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Weakly nonlinear propagation of small-wavelength, impulsive acoustic waves in a general atmosphere



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HIGHLIGHTS

- Derivation of the nonlinear, geometrical acoustics equations for a general atmosphere.
- Unification of sonic-boom and explosion problems.
- Governing equations as the basis for analysis.
- Multiple-scale asymptotic analysis.
- Results for an explosion.

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1. Introduction

ABSTRACT

Multiple-scale asymptotic analysis is applied to small-wavelength, weakly nonlinear propagation of an impulsive acoustic wave in a general (3D, in-motion and time dependent) atmosphere. In keeping with previous work on sonic booms and nonlinear acoustics in general, the result is a combination of ray tracing and a generalised Burgers equation describing evolution of the waveform carried by a ray. This is nonetheless, to our knowledge, the first derivation of such a model based on asymptotic analysis of the governing equations for a general atmosphere. Results are given, discussed and compared with measurements for the particular example of the test explosion known as Misty Picture.

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Supersonic aircraft and explosions are examples of high-intensity sources of impulsive acoustic waves, whose atmospheric propagation can involve important nonlinear effects. In addition to direct arrivals near the source, such waves may propagate to high altitude and be reflected in either the stratosphere or thermosphere, before returning to the ground as long-range arrivals, a phenomenon known as secondary sonic boom in the aircraft case [1]. Modelling of such arrivals requires consideration of the entire propagation process through the upper atmosphere, during which changes in atmospheric properties, such as density and acoustic attenuation, have significant effects on the wave which comes back to the ground.

Sonic-boom propagation modelling has a long history [2–4]. It is generally based on a combination of geometrical acoustics, *i.e.* ray theory, which requires that acoustic wavelengths be small compared with atmospheric scales and accounts for the principal effects of wind and atmospheric nonuniformities, and nonlinear acoustics, assuming weak nonlinearity and

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attenuation, which describes the propagation of the waveform carried by a ray. Ray theory, which itself has an extensive literature (see e.g. [5–7]), is independent of nonlinearity and attenuation and expresses the idea that the motion of a sonic wavefront is mainly due to a combination of acoustic propagation and convection by wind. On the other hand, the other ingredients of the model, namely nonlinearity and attenuation, contribute most of the physics: nonlinearity leads to propagation at a speed slightly different from its small-amplitude value, but whose cumulative consequences are important, and attenuation arises from a combination of thermoviscous diffusion and relaxation. These effects are expressed by a generalised Burgers equation (see e.g. [8] and references therein), which can be integrated alongside the ray equations to obtain the final result: the acoustic signal at the receiver.

Fewer studies of the explosive-source problem have appeared. Results obtained using the present model have been reported in *e.g.* [9–11]. More recently, [12] have examined the effects of horizontal variations of atmospheric properties on the propagation of explosion induced waves in a time-independent atmosphere.

Our aim here is to provide a single model, unifying the sonic boom and explosion problems, which allows for a general atmosphere (three dimensional, in-motion and which may also be time dependent) and arbitrary source motion in the sonic-boom case.

Sections 2 and 3 present the work, which is based on multiple-scale asymptotic analysis, starting from the governing equations of a Newtonian, compressible fluid. In Section 2, the ray-tracing equations arise via solvability conditions at first order. In Section 3, solvability at second order leads to a generalised Burgers equation which describes evolution of the waveform carried by the rays. For simplicity's sake, the second-order analysis is first given for the case of a thermodynamically simple fluid, the effects of relaxation and variable atmospheric composition being later described in Section 3.3 and Appendix D. Caustics (which are problematic for geometrical-acoustics models, such as the present one, because the model does not apply in a small region near the caustic) are briefly discussed in Section 3.4. Finally, Section 4 gives some results obtained using the model.

2. Basic equations and first-order asymptotics

2.1. Basic equations

The momentum and mass equations are:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{\nabla \cdot \boldsymbol{\sigma}}{\rho} + \mathbf{g} - 2 \, \boldsymbol{\Omega} \times \mathbf{v}, \tag{2.1}$$
$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

where \mathbf{v} , ρ and $\boldsymbol{\sigma}$ are the velocity, density and stress tensor, \mathbf{g} combines gravitational and centrifugal acceleration, and $\boldsymbol{\Omega}$ is the rotational angular velocity of the earth. Although gravity and rotation have been included here, because they are certainly important in the dynamics of the underlying atmosphere, as we shall see they turn out to have negligible direct effects on the acoustic perturbation at the orders to which the asymptotic analysis is carried out.

