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On the Refraction Law for a Sound Ray in a Moving Medium

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Summary
A recent paper by D. Hohenwarter and F. Jelinek (Acustica - acta acustica **86** (2000) 1-14) states that the refraction law known in the literature for a sound ray in a two-dimensional stratified moving medium is approximate. In that paper, using Fermat's principle of least time, a refraction law for a sound ray is derived which is claimed to be different from those known in the literature. In the present paper, we show that, in fact, the refraction law obtained by D. Hohenwarter and F. Jelinek coincides with those known in the literature. However, these authors were most likely the first to derive the refraction law for a sound ray in a moving medium by using Fermat's principle of least time.

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1. Introduction

Sound propagation in a moving medium is a complicated phenomenon since motion of the medium results in anisotropy of its acoustical properties. Usually qualitative approaches in studies of sound propagation in moving media have led to incorrect results (also see p. xi from [1]). This statement can be illustrated by the history of derivations for refraction laws for a sound ray and the normal to a wavefront in a stratified (layered) moving medium.

In a moving medium, the unit vector $\mathbf{s}$ in the direction of a sound ray, in a general case, does not coincide with the unit vector $\mathbf{n}$ in the direction normal to a wavefront. The refraction law for the normal to a wavefront (for the vector $\mathbf{n}$) was derived by Rayleigh [2]. However, Rayleigh did not distinguish between vectors $\mathbf{n}$ and $\mathbf{s}$. In 1901, Barton [3] was the first to understand this difference and to derive the refraction law for a sound ray (for the vector $\mathbf{s}$) for the case of a two-dimensional (2D) stratified moving medium.

In 1953 Kornhauser [4], probably not aware of results obtained by Barton [3], obtained the same refraction law for a sound ray in a 2D stratified moving medium. Thompson [5] derived this refraction law by using a different approach. Despite these papers by Barton, Kornhauser, and Thompson, there were also several papers (e.g. [6, 7, 8, 9, 10, 11]) which have used incorrect refraction laws for a sound ray in a stratified moving medium. Several references (e.g. [12, 13, 14]) have pointed out that these formulations of a refraction law are incorrect. Finally, Ostashev [15, 16, 1] generalized the refraction law for a sound ray to the case of a three-dimensional (3D) stratified moving medium and considered incorrect formulations of a refraction law used in the literature in detail.

Thus, a combined effort of many researchers has established a correct form of the refraction law for a sound ray in a stratified moving medium. This refraction law is one of the most important results in the acoustics in moving media and has been successfully used in the literature for studies of sound propagation in the atmosphere and ocean.

Nevertheless, the recent paper by Hohenwarter and Jelinek [17] states that previously derived refraction laws for a sound ray in a stratified moving medium are approximate. Using Fermat's principle of least time, they derived a refraction law for a sound ray in a 2D stratified moving medium and claimed that it is different from those known in the literature.

In the present paper we discuss and clarify the abovementioned results and statements made in [17]. This will help specialists in the field to choose the correct refraction law for a sound ray in a moving medium. Note that we will only comment on the most important results and statements in [17] and will not consider others.

The paper is organized as follows. In section 2, using previously derived results [1], we present a correct refraction law for a sound ray in a 3D stratified moving medium. In

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section 3, a particular form of this law, valid for a 2D stratified moving medium, will be given. We will also show that the latter refraction law coincides with those adopted in the literature and also coincides with the refraction law obtained in [17]. In section IV, the refraction law for the normal to a wavefront is considered briefly. The main results of the present paper are summarized in the Conclusion.

2. The refraction law for a sound ray in a 3D stratified medium

A sound ray path in a 3D stratified moving medium is shown in Figure 1. In the medium, the adiabatic sound speed c and medium velocity $\mathbf{v}$ depend on the vertical coordinate z. For simplicity, we assume that the vertical component of $\mathbf{v}$ is zero. The unit vectors $\mathbf{s}$ and $\mathbf{n}$ indicate the directions of the ray path and the normal to a wavefront at the origin of the coordinate system x, y, z. α and θ are the grazing angles of the vectors $\mathbf{s}$ and $\mathbf{n}$. Furthermore, $\mathbf{m}$ and $\mathbf{e}$ are unit vectors in the directions of horizontal projections of $\mathbf{s}$ and $\mathbf{n}$. Finally, ψ is the angle between vectors $\mathbf{m}$ and $\mathbf{e}$.

