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Physically Admissible Impedance Models for Time-Domain Computations of Outdoor Sound Propagation

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Summary
Conditions for an impedance model to be physically admissible are checked for some popular models in the outdoor sound propagation community. They require that the definition of the impedance model is extended in the whole complex plane and that its inverse Fourier transform is real, causal and passive. For the many impedance models that are written as the square-root of a rational function, such as the Zwikker and Kosten model, the four-parameter Attenborough model and the Hamet and Bérengier model, these conditions are shown to be satisfied for a semi-infinite ground and for a rigidly backed layer. The case of polynomial type models is then investigated. The Delany and Bazley model is not physically admissible as it is real or causal depending on its extension in the complex plane, but it can not simultaneously fulfill both conditions. The Miki model for a rigidly backed layer does not satisfy also the passivity condition as its real part is negative for low frequencies. A new polynomial model is thus proposed and is shown to be physically admissible.
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1. Introduction

Reflection of acoustic waves over a natural ground is typically modelled with an impedance boundary condition, which relates the Fourier transforms of the acoustic pressure and the normal velocity to the ground through the surface impedance. This quantity accounts for the ground properties, such as the air flow resistivity, the porosity and the tortuosity. Up until 10 years ago, analytical and numerical models for outdoor sound propagation were developed in the frequency domain. Consequently, most of the impedance models used for natural grounds have been proposed in this context. Nowadays, time-domain methods have become increasingly popular for studying outdoor propagation in complex meteorological conditions [1, 2]. Accounting for ground effects in time domain models has proved challenging and, therefore, two main strategies have been adopted. In the first one, propagation in the ground medium is explicitely modelled through propagation equations, using usually those proposed by Zwikker and Kosten [1, 2] as it does not lead to the computation of convolutions, which is cumbersome for long-range sound propagation [3]. More complex propagation equations, which involve convolutions, such as those based on the Wilson's relaxation model [4] have been proposed but related numerical solvers are limited up to now to one-dimensional and two-dimensional configurations [5, 6, 7], due to the high computational cost of the direct evaluation of convolution integrals. In the second strategy, the impedance boundary condition is translated into the time domain, using the local reaction assumption, which is generally valid for natural grounds [8]. The surface impedance is usually approximated by polynomials [9] or rational functions [10, 11, 12, 13, 14, 15] to avoid the computation of convolutions. In the proposed applications, many impedance models have been used such as the Delany and Bazley model [9], the Miki model [12, 14], the variable porosity model [13], the Zwikker and Kosten model [11], the Attenborough model [11] or the Hamet and Bérengier model [16].

However, it is not clear if the surface impedance models proposed in the frequency domain are adapted to time-domain computations. Indeed, the translation of the impedance boundary condition into the time domain leads to some restrictions on the possible analytical expressions of the impedance [17]. First of all, the definition of the impedance must be extended to the whole complex plane. Then, the inverse Fourier transform of the impedance must be real-valued and causal. In addition, the ground must also be passive, because no acoustic energy comes from the ground. Rienstra [17] has proposed three conditions in the frequency domain for an impedance model to be physically admissible, which have been checked for some models used in the duct acoustics community. The causality condition has already been investigated by Berthelot [18] by invoking the Kramers-Krönig relations. Besides, other

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studies have been concerned with the causality of propagation equations in porous media for various models such as the Johnson and Allard model [19] or the power-law attenuation model [20].

The main objective of this paper is to check some widely spread impedance models in the outdoor sound propagation studies against Rienstra's conditions. A recent study of Attenborough *et al* [21] presents the impedance models available to characterize natural grounds. In this paper, two types of impedance models which are those, which can be written as a square root of a rational function and which are referred to as the square-root type models, and the polynomial-type models are investigated for both semi-infinite ground media and for rigidly backed layer media.

The paper is organized as follows. In section II, the conditions proposed by Rienstra are reviewed and discussed. In section III, these conditions are considered for the square-root type impedance models. Section IV is concerned with the polynomial models, and a new model which satisfies the Rienstra's conditions is proposed.

2. Sufficient conditions

The impedance boundary condition is classically expressed as $P(\omega) + Z_S(\omega)V_n(\omega) = 0$, where ω is the angular frequency, Z_S is the surface impedance and $P(\omega)$ and $V_n(\omega)$ are the Fourier transforms of the acoustic pressure and acoustic velocity normal to the ground, respectively. Throughout the paper, the convention $e^{-i\omega t}$ is used for the Fourier transform. The impulse response corresponding to the inverse Fourier transform of the surface impedance is thus given by

$$z_s(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Z_S(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{1}$$

Consequently, the impedance boundary condition is written in the time domain as the convolution,

$$p(t) = -\int_{-\infty}^{\infty} z_S(t-t')\,v_n(t')\,\mathrm{d}t'. \tag{2}$$

2.1. Rienstra's conditions

Rienstra has proposed three conditions in the frequency domain for a surface impedance model to be physically admissible:

- reality condition: $\overline{Z_S(\omega)} = Z_S(-\omega)$,
- passivity condition: $\mathrm{Re}[Z_S(\omega)] \geq 0$ for $\omega > 0$,
- causality condition: $Z_S(\omega)$ is analytic in $\mathrm{Im}(\omega) \geq 0$, $|Z_S(\omega)|$ is square integrable over the real ω-axis and there is a real t_0 such that $Z_S(\omega)\mathrm{e}^{-\mathrm{i}t_0} \to 0$ uniformly with regard to $\arg(\omega)$ for $|\omega| \to \infty$ in $\mathrm{Im}(\omega) \geq 0$,

where $\overline{Z}$ denotes the complex conjugate of Z. The first condition expresses that, as $p(t)$ and $v_n(t)$ are real quantities, $z_S(t)$ must be real. Therefore, the surface impedance $Z_S(\omega)$ must be an Hermitian function of ω. The second condition implies that the acoustic intensity into the ground is positive, as the ground is a passive medium. The third condition is a sufficient condition for an impedance model to be causal [22, 17] implying that $z_S(t)$ vanishes for $t < t_0$. Rienstra has also proposed an other causality condition, requiring only that $Z_S(\omega)$ must be analytic in $\mathrm{Im}(\omega) > 0$. This is a necessary but not sufficient condition as, for instance, the Gaussian function $Z_S(\omega) = \mathrm{e}^{-\omega^2/\omega_0^2}$, with ω_0 a real parameter, is analytic in $\mathrm{Im}(\omega) > 0$, but its inverse Fourier transform $z_S(t) \propto \mathrm{e}^{-t^2\omega_0^2/4}$ is not causal. Moreover, as expected the Gaussian function does not satisfy the sufficient causality condition, as its modulus does not converge to 0 in $\mathrm{Im}(\omega) \geq 0$.

A causal model has also other properties. If $|Z(\omega)|$ is square-integrable over the real ω-axis, the causality of $z_S(t)$ is equivalent to the well-known Kramers-Krönig relations [23], which relate the real and imaginary parts of $Z_S(\omega)$ through Hilbert transforms. Berthelot [18] has investigated these relations for various impedance models by evaluating numerically the Hilbert transforms.