Denote by \mathbf{v}_0 , ρ_0 , σ_0 the flow field in the absence of the source, referred to as the underlying atmosphere (or underlying flow). Next, let

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}', \qquad \rho = \rho_0 + \rho', \qquad \mathbf{\sigma} = \mathbf{\sigma}_0 + \mathbf{\sigma}' \tag{2.3}$$

be the flow field with the source. Thus, \mathbf{v}' , ρ' , σ' represent the perturbation due to the source. Since both the underlying and perturbed flows satisfy (2.1) and (2.2), subtraction yields the perturbation equations

$$\frac{\partial \mathbf{v}'}{\partial t} + (\mathbf{v}_0 \cdot \nabla) \mathbf{v}' - \frac{\nabla \cdot \boldsymbol{\sigma}'}{\rho_0} = -\frac{\rho'}{\rho_0(\rho_0 + \rho')} \nabla \cdot \left(\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}'\right) - (\mathbf{v}' \cdot \nabla) \mathbf{v}_0 - (\mathbf{v}' \cdot \nabla) \mathbf{v}' - 2\,\boldsymbol{\Omega} \times \mathbf{v}', \tag{2.4}$$

$$\frac{\partial \rho'}{\partial t} + \mathbf{v}_0 \cdot \nabla \rho' + \rho_0 \nabla \cdot \mathbf{v}' = -\mathbf{v}' \cdot \nabla \rho_0 - \rho' \nabla \cdot \mathbf{v}_0 - \mathbf{v}' \cdot \nabla \rho' - \rho' \nabla \cdot \mathbf{v}', \qquad (2.5)$$

where terms on the left-hand sides are those which contribute at leading order in the asymptotic analysis of the following sections. This is because they: (a) are linear in the perturbation, and (b) contain derivatives of the perturbation. The asymptotics of small wavelength and weak nonlinearity made henceforth imply asymptotic dominance of such terms.

2.2. Short-wave asymptotics

From here on, the source size and acoustic wavelength are assumed much smaller than the length scale for variation of the properties of the underlying atmosphere, the disparity in scales being formally expressed by a small parameter $\epsilon \rightarrow 0$. In the near field of the source, it is as if the atmosphere were uniform, whereas at the propagation distances considered here, comparable with the atmospheric scale, the acoustic perturbation is localised near a surface (the wavefront) $\Phi(\mathbf{x}, t) = 0$, whose determination forms part of the problem.

The presence of asymptotically distinct scales suggests a multiple-scaling approach with fast variable $\eta = \Phi(\mathbf{x}, t)/\epsilon$, representing distance from the wavefront scaled appropriately for the acoustic perturbation, and slow variables \mathbf{x} , t. The asymptotic expansions

$$\mathbf{v}' = \epsilon \mathbf{v}'_1 + \epsilon^2 \mathbf{v}'_2 + \cdots$$

$$\rho' = \epsilon \rho'_1 + \epsilon^2 \rho'_2 + \cdots$$

$$\sigma' = \epsilon \sigma'_1 + \epsilon^2 \sigma'_2 + \cdots$$
(2.6)

are used to describe the acoustic perturbation, where the coefficients are functions of η , **x** and *t*. On the other hand, Φ and atmospheric properties, such as **v**₀, ρ_0 and σ_0 , are functions of **x** and *t* alone. Multiple scaling means that $\partial/\partial t$ and ∇ are replaced by $\partial/\partial t + \epsilon^{-1}\partial \Phi/\partial t \partial/\partial \eta$ and $\nabla + \epsilon^{-1}\nabla \Phi \partial/\partial \eta$ in equations such as (2.4) and (2.5).

The scaling, $\mathcal{O}(\epsilon)$, of the leading-order terms in (2.6) is such that nonlinearity first appears at second order and hence contributes to the generalised Burgers equation which arises as a solvability condition at that order. Similar scaling assumptions are later made for the attenuation coefficients. Rather than being restrictive, as may appear at first sight, these assumptions express formal asymptotic bookkeeping, aimed at incorporating both nonlinearity and attenuation into the Burgers equation (in accord with the general principle of "distinguished" scalings of asymptotic analysis). Should it turn out that one or other of these scaling assumptions is inappropriate, the corresponding term in the Burgers equation will reflect this by inducing either small or large changes in the waveform during propagation. The model remains valid in the short-wave limit provided that nonlinearity and attenuation are small in the sense that they do not enter at first order. This requires a small acoustic Mach number and that the distance for attenuation be large compared with the wavelength, the usual assumptions of nonlinear acoustics.