The refraction law for a sound ray in a 3D stratified moving medium, derived in [15], is given by (see also [1] equation 3.53):

$$c\left[\sqrt{\cos^2\alpha + \left[(\mathbf{e}\cdot\mathbf{v}/c)^2\sin^2\alpha - v^2/c^2\right]\sin^2\alpha} - (\mathbf{e}\cdot\mathbf{v}/c)\sin^2\alpha\right]^{-1} + \mathbf{e}\cdot\mathbf{v} = \text{const}, \qquad (1)$$

where $v = |\mathbf{v}|$. (Note that equation (3.53) in [1] has a misprint: α on the right-hand side of this equation should be replaced by a). Given vertical profiles $c(z)$ and $\mathbf{v}(z)$, equation (1) allows us to calculate the grazing angle α of the sound ray. The vector $\mathbf{e}$ in equation (1) does not depend on z, i.e., it remains constant along the ray.

On the other hand, the angle ψ changes along the ray, i.e. it depends on z. Equations (3.51) and (3.52) in [1] express ψ in terms of $c(z)$ and $\mathbf{v}(z)$. Finally, note that equations for $\alpha(z)$ and $\psi(z)$ in a 3D stratified moving medium have also been obtained for the case when the vertical component of the vector $\mathbf{v}$ is nonzero [15, 1].

3. The refraction law for a sound ray in a 2D stratified medium

Most of the references mentioned in the Introduction consider a refraction law for a sound ray in a 2D stratified moving medium. In such a medium, the vectors $\mathbf{v}$, $\mathbf{e}$, and $\mathbf{m}$ are located in one vertical plane which will be chosen as the zx-plane without loss of generality. In this case, the geometry of the ray path and the vectors $\mathbf{n}$ and $\mathbf{s}$ is shown in Figure 2.

In equation (1), we denote $\mathbf{e}\cdot\mathbf{v} = u$. For a 2D stratified moving medium, $u = v$ if sound propagates in the direction of the vector $\mathbf{v}$, and $u = -v$ if sound propagates in the opposite direction. With this notation, equation (1) can be written as:

$$\frac{c}{\sqrt{\cos^2\alpha - (u/c)^2\cos^2\alpha\sin^2\alpha} - (u/c)\sin^2\alpha} + u = \text{const}. \qquad (2)$$

This equation expresses the refraction law for a sound ray in a 2D stratified moving medium. Equation (2) was not directly presented in [1]. However, an explanation following equation (3.53) from [1] unambiguously shows how to obtain equation (2) from equation (1).

Let us rewrite equation (2) in a form appropriate for comparison with equations known in the literature. We combine the two terms on the left-hand side of equation (2) in one fraction, factor out $c/\cos\alpha$ from the fraction, and denote

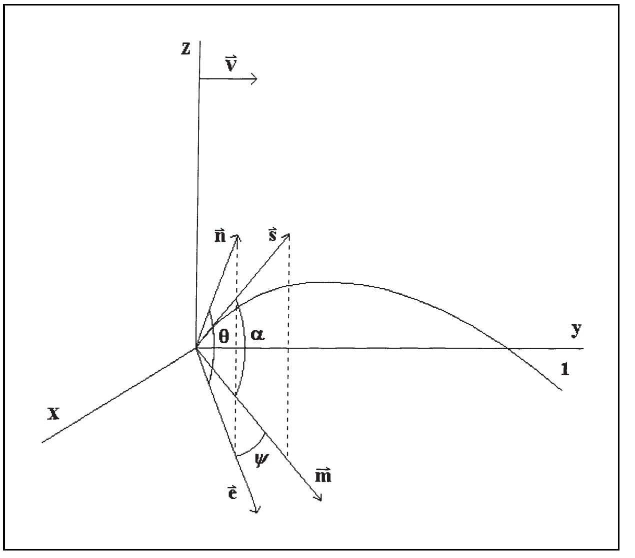

Figure 1. The geometry of a sound ray path (1) and the unit vectors $\mathbf{s}$ and $\mathbf{n}$ in the directions of the sound ray and the normal to a wavefront in a 3D stratified moving medium.

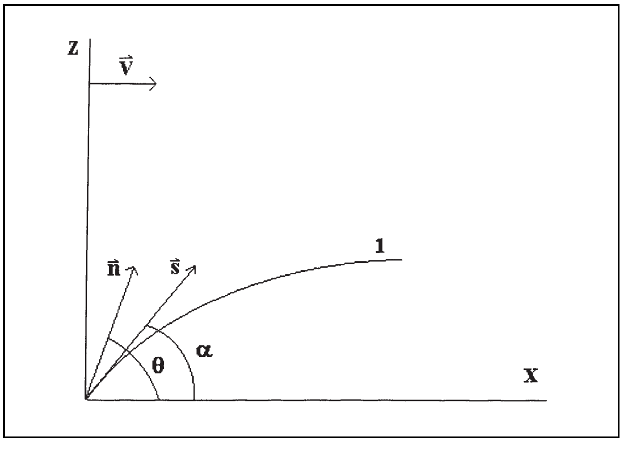

Figure 2. The geometry of a sound ray path (1) and the unit vectors $\mathbf{s}$ and $\mathbf{n}$ in the directions of the sound ray and the normal to a wavefront in a 2D stratified moving medium.