Additionally, impedance models which satisfy the reality, the passivity and the causality conditions are called positive real functions in electrical network analysis [22].

Two types of surface impedances, which are typically used in outdoor sound propagation studies, are investigated hereafter,

- a semi-infinite ground:

$$Z_{S,\infty} = Z_c, \tag{3}$$

  with the characteristic impedance of the medium Z_c.
- a rigidly backed layer [24]:

$$Z_{S,d} = Z_c \coth\big(-\mathrm{i}k_c d\big), \tag{4}$$

  with the wavenumber in the medium k_c and the layer thickness d.

More complex models are possible by considering for instance a finite-impedance backed layer or a multilayered porous medium. However, characterization of natural grounds is often limited, because in-situ measurements are complex and costly. Therefore, the two types of surface impedances presented in equations (3) and (4) are used in the large majority of the studies.

2.2. Additional remarks for a rigidly backed layer model

For a rigidly backed layer, because the causality condition requires that $Z_{S,d}$ is analytic in $\mathrm{Im}(\omega) \geq 0$, $\coth(-\mathrm{i}k_c d)$ must be analytic in $\mathrm{Im}(\omega) \geq 0$. Because the complex function $\coth(-\mathrm{i}x)$ has poles for $x = n\pi$, with n integer, one must have $\mathrm{Im}(k_c) > 0$ in $\mathrm{Im}(\omega) \geq 0$, so that no poles are crossed. In that case, the function $\coth(-\mathrm{i}k_c d)$ can be expanded as an infinite sum [17],

$$\coth(-\mathrm{i}k_c d) = 1 + 2\sum_{n=1}^{+\infty} \mathrm{e}^{2\mathrm{i}nk_c d}. \tag{5}$$

Table I. Parameters α, β, ω_1, ω_2 and ω_3 for several square-root type impedance models as a function of the ground properties. The parameter λ is defined in equation (10).

Impedance model	α	β	ω_1	ω_2	ω_3
Zwikker and Kosten [25]	1	1	$\sigma_0\Omega/(\rho_0 q^2)$	0	0
Modified Zwikker and Kosten [11]	1	–	$\sigma_0\Omega/(\rho_0 q^2\gamma)$	0	0
Taraldsen and Jonasson [27]	$\gamma^{-1/2}$	$\gamma^{1/2}$	$\sigma_0\Omega/(\rho_0 q^2)$	0	0
Four-parameter Attenborough [29]	$4(\lambda\gamma)^{-1/2}/3$	$(\lambda\gamma)^{1/2}$	$3s_f^2\sigma_0\Omega/(4\rho_0 q^2)$	ω_1	$s_f^2\sigma_0\Omega/(\rho_0\lambda q^2)$
Hamet and Bérengier [31]	1	1	$\sigma_0\Omega/(\rho_0 q^2\gamma)$	$\gamma\sigma_0/(\rho_0\mathrm{Pr})$	$\sigma_0/(\rho_0\mathrm{Pr})$

The model of a rigidly backed layer can then be written as

$$Z_{S,d} = Z_{S,\infty} + 2\sum_{n=1}^{+\infty} Z_{S,d}^{(n)}, \qquad (6)$$

with $Z_{S,d}^{(n)} = Z_{S,\infty}\mathrm{e}^{2\mathrm{i}nk_c d}$. Therefore, if $\mathrm{Im}(k_c) > 0$ in $\mathrm{Im}(\omega) \geq 0$, the causality condition for the rigidly backed layer model is equivalently ensured if $Z_{S,\infty}$ and $Z_{S,d}^{(n)}$ satisfy the causality condition for $n > 1$.

In what follows, the relation for a complex number z with positive real and imaginary parts,

$$\mathrm{Arg}\big[-\mathrm{i}z\big] < \mathrm{Arg}\big[\coth(-\mathrm{i}z)\big] < -\mathrm{Arg}\big[-\mathrm{i}z\big], \qquad (7)$$

is used. In the preceding inequality, Arg denotes the argument of a complex number, which takes its values in $]-\pi,\pi]$. The inequality (7) is demonstrated in the Appendix.

3. Square-root type impedance models

A lot of impedance models can be written as a square-root of rational functions of the angular frequency ω,

$$Z_c = \frac{\rho_0 c_0 q}{\Omega}\,\alpha\left[\frac{(\omega_1-\mathrm{i}\omega)(\omega_2-\mathrm{i}\omega)}{-\mathrm{i}\omega(\omega_3-\mathrm{i}\omega)}\right]^{1/2}, \qquad (8)$$

$$k_c = \frac{\omega}{c_0}\,q\beta\left[\frac{(\omega_1-\mathrm{i}\omega)(\omega_3-\mathrm{i}\omega)}{-\mathrm{i}\omega(\omega_2-\mathrm{i}\omega)}\right]^{1/2}, \qquad (9)$$

where α, β, ω_1, ω_2 and ω_3 are all positive real numbers. These parameters are given in Table I and depend on the properties of the ground, which are the tortuosity q, the porosity Ω, the air flow resistivity σ_0 and the pore shape factor ratio s_f and on the ratio of specific heats in air γ. Among the models whose analytical expressions are given by equations (8) and (9), one can cite the Zwikker and Kosten (ZK) model, which can be derived from the acoustic equations in the porous media layer proposed by Zwikker and Kosten [25, 26]. This model, as $\omega_2 = \omega_3 = 0$, has only one parameter ω_1 which is a function of q, Ω and σ_0. As the semi-infinite medium model depends only on the ratios Ω/q and $\sigma_0\Omega/q^2$, different sets of parameters can lead to the same values of the impedance. A modification of the pulsation ω_1 has been proposed by Ostashev *et al.* [11] to improve the prediction of the ZK model at high frequencies. More recently, Taraldsen and Jonasson [27] have developed a similar model, based on the Darcy's law.

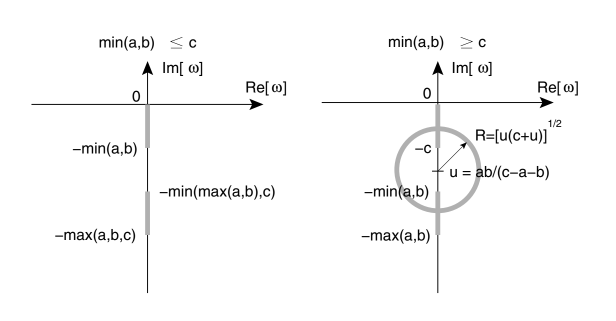

Figure 1. Branch cuts in the ω-plane of the function $[(a-\mathrm{i}\omega)(b-\mathrm{i}\omega)/(-\mathrm{i}\omega(c-\mathrm{i}\omega))]^{1/2}$, where a, b and c are all real positive coefficients.

