\}
\]

\[
= \frac{k_1^2}{x^2} \left\{ \sqrt{1 - \mu^2} P_m(x^2 F_2'') + F_2 \left[ (1 - \mu^2)(\sqrt{1 - \mu^2} P_m)'' - 2\mu(\sqrt{1 - \mu^2} P_m) + 2\mu P_m^1 - 2\sqrt{1 - \mu^2} P_m - 2\mu \sqrt{1 - \mu^2} P_m \right] \right\}
\]

\[
= \frac{k_1^2}{x^2} \left\{ \sqrt{1 - \mu^2} P_m \left( F_2'' + \frac{2}{x} F_2' - \frac{m^2 + m + 2}{x^2} F_2 \right) + 2m(m + 1) \sqrt{1 - \mu^2} P_m F_2 \right\}.
\]  
(C4)

Here, we have used the following equations:

\[
(1 - \mu^2)P_m''(\mu) - 2\mu P_m'(\mu) = -m(m + 1)P_m(\mu),
\]  
(C5)

\[
\sqrt{1 - \mu^2} P_m'(\mu) = -P_m(\mu).  
\]  
(C6)

Applying the operator \(D\) to Eq. (C1), one has

\[
D^2[\mu P_m^1(\mu)F_1(x)] = k_1^2 D[\mu P_m^1 H_1] + 2m(m + 1)k_1^2 D[\sqrt{1 - \mu^2} P_m H_2],
\]  
(C7)

where

\[
H_1 = F_1'' + \frac{2}{x} F_1' - \frac{m(m + 1)}{x^2} F_1, \quad H_2 = \frac{F_1}{x^2}.
\]

By using Eqs. (C1) and (C4), one obtains

\[
D^2[\mu P_m^1(\mu)F_1(x)] = k_1^2 \left\{ \mu P_m^1 \left[ H_1'' + \frac{2}{x} H_1' - \frac{m(m + 1)}{x^2} H_1 \right] + 2m(m + 1) \sqrt{1 - \mu^2} P_m \frac{H_1}{x^2} \right\}
\]

\[
+ 2m(m + 1)k_1^2 \left\{ \sqrt{1 - \mu^2} P_m \left( H_2'' + \frac{2}{x} H_2' - \frac{m^2 + m + 2}{x^2} H_2 \right) + 2\mu P_m^1 \frac{H_2}{x^2} \right\}.  
\]  
(C9)

Substitution of Eq. (C8) into Eq. (C9) yields

\[
D^2[\mu P_m^1(\mu)F_1(x)] = 4m(m + 1)k_1^4 \sqrt{1 - \mu^2} P_m \left[ \frac{F_1''}{x^2} - \frac{m(m + 1)}{x^4} F_1 \right]
\]

\[
+ k_1^4 \mu P_m^1 \left[ F_1'' + \frac{2}{x} F_1' - \frac{2m(m + 1)}{x^2} F_1'' + \frac{m(m + 1)(m^2 + m + 2)}{x^4} F_1 \right].
\]  
(C10)

Applying the operator \(D\) to Eq. (C4), one has

\[
D^2[\sqrt{1 - \mu^2} P_m(\mu)F_2(x)] = k_1^2 D[\sqrt{1 - \mu^2} P_m J_1] + 2k_1^2 D[\mu P_m^1 J_2],
\]  
(C11)

where

\[
J_1 = F_2'' + \frac{2}{x} F_2' - \frac{m^2 + m + 2}{x^2} F_2, \quad J_2 = \frac{F_2}{x^2}.
\]  
(C12)
By using Eqs. (C1) and (C4), one obtains
\[
D^2 \left[ \sqrt{1 - \mu^2 P_m(\mu) F_2(x)} \right] = 2k_0^4 P_m \left[ \frac{J''_2}{x^2} - \frac{m(m+1)}{x^2} J_2 + \frac{1}{x^2} J_1 \right] 
+ k_0^4 \sqrt{1 - \mu^2 P_m} \left[ \frac{J''_1}{x^2} - \frac{2(m^2 + m + 2)}{x^2} J_1 + \frac{4m(m+1)}{x^2} J_1 \right].
\] (C13)

Substitution of Eq. (C12) into Eq. (C13) results in
\[
D^2 \left[ \sqrt{1 - \mu^2 P_m(\mu) F_2(x)} \right] = 4k_0^4 P_m \left[ \frac{1}{x^2} F''_2 - \frac{m(m+1)}{x^2} F_2 \right] 
+ k_0^4 \sqrt{1 - \mu^2 P_m} \left[ F''_1 + \frac{4}{x^2} F''_2 - \frac{2(m^2 + m + 2)}{x^2} F_2 + \frac{m(m+1)(m^2 + m + 6)}{x^4} F_2 \right].
\] (C14)