$M = u/c$. As a result, we obtain

$$\frac{c}{\cos\alpha}\,\frac{1 + M\cos\alpha\sqrt{1 - M^2\sin^2\alpha} - M^2\sin^2\alpha}{\sqrt{1 - M^2\sin^2\alpha} - M\sin^2\alpha/\cos\alpha} = \text{const.} \qquad (3)$$

If terms of order M^2 and higher order terms are ignored, equation (3) can be written as $(c/\cos\alpha)(1 + M/\cos\alpha) = \text{const.}$

3.1. Refraction law obtained by Barton [3]

In reference [3], the following refraction law for a sound ray was presented (see equation (1) from [3] or equation (3.54) from [1])

$$\cot\alpha = \cot\theta + u/(c\sin\theta)\,. \qquad (4)$$

Using the refraction law for the normal to a wavefront obtained by Rayleigh [2] (see equation (13) below), $\cos\theta$ can be expressed in terms of c and u: $\cos\theta = c/(K - u)$, where K is a positive constant. Using this formula in equation (4), we have

$$\sqrt{(K-u)^2 - c^2} = \left[c + M(K-u)\right]\tan\alpha. \qquad (5)$$

Squaring both sides of equation (5) and solving the resulting equation for $(K - u)$ yields

$$\frac{u\tan^2\alpha \pm \sqrt{1 - M^2\sin^2\alpha}(c/\cos\alpha)}{1 - M^2\tan^2\alpha} + u = K. \qquad (6)$$

A sign in front of the square root in this equation can be chosen from considering the limiting case $u = 0$. In this case, it follows from equation (6) that $\pm c/\cos\alpha = K$. Since $K > 0$, we must keep only the sign $+$ in front of the square root in this equation and in equation (6). Then, we combine the two terms on the left-hand side of equation (6) in one fraction, and factor out the term $c/\cos\alpha$. The result is:

$$\frac{c}{\cos\alpha}\,\frac{\frac{M}{\cos\alpha} + \sqrt{1 - M^2\sin^2\alpha} - M^3\frac{\sin^2\alpha}{\cos\alpha}}{1 - M^2\tan^2\alpha} = K. \qquad (7)$$

If we multiply the numerator and denominator of the second fraction on the left-hand side of equation (3) by the factor $\sqrt{1 - M^2\sin^2\alpha} + M\sin^2\alpha/\cos\alpha$, equation (7) results immediately. Thus, the refraction law for a sound ray derived by Barton [3] coincides with equation (3) and, hence, with equation (2).

3.2. Refraction law obtained by Kornhauser [4]

Equation (35) from [4] presents the refraction law for a sound ray in a moving medium. In our notation, this refraction can be written as

$$\frac{\frac{c_0}{c}\cos\alpha\sqrt{1 - M^2\sin^2\alpha} - \frac{c_0}{c}M\sin^2\alpha}{1 + M\cos\alpha\sqrt{1 - M^2\sin^2\alpha} - M^2\sin^2\alpha} = \text{const}, \quad (8)$$

where c_0 is a reference value of c.

Let us divide both sides of equation (8) by c_0. Since c_0 does not depend on z, the right-hand side of the equation obtained is a constant. Furthermore, on the left-hand side of this equation, we factor out the term $\cos\alpha/c$. As a result, we have

$$\frac{\cos\alpha}{c}\,\frac{\sqrt{1 - M^2\sin^2\alpha} - M\frac{\sin^2\alpha}{\cos\alpha}}{1 + M\cos\alpha\sqrt{1 - M^2\sin^2\alpha} - M^2\sin^2\alpha} = \text{const.} \qquad (9)$$

This is the inverse of equation (3), so equation (8) derived in [4] coincides with equation (3) and, hence, with equation (4) obtained by Barton [3]. Note that equation (8) has been used in many references, e.g. [12, 18].