They provide also a semi-empirical relation relating Ω, q and σ_0. The model proposed by Heutschi [28] for ballast surfaces, which is based on an electrical network analysis, is also of the same type. Moreover, the four-parameter Attenborough model [29] can be written using the analytical expressions in equations (8) and (9), with $\omega_1 = \omega_2$. It has been obtained from a low frequency and/or a high flow resistivity approximation of a more complex model, accounting for porous media with random orientation, non-aligned and non-circular cross shape pores [29, 30], which is used in the study of Ostashev *et al.* [11]. It has an additional parameter which is the pore shape factor ratio s_f. It depends also on the Prandtl number, denoted by Pr, through the parameter λ defined by

$$\lambda = \left(\frac{4}{3} - \frac{\gamma-1}{\gamma}\right)\mathrm{Pr}. \qquad (10)$$

Finally, the Hamet and Bérengier model [31] is a phenomenological model which has been proposed initially for porous road pavement. Recently, it has also been used to characterize ballast surfaces [16].

The branch cut of the complex square root function is chosen as the negative real axis. Corresponding branch cuts in the ω-plane lie in $\mathrm{Im}(\omega) < 0$ as shown in Figure 1.

3.1. Semi-infinite ground layer

Rienstra's conditions are now checked for the semi-infinite ground layer model. It is first noted that the impulse response can be calculated for the Zwikker and Kosten type model [5], yielding

$$z^{\mathrm{ZK}}_{S,\infty}(t) = \frac{\rho_0 c_0 q}{\Omega}\alpha\Big[\delta(t) + \frac{\omega_1}{2}\mathrm{e}^{-\omega_1 t/2}\Big(\mathrm{I}_0\Big(\frac{\omega_1 t}{2}\Big) + \mathrm{I}_1\Big(\frac{\omega_1 t}{2}\Big)\Big)\mathrm{H}(t)\Big], \tag{11}$$

which shows that the model is real and causal. The passivity, as for the other models, comes from the choice of the branch cut for the complex square root function, which provides $\mathrm{Re}[Z_{S,\infty}] \geq 0$ in the whole complex ω-plane. For the other models, the reality condition is fulfilled, since $\overline{Z_{S,\infty}(\omega)} = Z_{S,\infty}(-\omega)$. Concerning the causality condition, it can be noticed that $|Z_{S,\infty}|$ is not square-integrable over the real-axis, because $|Z_{S,\infty}|$ tends to $Z_\infty = \rho_0 c_0 q\alpha/\Omega$ for large $|\omega|$, and $Z^2_{S,\infty}$ has a pole at $\omega = 0$. Therefore, the function

$$A(\omega) = Z_{S,\infty} - Z_\infty\left(1 + \frac{\omega_1}{\omega_1 - \mathrm{i}\omega}\sqrt{\frac{\omega_1\omega_2}{-\mathrm{i}\omega\omega_3}}\right) \tag{12}$$

is considered instead. The second and last term in equation (12) aim to have $A(\omega) \to 0$ for $|\omega| \to \infty$ and to remove the singularity at $\omega = 0$, respectively. The function $A(\omega)$ is analytic in $\mathrm{Im}(\omega) \geq 0$ and $|A(\omega)|$ is square-integrable over the real-axis as $|A(\omega)|^2$ decreases as ω^{-2} for large $|\omega|$. It is also uniformly converging to zero in $\mathrm{Im}[\omega] \geq 0$ for large $|\omega|$. To demonstrate this point, introducing $\omega = R\mathrm{e}^{\mathrm{i}\theta}$ with $R > 0$ and $\theta \in [0,\pi]$, the inequality

$$\big|A(R,\theta)\big| \leq \big|Z_{S,\infty} - Z_\infty\big| + \frac{Z_\infty\omega_1}{|R-\omega_1|}\sqrt{\frac{\omega_1\omega_2}{R\omega_3}} \tag{13}$$

is obtained. The second term in the preceding equation tends to zero for large $|\omega|$ independently of θ in the upper half-plane. For the first term, the inequality $|z-1|^2 \leq |z^2-1|$, which is true for a complex number with a positive real part, is used. It can be readily demonstrated using the relations $(z^2-1) = (z-1)(z+1)$ and $|z-1| \leq |z+1|$ for $\mathrm{Re}(z) \geq 0$. This leads to

$$\big|Z_{S,\infty} - Z_\infty\big| \leq Z_\infty\left|\frac{-\mathrm{i}\omega(\omega_1+\omega_2-\omega_3) + \omega_1\omega_2}{-\mathrm{i}\omega(\omega_3 - \mathrm{i}\omega)}\right|^{1/2}. \tag{14}$$

Introducing the polar form $\omega = R\mathrm{e}^{i\theta}$ yields

$$\big|Z_{S,\infty} - Z_\infty\big| \leq \frac{Z_\infty}{R^{1/2}}\left(\frac{(\omega_1+\omega_2-\omega_3)R + \omega_1\omega_2}{|R-\omega_3|}\right)^{1/2}. \tag{15}$$

From equations (13) and (15), it is shown that $A(\omega) \to 0$ uniformly with regard to $\mathrm{Arg}(\omega)$ for $|\omega| \to \infty$ in $\mathrm{Im}(\omega) \geq 0$. Therefore, $A(\omega)$ is a causal transform. As the inverse Fourier transform of a constant is a Dirac delta function and as the inverse Fourier transform of the last term in equation (12) is causal [32], $Z_{S,\infty}$ is also a causal transform. Note, that in the study of Berthelot [18], it is concluded that the Attenborough model is not causal at high frequencies, which is in contradiction with the present results. Finally, all models for a semi-infinite medium satisfy the three conditions proposed by Rienstra and, hence, are physically admissible.

3.2. Rigidly backed layer

The model for a rigidly backed layer is now investigated. It is easily verified that the model is real. Concerning the passivity condition, the argument of the wavenumber k_c,

$$\mathrm{Arg}(k_c) = \frac{1}{2}\Big[\mathrm{Arg}(\omega_1 - \mathrm{i}\omega) + \frac{\pi}{2} - \mathrm{Arg}(\omega_2 - \mathrm{i}\omega) + \mathrm{Arg}(\omega_3 - \mathrm{i}\omega)\Big] \tag{16}$$

is first considered for $\omega > 0$ in order to use equation (7). For the Zwikker and Kosten model, as $\omega_2 = \omega_3 = 0$, one obtains $0 < \mathrm{Arg}(k_c) \leq \pi/2$. The same inequality is satisfied for the Attenborough model, as $\omega_2 = \omega_1$. Finally, for the Hamet and Bérengier [31] model, $\omega_3 = \gamma\omega_2$ and hence $\omega_3 > \omega_2$, which implies

$$0 \leq -\mathrm{Arg}\big(\omega_2 - \mathrm{i}\omega\big) + \mathrm{Arg}\big(\omega_3 - \mathrm{i}\omega\big) \leq \frac{\pi}{2}, \tag{17}$$

and consequently $0 < \mathrm{Arg}(k_c) \leq \pi/2$ for $\omega > 0$. For all models, the wavenumber k_c has positive real and imaginary parts, which allows one to use equation (7). Along with the relation $\mathrm{Arg}[Z_{S,d}] = \mathrm{Arg}[Z_c] + \mathrm{Arg}[\coth(-ik_c d)]$, this leads to

$$\mathrm{Arg}\big[Z_{S,d}\big] \leq \mathrm{Arg}\big[Z_c\big] - \mathrm{Arg}\big[-\mathrm{i}k_c d\big], \tag{18}$$

which shows that

$$\mathrm{Arg}\big[Z_{S,d}\big] \leq \frac{\pi}{2} + \mathrm{Arg}\big(\omega_2 - \mathrm{i}\omega\big) - \mathrm{Arg}\big(\omega_3 - \mathrm{i}\omega\big) \leq \frac{\pi}{2}. \tag{19}$$