**APPENDIX D: EXPRESSIONS FOR \( S'_1(\bar{x}) \) AND \( S'_2(\bar{x}) \)**

This Appendix provides expressions for \( S'_1(\bar{x}) \) and \( S'_2(\bar{x}) \), which appear in Eqs. (B61)–(B64). They are calculated by differentiating Eqs. (B56) and (B57).

\[
S'_1(\bar{x}) = \left[ \frac{6}{\bar{x}^2} h'_1(\bar{x}) - \frac{14}{\bar{x}} h''_1(\bar{x}) + \frac{3}{\bar{x}^3} h'''_1(\bar{x}) \right]^* \left[ \frac{\bar{x}^2 h''_1(\bar{x}) - (m^2 + m - 2) h''_1(\bar{x}) - \bar{x} h'''_1(\bar{x})}{2(m+2)} \right] 
+ \left[ \frac{2 h''_1(\bar{x}) - h''_1(\bar{x})}{2 m} \right]^* \left[ \frac{m(m+1)}{2m} h''_1(\bar{x}) - h''_1(\bar{x}) - \frac{3}{2} \bar{x} h'''_1(\bar{x}) \right] 
+ \left[ \frac{4}{\bar{x}^2} h''_1(\bar{x}) - \frac{6}{\bar{x}} h'''_1(\bar{x}) + \frac{12}{\bar{x}^3} h''''_1(\bar{x}) \right]^* \left[ \frac{3}{2} \bar{x}^2 h''_1(\bar{x}) + \bar{x} h'''_1(\bar{x}) - \frac{m(m+1)}{2} h''_1(\bar{x}) \right] 
- \left[ h''_1(\bar{x}) + \frac{6}{\bar{x}^2} h'''_1(\bar{x}) \right]^* \left[ \frac{\bar{x}^2 h''_1(\bar{x}) - m - 2}{2} \bar{x} h'''_1(\bar{x}) - h''_1(\bar{x}) + \frac{(m+2)(m^2 + m - 2)}{2 \bar{x}} h''_1(\bar{x}) \right].
\] (D1)

\[
S'_2(\bar{x}) = \frac{m+1}{2} \left[ \left[ \frac{6}{\bar{x}^2} h''_1(\bar{x}) - \frac{6}{\bar{x}} h'''_1(\bar{x}) - \frac{3}{\bar{x}^3} h''''_1(\bar{x}) \right]^* \left[ \frac{\bar{x}^2 h''_1(\bar{x})}{m+2} + (m+1) h''_1(\bar{x}) \right] 
+ \left[ \frac{9 h''_1(\bar{x}) + 6 h'''_1(\bar{x}) - \frac{6}{\bar{x}^2} h''''_1(\bar{x})}{2 \bar{x}} \right]^* \left[ \frac{(m-2)(m+1)}{2 \bar{x}} h''_1(\bar{x}) + m h''_1(\bar{x}) - \frac{5}{2} h''_1(\bar{x}) \right] 
+ \left[ \frac{6}{\bar{x}^3} h''_1(\bar{x}) - \frac{5}{\bar{x}^2} h'''_1(\bar{x}) \right]^* \left[ \frac{(m+1)(\bar{x}^2 h''_1(\bar{x}) - m^2 + m - 2) h''_1(\bar{x})}{m+2} - m \bar{x} h'''_1(\bar{x}) \right] 
+ \left[ h''_1(\bar{x}) - \frac{6}{\bar{x}} h'''_1(\bar{x}) - \frac{6}{\bar{x}^2} h''''_1(\bar{x}) \right]^* \left[ \frac{(m+1)(m^2 + m - 2) h''_1(\bar{x})}{2 \bar{x}} - \frac{3m+1}{2} \bar{x} h'''_1(\bar{x}) \right] \right].
\] (D2)


[5] See Supplemental Material at [http://link.aps.org/supplemental/10.1103/PhysRevE.100.033105](http://link.aps.org/supplemental/10.1103/PhysRevE.100.033105) for a MATLAB code that implements the calculation of the Lagrangian streaming velocity for the \( 1 \rightarrow 1 \) case (MainProgramm_Case_11.m) and for the \( 1 \rightarrow m \) case (MainProgramm_Case_1m.m).