3.3. Refraction law obtained by Thompson [5]

Equation (12) from reference [5] gives the refraction law for a sound ray in a moving medium. This equation was derived by an approach different from those used in references [3, 4]. In our notation, equation (12) from [5] is given by:

$$\frac{c\sec\alpha}{\sqrt{1 - (u/c)^2\sin^2\alpha} - \frac{u}{c}\sin^2\alpha\sec\alpha} + u = \text{const.} \quad (10)$$

Multiplying numerator and denominator on the left-hand side of equation (10) by $\cos\alpha$, we reveal that the equation obtained coincides with equation (2). Thus, the refraction law for a sound ray derived by Thompson [5] coincides with equation (2) and, hence, with Equations (4) and (8) derived by Barton [3] and Kornhauser [4], respectively.

3.4. Refraction law obtained by Hohenwarter and Jelinek [17]

Finally, let us consider the refraction law for a sound ray presented in reference [17], see equation (3) from that reference. In our notation, that equation can be written as

$$\frac{1}{c\left(1 - \left(\frac{u}{c}\right)^2\right)}\left[\frac{\sin(90^\circ - \alpha)}{\sqrt{1 - \left(\frac{u}{c}\right)^2\cos^2(90^\circ - \alpha)}} - \frac{u}{c}\right] = \text{const.} \quad (11)$$

In this equation, we denote $M = u/c$, combine the two terms in the square brackets in one fraction, invert the result, and factor out the term $c/\cos\alpha$. The resulting equation is

$$\frac{c}{\cos\alpha}\,\frac{(1 - M^2)\sqrt{1 - M^2\sin^2\alpha}}{1 - (M/\cos\alpha)\sqrt{1 - M^2\sin^2\alpha}} = \text{const.} \qquad (12)$$

It can be shown that the second fraction on the left-hand side of this equation coincides with the second fraction on the left-hand side of equation (3) to any order in M if $0 \le M < 1$. (To obtain this result, one can equate these fractions, multiply both sides of the equation obtained by the product of denominators, and show that the resulting equation is an identity.) Therefore, equation (11) derived in reference [17] coincides with equation (3) and, hence, with Equations (4), (8), and (10), obtained in references [3, 4, 5], respectively.

Thus, the refraction law for a sound ray obtained in [17] is the well known refraction law in a 2D stratified moving medium. What is new in [17] is that this refraction law was derived by using Fermat's principle.

In [17], equation (11) is compared with a refraction law obtained in [5], see equation (10) in the present paper. As mentioned above, these refraction laws coincide. However, it is stated in [17] that they are different. The reason for this conclusion is that in reference [17], when considering equation (10), the grazing angle α of the vector $\mathbf{s}$ was misinterpreted as the grazing angle θ of the vector $\mathbf{n}$.

Hohenwarter and Jelinek [17] correctly point out that the refraction law for a sound ray was explicitly presented in [1] only for the case of a 3D stratified moving medium. As mentioned earlier, in [1] it was explained how to obtain equation (2) from equation (1), but the derivation was not presented.

4. The refraction law for the normal to a wavefront

For the geometry shown in Figure 2, the correct refraction law for the normal to a wavefront, i.e., the refraction law for the angle θ, was obtained by Rayleigh [2]:

$$\frac{c}{\cos\theta} + u = K, \tag{13}$$

where K is a positive constant. This law has been rederived several times by making use of different approaches and has been also generalized to the case of a 3D stratified moving medium, e.g., see equation (3.48) from [1] and [4, 13, 19].

In [17], equation (13) with θ replaced by α is referred to as “classical law of refraction" for a sound ray, “classical calculation",“usual approximation", “classical idea". This terminology might be somewhat misleading. In fact, equation (13) with θ replaced by α was used as a refraction law for a sound ray only in a few papers. For example, among the papers [6, 7, 8, 9, 10, 11] mentioned in the Introduction, where incorrect refraction laws were used, equation (13) with θ replaced by α was used only in reference [7].

Finally, it should be pointed out that equation (13) with θ replaced by α does not agree with equation (4) from [17], despite the statement in the paragraph following the latter equation.

5. Conclusion

Using results from reference [1], we have presented refraction laws for a sound ray in 3D and 2D stratified moving media, i.e. Equations (1) and (2), respectively. We have shown that equation (2) coincides with previously published refraction laws for a sound ray in 2D stratified moving media, e.g., [3, 4, 5, 12, 18]. It also coincides with the refraction law recently obtained in reference [17] where, most likely for the first time, it was derived by using Fermat's principle of least time.

Also we have presented the well-known refraction law for the normal to a wavefront of a sound wave in a 2D stratified moving medium, equation (13).

Thus, the present paper clarifies which equations should be used as refraction laws for a sound ray and the normal to a wavefront in a stratified moving medium and thus removes the possibility of confusing following publication of reference [17].

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