Similarly, one has

$$\mathrm{Arg}\big[Z_{S,d}\big] \geq \mathrm{Arg}\big[Z_c\big] + \mathrm{Arg}\big[-\mathrm{i}k_c d\big], \tag{20}$$

which leads to

$$\mathrm{Arg}\big[Z_{S,d}\big] \geq \mathrm{Arg}\big(\omega_1 - \mathrm{i}\omega\big) \geq -\frac{\pi}{2}. \tag{21}$$

Then, for all cases considered, as $\mathrm{Re}[Z_{S,d}] \geq 0$, the passivity condition is fulfilled for rigidly backed layer models.

The causality condition is now considered. First, it must be shown that $\mathrm{Im}(k_c) > 0$ in $\mathrm{Im}(\omega) > 0$. For that, using the polar form $\omega = R\mathrm{e}^{i\theta}$ with $R > 0$ and $\theta \in [0,\pi]$, the argument of k_c is written as

$$\mathrm{Arg}(k_c) = \frac{1}{2}\Big[\theta + \mathrm{Arg}\big(\omega_1 - \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\big) + \frac{\pi}{2} - \mathrm{Arg}\big(\omega_2 - \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\big) + \mathrm{Arg}\big(\omega_3 - \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\big)\Big]. \tag{22}$$

For the Zwikker and Kosten type model, the relation $\omega_2 = \omega_3 = 0$ implies that

$$\mathrm{Arg}(k_c) = \frac{1}{2}\left[\theta + \mathrm{Arg}\big(\omega_1 - R\mathrm{e}^{\mathrm{i}\theta}\big) + \frac{\pi}{2}\right]. \tag{23}$$

As $\omega_1 > 0$, one obtains the inequalities

$$-\frac{\pi}{2} < \mathrm{Arg}\big(\omega_1 - \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\big) \le 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right], \quad (24)$$

$$0 \le \mathrm{Arg}\big(\omega_1 - \mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}\big) < \frac{\pi}{2} \text{ for } \theta \in \left[\frac{\pi}{2}, \pi\right], \quad (25)$$

which lead to $0 < \mathrm{Arg}(k_c) < \pi$ for $\theta \in [0, \pi]$. The imaginary part of k_c is then positive in $\mathrm{Im}(\omega) \ge 0$. For the Attenborough model, because $\omega_1 = \omega_2$, the same conclusions are straightforwardly deduced. For the Hamet and Bérengier model, as $\omega_3 > \omega_2$, the inequality

$$0 \le \mathrm{Arg}\big(\omega_3 - \mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}\big) - \mathrm{Arg}\big(\omega_2 - \mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}\big) \le \frac{\pi}{2}, \quad (26)$$

is obtained for $\theta \in [0, \pi/2]$, yielding

$$0 < \mathrm{Arg}(k_c) \le \frac{\theta}{2} + \frac{\pi}{2} < \pi. \quad (27)$$

Similarly for $\theta \in [\pi/2, \pi]$, the inequality

$$-\frac{\pi}{2} \le \mathrm{Arg}\big(\omega_3 - \mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}\big) - \mathrm{Arg}\big(\omega_2 - \mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}\big) \le 0, \quad (28)$$

implies that

$$0 < \frac{\theta}{2} < \mathrm{Arg}(k_c) < \frac{\theta}{2} + \frac{\pi}{2} < \pi. \quad (29)$$

Therefore, it has been shown that $\mathrm{Im}(k_c) > 0$ for $\mathrm{Im}(\omega) \ge 0$ for all impedance models considered in this section. As discussed in section II.B, the causality condition can then be equivalently checked for $Z_{S,d}^{(n)} = Z_{S,\infty}\mathrm{e}^{2inkc_d}$. However, $|Z_{S,d}^{(n)}|$ is not square-integrable on the real axis as it is singular at $\omega = 0$ and as its limit is $Z_\infty \mathrm{e}^{-\omega^+ t_n}$ for large ω, with $t_n = 2ndq\beta/c_0$ and $\omega^+ = (\omega_1 + \omega_3 - \omega_2)/2$. Therefore, the function $B(\omega) = B_1(\omega) - B_2(\omega)$ with

$$B_1(\omega) = Z_{S,d}^{(n)}\mathrm{e}^{-\mathrm{i}\omega t_n} - Z_{S,\infty}\mathrm{e}^{-\omega^+ t_n}, \quad (30)$$

$$B_2(\omega) = Z_\infty \frac{\omega_1}{\omega_1 - \mathrm{i}\omega}\sqrt{\frac{\omega_1\omega_2}{-\mathrm{i}\omega\omega_3}}\left[1 - \mathrm{e}^{-\omega^+ t_n}\right], \quad (31)$$

is considered instead. $B(\omega)$ is analytic in $\mathrm{Im}(\omega) > 0$ and $|B(\omega)|$ is square-integrable over the real-axis as $|B(\omega)|^2$ decays as $1/\omega^2$ for large ω. It remains to show that $B(\omega)$ is uniformly converging to zero in the upper half-plane. First, note that this is the case for the function $B_2(\omega)$, as shown in equation (13). In addition, the function $B_1(\omega)$ can be rewritten as a simpler form

$$B_1(\omega) = Z_{S,\infty}\mathrm{e}^{\omega^+ t_n}\left[\mathrm{e}^{\mathrm{i}\omega x t_n} - 1\right], \quad (32)$$

with

$$x = \left[\frac{(\omega_1 - \mathrm{i}\omega)(\omega_3 - \mathrm{i}\omega)}{-\mathrm{i}\omega(\omega_2 - \omega)}\right]^{1/2} - 1 - \frac{\omega^+}{-\mathrm{i}\omega}. \quad (33)$$

Using the inequality $|\mathrm{e}^z - 1| \le |z|\mathrm{e}^{|z|}$ [32] leads to the estimate

$$\big|B_1(\omega)\big| \le \mathrm{e}^{\omega^+ t_n}\big|Z_{S,\infty}\big|R|x|t_n\mathrm{e}^{R|x|t_n}. \quad (34)$$

As shown in the preceding section, $Z_{S,\infty}$ tends uniformly to Z_∞ as $R \to \infty$. It is then sufficient to show that $R|x|$ tends uniformly to zero as $R \to \infty$. For that, x is rewritten as

$$x(z) = \left[\frac{(1 + \omega_1 z)(1 + \omega_3 z)}{1 + \omega_2 z}\right]^{1/2} - 1 - \omega^+ z, \quad (35)$$

with $z = 1/(-\mathrm{i}\omega)$. As the function $x(z)$ is analytic for $|z| < A$, where A is the radius of convergence which is at least the distance to the nearest singularity. it can be written as a power series which is absolutely convergent [33, 34]. Thus, for large $|\omega|$ and hence for small $|z|$, $x(z)$ can be expanded as

$$x(z) = \sum_{m=2}^{\infty} a_m z^m, \quad (36)$$

as the relations $a_0 = a_1 = 0$ are obtained from a Taylor expansion of x around $z = 0$. Introducing $\omega = R\mathrm{e}^{i\theta}$ leads to

$$|x| \le \frac{1}{R^2}\sum_{m=2}^{\infty} |a_m||R|^{2-m}, \quad (37)$$

which shows that $R|x|$ tends uniformly to zero as $R \to \infty$. Consequently, $B(\omega)$ is a causal transform. This is also the case for $Z_{S,d}^{(n)}\mathrm{e}^{-i\omega t_n}$, because $Z_{S,\infty}$ and $B_2(\omega)$ are causal transforms, as shown in the preceding section. As its inverse Fourier transform is $z_{S,d}^{(n)}(t + t_n)$, it is deduced that $z_{S,d}^{(n)}(t)$ is null for $t < t_n$ and hence is causal. Finally, $Z_{S,d}$ is also a causal transform by linearity. As a consequence, the impedance models considered in this section are physically admissible.

3.3. Variable porosity impedance model

Similar to the square-root type, the variable porosity model is used in the American National Standard [35] for ground impedance measurements. It is based on a low frequency/high flow resistivity limit of the surface impedance of a semi-infinite ground with an exponentially decreasing porosity. The surface impedance is

$$Z_S = \rho_0 c_0\left(\sqrt{\frac{\omega_0}{-\mathrm{i}\omega}} + \frac{\omega_1}{-\mathrm{i}\omega}\right), \quad (38)$$

with $\omega_0 = 4\sigma_0/(\gamma\rho_0)$ and $\omega_1 = 4\gamma/(c_0\alpha_0)$, where α_0 is the effective rate of change of porosity. The inverse Fourier transform of Z_S is straightforwardly calculated, and is given by [32],

$$z_S(t) = \rho_0 c_0\left(\sqrt{\frac{\omega_0}{\pi t}} + \omega_1\right)\mathrm{H}(t). \quad (39)$$

This surface impedance model is then real and causal. It is also passive, because the real part of equation (38) is positive for $\omega > 0$. It is then well suited for time-domain computations.

Table II. Coefficients of the polynomial models in equations (40-41) and (57-58) computed using an air density $\rho_0 = 1.2$ kg m-3.

	a	c	b	d	μ	p	r	q	s	ν
Delany and Bazley	0.232	0.336	0.75	0.73	–	0.353	0.576	0.70	0.59	–
Miki	0.251	0.384	0.632		0.459	0.380	0.557	0.618		0.673
modified Miki						0.351	0.539	0.632		0.643

4. Polynomial type impedance model

The other important family of impedance models is the polynomial type. They were initially proposed by Delany and Bazley [36], who measured the characteristic impedance and the wave number of a large set of fibrous and highly porous materials. Use of polynomial models is then restricted to medium whose porosity and tortuosity are close to 1, and hence $\Omega = q = 1$ thereafter. As they are simple models, it is not expected that predictions are accurate over a very wide frequency band. Thus, Delany and Bazley have suggested that their model is valid only for frequencies f satisfying $0.01 < f/\sigma_0 < 1$. Moreover, Attenborough *et al.* [21] have compared a lot of impedance models for in situ determination of ground impedance. It was shown that the Delany and Bazley model results in poorer fitting to short range data over many ground surfaces than alternative 2 (or 3 for rigid backed layer) parameter models. Polynomial models are however extensively used in the literature, mainly because they allow us to characterize the behaviour of porous media with only one parameter, which is the air flow resistivity. This is particularly of interest for outdoor sound propagation studies, because direct measurements of the acoustical properties of natural grounds are difficult to carry out. The model is given by the formula

$$Z_c(\omega > 0) = \rho_0 c_0 \left[1 + a\left(\frac{\omega_0}{\omega}\right)^b + \mathrm{i}c\left(\frac{\omega_0}{\omega}\right)^d\right], \quad (40)$$

$$k_c(\omega > 0) = \frac{\omega}{c_0}\left[1 + p\left(\frac{\omega_0}{\omega}\right)^q + \mathrm{i}r\left(\frac{\omega_0}{\omega}\right)^s\right], \quad (41)$$

with the angular frequency $\omega_0 = \sigma_0/\rho_0$ and where the parameters a, b, c, d, p, q, r and s are all real positive numbers. For medium whose porosity is lower than 1, it has been proposed in the literature [37, 30, 38] to use an effective flow resistivity equal to $\sigma_e = \Omega\sigma_0$. Similarly, if the tortuosity of the medium is larger than one, the effective flow resistivity can be written as $\sigma_e = \Omega\sigma_0/q^2$ [39]. In that case, the parameter ω_0 corresponds to ω_1 in the Zwikker and Kosten, Taraldsen and Jonasson and Hamet and Bérengier models. Miki [39] has also proposed an expression for the effective flow resistivity including the pore shape factor ratio.

4.1. Delany and Bazley model

4.1.1. Semi-infinite ground layer

The coefficients in equation (40) proposed by Delany and Bazley [36] are given in Table II for $\rho_0 = 1.2$ kg m-3. The passivity condition is satisfied as $\mathrm{Re}[Z_{S,\infty}] \geq 0$ for $\omega > 0$. The Delany and Bazley model is an interesting case, because it has been obtained by fitting experimental results for positive frequencies. It fulfills different conditions depending on its extension in the complex ω-plane. Following Miki, the reality condition can be imposed by using the following extension of the polynomial model:

$$Z_c^{[1]} = \rho_0 c_0 \left[1 + \frac{a}{2\cos(b\pi/2)}\left(\left[\frac{\omega_0}{-\mathrm{i}\omega}\right]^b + \left[\frac{\omega_0}{i\omega}\right]^b\right) + \frac{c}{2\sin(d\pi/2)}\left(\left[\frac{\omega_0}{-\mathrm{i}\omega}\right]^d - \left[\frac{\omega_0}{\mathrm{i}\omega}\right]^d\right)\right], \quad (42)$$

$$k_c^{[1]} = \frac{\omega}{c_0}\left[1 + \frac{p}{2\cos(q\pi/2)}\left(\left[\frac{\omega_0}{-i\omega}\right]^q + \left[\frac{\omega_0}{i\omega}\right]^q\right) + \frac{r}{2\sin(s\pi/2)}\left(\left[\frac{\omega_0}{-\mathrm{i}\omega}\right]^s - \left[\frac{\omega_0}{\mathrm{i}\omega}\right]^s\right)\right], \quad (43)$$

where the branch cut of the power functions is chosen as the negative real axis. For ω real, the preceding formula reduce to

$$Z_c^{[1]}(\omega \text{ real}) = \rho_0 c_0 \left[1 + a\left|\frac{\omega_0}{\omega}\right|^b + \mathrm{i}\frac{\omega}{|\omega|}c\left|\frac{\omega_0}{\omega}\right|^d\right], \quad (44)$$

$$k_c^{[1]}(\omega \text{ real}) = \frac{\omega}{c_0}\left[1 + p\left|\frac{\omega_0}{\omega}\right|^q + \mathrm{i}\frac{\omega}{|\omega|}r\left|\frac{\omega_0}{\omega}\right|^s\right]. \quad (45)$$

Using the Fourier transforms [40]

$$\int_{-\infty}^{+\infty} \frac{1}{\pi}\Gamma(1-\nu)|t|^{\nu-1}\sin\left(\frac{\nu\pi}{2}\right)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = |\omega|^{-\nu}, \quad (46)$$

$$\int_{-\infty}^{+\infty} \frac{1}{\pi}\Gamma(1-\nu)|t|^{\nu-1}\cos\left(\frac{\nu\pi}{2}\right)\mathrm{sign}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \mathrm{i}|\omega|^{-\nu}\mathrm{sign}(\omega), \quad (47)$$

valid for $0 < \nu < 1$, the impulse response obtained from equation (42) is then given by

$$z_{S,\infty}^{[1]}(t) = \rho_0 c_0 \left[\delta(t) + \frac{a\omega_0}{\pi}\sin\left(\frac{b\pi}{2}\right)\frac{\Gamma(1-b)}{|\omega_0 t|^{1-b}} + \frac{c\omega_0}{\pi}\cos\left(\frac{d\pi}{2}\right)\frac{\Gamma(1-d)}{|\omega_0 t|^{1-d}}\mathrm{sign}(t)\right], \quad (48)$$

where $\mathrm{sign}(t < 0) = -1$, $\mathrm{sign}(t = 0) = 0$ and $\mathrm{sign}(t > 0) = 1$. It is represented as a function of the normalized time $\omega_0 t$ for the set of coefficients proposed by Delany and Bazley in Figure 2. It is seen that the impulse response is non-zero for $t < 0$, which shows that the Delany and Bazley model is not causal for the first extension.

However, extension of the polynomial model in the complex plane can be carried out in other ways. Thus, without imposing the reality condition, a second extension of the polynomial model into the complex plane,

$$Z_c^{[2]} = \rho_0 c_0 \left[1 + a\Big(\frac{\omega_0}{\omega}\Big)^b + \mathrm{i}c\Big(\frac{\omega_0}{\omega}\Big)^d\right], \qquad (49)$$

$$k_c^{[2]} = \frac{\omega}{c_0}\left[1 + p\Big(\frac{\omega_0}{\omega}\Big)^q + \mathrm{i}r\Big(\frac{\omega_0}{\omega}\Big)^s\right], \qquad (50)$$

where the branch cut of the power functions is chosen as the negative imaginary axis, can be proposed. Using the Fourier transform, [40]

$$\int_{-\infty}^{+\infty} t^{\nu-1}\mathrm{H}(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t = \Gamma(\nu)(-i\omega)^{-\nu}, \qquad (51)$$

valid for $\nu > 0$, leads to the impulse response

$$z_{S,\infty}^{[2]}(t) = \rho_0 c_0 \Bigg[\delta(t) + (-\mathrm{i})^b \frac{a}{\Gamma(b)}\frac{\omega_0}{(\omega_0 t)^{1-b}}\mathrm{H}(t) - (-\mathrm{i})^{d+1}\frac{c}{\Gamma(d)}\frac{\omega_0}{(\omega_0 t)^{1-d}}\mathrm{H}(t)\Bigg]. \qquad (52)$$

It is also plotted as a function of the normalized time $\omega_0 t$ for the set of coefficients proposed by Delany and Bazley in Figure 2. It is observed that this extension is causal but is not real.

In conclusion, the impulse response related to the Delany and Bazley model for a semi-infinite ground can not be causal and real at the same time. Therefore, the Delany and Bazley model is not suited for time-domain computations.

4.1.2. Rigidly backed layer

It is well known that the Delany and Bazley model for a rigidly backed layer does not satisfy the passivity condition [41, 12]. As an example, the real and imaginary parts of the Delany and Bazley model are plotted in Figure 3 as a function of frequency for $\sigma_0 = 100$ kPa s m^{-2} and $d = 0.01$ m. It is clearly seen that $\mathrm{Re}[Z_{S,d}] < 0$ for frequencies below 300 Hz. Moreover, as for the semi-infinite ground case, the Delany and Bazley model does not fulfill both reality and causality conditions. To check the causality condition for the second extension in the complex ω-plane proposed in the preceding paragraph, it is first shown that $\mathrm{Im}(k_c^{[2]}) > 0$ in $\mathrm{Im}(\omega) \geq 0$. For that, using the polar notation $\omega = R\mathrm{e}^{i\theta}$ with $R > 0$ and $\theta \in [0, \pi]$, it is straightforwardly obtained that

$$\mathrm{Im}(k_c^{[2]}) = R\sin\theta + p\sin\big[(1-q)\theta\big]R^{1-q} + r\cos\big[(1-s)\theta\big]R^{1-s}, \qquad (53)$$

which shows that $\mathrm{Im}(k_c^{[2]}) > 0$ because all terms in the preceding equation are positive for $\theta \in [0, \pi]$. As shown in section II.B, the function $Z_{S,d}^{(n)} = Z_{S,\infty}^{[2]}\mathrm{e}^{2ink_c^{[2]}d}$ can then be considered to show that the extension of the Delany and Bazley model of a rigidly backed layer is causal. As $Z_{S,d}^{(n)}$ is singular at $\omega = 0$, we consider the function

$$C(\omega) = Z_{S,d}^{(n)} - \rho_0 c_0\left(a\left(\frac{\omega_0}{\omega}\right)^b + \mathrm{i}c\left(\frac{\omega_0}{\omega}\right)^d\right), \qquad (54)$$

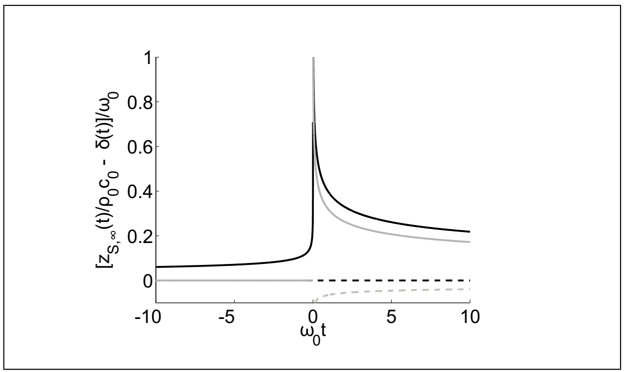

Figure 2. Continuous part of the normalized impulse response $[z_{S,\infty}(t)/\rho_0 c_0 - \delta(t)]/\omega_0$ as a function of the normalized time $\omega_0 t$: (solid line) real and (dashed line) imaginary parts of the first extension (in black) in equation (48) and second extension (in grey) in equation (52) of the Delany and Bazley impedance model of a semi-infinite ground.

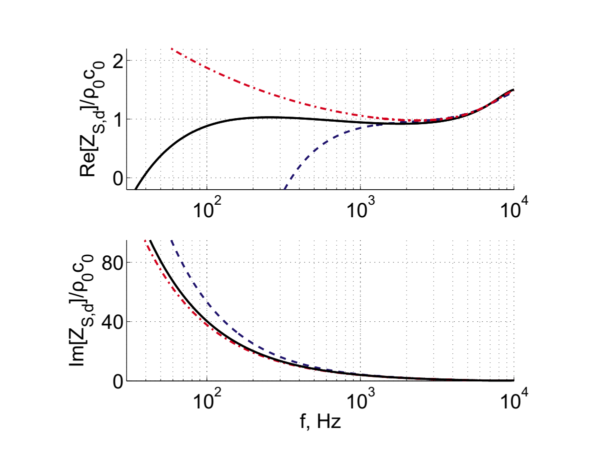

Figure 3. Real and imaginary parts of a rigidly backed layer of thickness $d = 0.01$ m and of air flow resistivity $\sigma_0 = 100$ kPa s m^{-2} as a function of the frequency f: (dashed line) Delany and Bazley model, (solid line) Miki model and (dash-dotted line) modified Miki model.

whose modulus is square-integrable over the real-axis as $|Z_{S,d}^{(n)}|$ decreases exponentially for large ω as $\mathrm{Im}(k_c^{[2]}) > 0$. It is also uniformly converging to 0 in $\mathrm{Im}(\omega) \geq 0$, because, as $2nd\,|\mathrm{Im}(k_c^{(2)})| \geq g = 2ndr\cos[(1-s)\pi]R^{1-s}$ in $\theta \in [0, \pi]$, the following estimate:

$$\big|C(\omega)\big| \leq \rho_0 c_0 \Bigg[\bigg(1 + a\Big(\frac{\omega_0}{R}\Big)^b + c\Big(\frac{\omega_0}{R}\Big)^d\bigg)\mathrm{e}^{-g} + a\Big(\frac{\omega_0}{R}\Big)^b + c\Big(\frac{\omega_0}{R}\Big)^d\Bigg], \qquad (55)$$

is obtained. Therefore, $C(\omega)$ is a causal transform. As the two last terms in equation (54) are also causal transforms (see equation 47), the second extension of the Delany and Bazley model for a rigidly backed layer satisfies also the causality condition.

4.2. Miki model

4.2.1. Semi-infinite ground layer

Miki [41] has shown that the polynomial model for a semi-infinite-ground is physically admissible if the relations $b = d$ and $a\sin[(d\pi)/2] = c\cos[(b\pi)/2]$ are satisfied. In that case, using the relation $\Gamma(1-\nu)\Gamma(\nu) = \pi/\sin(\pi\nu)$ for ν real [40], it is easily verified that the impulse responses for the two extensions of the polynomial model in the complex plane in equations (48) and (52) are the same and are given by [12]

$$z_{S,\infty}(t) = \rho_0 c_0 \left[\delta(t) + \frac{\mu\omega_0}{\Gamma(b)(\omega_0 t)^{1-b}}\mathrm{H}(t)\right], \qquad (56)$$

with $\mu = a/\cos[(b\pi)/2]$. As expected, the impulse response is real and null for $t < 0$.

4.2.2. Rigidly backed layer

Corrections have been also proposed by Miki for the wavenumber in the ground medium, yielding $q = s$ and $p\sin[(s\pi)/2] = r\cos[(q\pi)/2]$. With these relations between the parameters, the polynomial model in equations (42)-(43) and (49)-(50) can be written in a simpler form [12, 42],

$$Z_c = \rho_0 c_0 \left[1 + \mu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right], \qquad (57)$$

$$k_c = \frac{\omega}{c_0}\left[1 + \nu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^q\right], \qquad (58)$$

with $\nu = p/\cos[(q\pi)/2]$.

The Miki model for a rigidly backed layer satisfies the reality condition, as it can be easily verified from equations (57) and (58). For the same reasons as those developed for the second extension of the Delany and Bazley model for a rigidly backed layer, it is a causal model. However, the Miki model of a rigidly backed layer is not passive at low frequencies. Indeed, the low-frequency limit of this model is

$$Z_{S,d}(\omega \to 0) = \frac{\rho_0\, c_0^2 \mu}{d\,\nu\,\omega_0^{-b+q}}(-\mathrm{i}\omega)^{-b+q-1}. \qquad (59)$$

With the parameters proposed by Miki, the real part of the impedance,

$$\mathrm{Re}\left[Z_{S,d}(\omega \to 0)\right] = -0.150\frac{\rho_0\, c_0^2}{d\omega_0^{-b+q}}\,\omega^{-b+q-1}, \qquad (60)$$

is then negative at low frequencies. This behaviour is illustrated in Figure 3 by using the same set of coefficients than in section 4.1.2. It is observed that the real part of the surface impedance is negative for frequencies below 30 Hz. Note that, contrary to the Delany and Bazley model, the passivity condition is violated only for very low frequencies, which are not of interest in practice. In addition, it can be noted that the Miki impedance model depends at low frequencies on the parameter ω_0 and, hence, on the air flow resistivity. However, this should not be the case, because as $Z_c = [\rho_g K_g]^{1/2}$ and $k_c = \omega[\rho_g/K_g]^{1/2}$, where $\rho_g(\omega)$ and $K_g(\omega)$ are the dynamic density and the dynamic bulk modulus of the ground medium, respectively, the relation

$$Z_{S,d}(\omega \to 0) = \frac{K_0}{-\mathrm{i}\omega d}, \qquad (61)$$

where $K_0 = K_g(\omega = 0)$ is the static bulk modulus, must be satisfied. As the propagation process in the pores of the medium is usually assumed to be isothermal at low frequencies, one has $K_0 = \rho_0 c_0^2/(\gamma\Omega)$. This is the case for all models presented in section III except for the Zwikker and Kosten model, for which adiabatic conditions are assumed [26, 21], yielding $K_0 = \rho_0 c_0^2/\Omega$.

4.3. Modified Miki model

A modified Miki model is now proposed to have a physically admissible impedance model. To fulfill the passivity condition at low frequencies, one chooses $q = b = 0.632$. Moreover, the low-frequency behaviour in equation (61) can be retrieved by setting $\nu = \gamma\mu$. Interestingly, in the Miki model, one finds $\nu/\mu = 1.47$ which is close to the value of γ. The real and imaginary parts of the modified Miki model are represented in Figure 3 as functions of the frequency f for the set of coefficients previously used in Secs. IV.A.2 and IV.B.2. It is observed that $\mathrm{Re}[Z_{S,d}] \geq 0$ for all frequencies contrary to the Delany and Bazley and Miki models.

The semi-infinite ground layer model is the same than that proposed by Miki and thus is physically admissible. For a rigidly backed layer, the modified Miki model is also real and causal for the same reasons indicated previously. It remains to show that it fulfills the passivity condition. For that, the argument of the wavenumber is written as

$$\mathrm{Arg}(k_c) = \mathrm{Arg}\left[1 + \nu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right], \qquad (62)$$

which leads to the inequality $0 \leq \mathrm{Arg}(k_c) \leq \pi/2$ for $\omega > 0$. This allows one to write, using equation (7), that

$$\begin{aligned}\mathrm{Arg}\left[Z_c\right] + \mathrm{Arg}\left[-\mathrm{i}k_c\right] &\leq \mathrm{Arg}\left[Z_{S,d}\right]\\ &\leq \mathrm{Arg}\left[Z_c\right] - \mathrm{Arg}\left[-\mathrm{i}k_c\right].\end{aligned} \qquad (63)$$

The left term in the preceding inequality can be expressed as

$$\begin{aligned}&\mathrm{Arg}\left[Z_c\right] + \mathrm{Arg}\left[-\mathrm{i}k_c\right] = \\ &-\frac{\pi}{2} + \mathrm{Arg}\left[1 + \nu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right] + \mathrm{Arg}\left[1 + \mu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right].\end{aligned} \qquad (64)$$

The two right terms have positive real and imaginary parts. Consequently, their argument is between 0 and $\pi/2$, which leads to the inequality

$$-\frac{\pi}{2} \leq \mathrm{Arg}\left[Z_c\right] + \mathrm{Arg}\left[-\mathrm{i}k_c\right] \leq \mathrm{Arg}\left[Z_{S,d}\right]. \qquad (65)$$

Similarly, the right term in the inequality (63) can be written as

$$\begin{aligned}&\mathrm{Arg}\left[Z_c\right] - \mathrm{Arg}\left[-\mathrm{i}k_c\right] = \\ &\frac{\pi}{2} + \mathrm{Arg}\left[1 + \mu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right] - \mathrm{Arg}\left[1 + \nu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right].\end{aligned} \qquad (66)$$

Table III. Conditions satisfied by the impedance models.

Impedance model	Semi-infinite ground			Rigidly backed layer		
	Reality	Causality	Passivity	Reality	Causality	Passivity
Delany and Bazley	yes/**no**		yes	yes/**no**		**no**
Miki	yes	yes	yes	yes	yes	**no**
Modified Miki	yes	yes	yes	yes	yes	yes
Zwikker and Kosten	yes	yes	yes	yes	yes	yes
Taraldsen and Jonasson	yes	yes	yes	yes	yes	yes
Attenborough	yes	yes	yes	yes	yes	yes
Hamet and Bérengier	yes	yes	yes	yes	yes	yes
Variable porosity	yes	yes	yes			

Because $\nu > \mu > 0$ and the real and imaginary parts of $e^{ib\pi/2}$ are positive, it is deduced that

$$\mathrm{Arg}\left[1+\nu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right] - \mathrm{Arg}\left[1+\mu\left(\frac{\omega_0}{-\mathrm{i}\omega}\right)^b\right] \geq 0. \qquad (67)$$

Therefore, the inequality

$$\mathrm{Arg}\big[Z_{S,d}\big] \leq \mathrm{Arg}\big[Z_c\big] - \mathrm{Arg}\big[-\mathrm{i}k_c\big] \leq \frac{\pi}{2} \qquad (68)$$

is obtained. Inequalities (65) and (68) show that the real part of $Z_{S,d}$ is positive. Thus, the modified Miki model of a rigidly backed layer is passive and is hence physically admissible.

5. Conclusion

Translation of impedance models defined in the frequency domain into the time domain is not straightforward. Indeed, the impedance model must be first extended into the complex plane and must obey some conditions to be physically admissible, as the time-domain counterpart must be real, passive and causal. These conditions were checked for popular models in the outdoor sound propagation community. Models that are written as the square-root of a rational function, such as the Zwikker and Kosten model, the four-parameter Attenborough model and the Hamet and Bérengier model, were first investigated. All these models were shown to be physically admissible for a semi-infinite ground medium and for a rigidly backed layer. The polynomial-type models were then studied. Depending on the extension of the model in the complex plane, the Delany and Bazley model fulfills different conditions.

However, the reality and causality conditions can not be simultaneously satisfied and consequently the Delany and Bazley model is not adapted for time-domain computations. In addition, the Miki model was shown to be physically admissible for a semi-infinite medium, but it was demonstrated that the corresponding model for a rigidly backed layer does not satisfy the passivity condition for very low frequencies. A new polynomial model was then proposed on the basis of the Miki model and was shown to be physically admissible. The results of the study are summarized in Table III.

Appendix

In this section, the inequality (7) is demonstrated. For that, the complex function coth is first written as

$$\coth(x+\mathrm{i}y) = \frac{1}{\tanh^2 x + \tan^2 y}\left[\frac{\tanh x}{\cos^2 y} - \mathrm{i}\frac{\tan y}{\cosh^2 x}\right], \qquad (A1)$$

where x and y are real numbers. For $x > 0$, the argument of $\coth(x+\mathrm{i}y)$ is then given by

$$\mathrm{Arg}\big[\coth(x+\mathrm{i}y)\big] = \arctan\left[\frac{\sin(-2y)}{\sinh(2x)}\right]. \qquad (A2)$$

As for $x > 0$ and $y < 0$, one has $\sin(-2y) < -2y$ and $\sinh(2x) > 2x > 0$, the inequality

$$\mathrm{Arg}\big[\coth(x+\mathrm{i}y)\big] < \arctan\left(\frac{-y}{x}\right) = -\mathrm{Arg}\big[x+\mathrm{i}y\big], \qquad (A3)$$

is deduced. Similarly, as for $x > 0$ and $y < 0$, one has $\sin(2y) < -2y$ and $\sinh(-2x) < -2x < 0$, one obtains

$$\mathrm{Arg}\big[\coth(x+\mathrm{i}y)\big] > \arctan\left(\frac{y}{x}\right) = \mathrm{Arg}[x+\mathrm{i}y]. \qquad (A4)$$

Thus, the inequality (7) is retrieved.

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