2.3. First-order analysis

At order $\mathcal{O}(\epsilon^0)$ (first order), (2.4) and (2.5) yield

$$\rho_0 \left(\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi\right) \frac{\partial \mathbf{v}_1'}{\partial \eta} - \nabla \Phi \cdot \frac{\partial \sigma_1'}{\partial \eta} = 0, \tag{2.7}$$

$$\left(\frac{\partial\Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla\Phi\right) \frac{\partial\rho_1'}{\partial\eta} + \rho_0 \nabla\Phi \cdot \frac{\partial\mathbf{v}_1'}{\partial\eta} = 0, \tag{2.8}$$

whose closure requires an expression for σ'_1 . At this order, we expect (an expectation justified later) the usual linear acoustics relation $\sigma'_1 = -c_0^2 \rho'_1 \mathbf{I}$ to hold, where $c_0(\mathbf{x}, t)$ is the sound speed of the underlying atmosphere and \mathbf{I} denotes the identity tensor. Replacing σ'_1 in (2.7) and taking the scalar product with $\nabla \Phi$ leads to a linear system for the quantities $\partial \rho'_1 / \partial \eta$ and $\rho_0 \nabla \Phi \cdot \partial \mathbf{v}'_1 / \partial \eta$. The condition for a non-zero solution yields the eikonal equation

$$\frac{\partial \Phi}{\partial t} + \mathbf{w} \cdot \nabla \Phi = 0, \tag{2.9}$$

where $\mathbf{w} = \mathbf{v}_0 + c_0 \mathbf{n}$ and $\mathbf{n} = \nabla \Phi / |\nabla \Phi|$ is a unit normal to the surface Φ = constant. A square root has been taken in deriving (2.9), implying a choice of signs. If necessary, Φ is replaced by $-\Phi$ to obtain (2.9). This has the effect of making acoustic propagation relative to the underlying atmosphere go in the direction of $\nabla \Phi$ and \mathbf{n} . (2.9) implies that the surfaces Φ = constant move at velocity \mathbf{w} , expressing the expected convection by the underlying velocity field and acoustic propagation normal to the locally plane wave.

Using (2.9), (2.7) becomes

$$\frac{\partial}{\partial \eta} \left\{ \mathbf{v}_1' - \frac{c_0 \mathbf{n}}{\rho_0} \rho_1' \right\} = \mathbf{0}.$$
(2.10)

Given the choice of sign made above, the wavefront $\Phi(\mathbf{x}, t) = 0$ propagates in the direction of increasing Φ relative to the fluid, *i.e.* in the direction of increasing η . As a result, the fluid at $\eta \to +\infty$, far ahead of the wavefront, has yet to see the effect of the source and the perturbation is zero there. Together with (2.10), this yields $\mathbf{v}'_1 = c_0 \rho'_1 \mathbf{n} / \rho_0$, the usual linear acoustics relation for a plane wave. This completes the first-order solution, in which ρ'_1 is an as yet undetermined function of \mathbf{x} , t and $\eta = \Phi / \epsilon$, and $\Phi(\mathbf{x}, t)$ is governed by (2.9).

2.4. Ray equations

Rays are here defined as points $\mathbf{X}(t)$ which move according to

$$\frac{d\mathbf{X}}{dt} = \mathbf{v}_0(\mathbf{X}, t) + c_0(\mathbf{X}, t)\mathbf{n}.$$
(2.11)



Fig. 1. Illustrations of the rays and wavefront, $\Phi = 0$, for: (a) an explosion, (b) a supersonic aircraft. In the latter case, the rays making up the wavefront at the given time, *t*, have different emission times, τ . Note that these figures do not show results of actual ray calculations for the atmosphere. They are simply sketches illustrating the source, rays and their relationship with the wavefront.

Taking the gradient of (2.9), setting $\mathbf{k} = \nabla \Phi$ and using $k_j \partial n_j / \partial x_i = 0$ (which follows from the gradient of $n_j n_j = 1$ and $n_j = k_j / |\mathbf{k}|$) leads to

$$\frac{dk_i}{dt} = -\left(\left.\frac{\partial v_{0j}}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}} + \left.\frac{\partial c_0}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}} n_j\right) k_j \tag{2.12}$$

following a ray. Note that, here and henceforth, the repeated-subscript summation convention applies. We also remark that (2.11) and (2.12) can be obtained from equations (105) and (106) of [13] using the dispersion relation $\omega = c_0 |\mathbf{k}| + \mathbf{v}_0 \cdot \mathbf{k}$.

Eq. (2.9) implies that Φ is constant following a ray; in particular, the wavefront $\Phi = 0$ consists of rays which originated at the source and we restrict attention to such rays from now on. For any such ray, define $\mathbf{K} = \mathbf{k}/|\mathbf{k}_s|$, where $\mathbf{k}_s = \mathbf{k}(\tau)$ is the value of \mathbf{k} at its emission time, τ . Since $|\mathbf{k}_s|$ is a constant for the ray, (2.12) implies

$$\frac{dK_i}{dt} = -\left(\left.\frac{\partial v_{0j}}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}} + \left.\frac{\partial c_0}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}} n_j\right) K_j.$$
(2.13)

Eqs. (2.11) and (2.13), together with $\mathbf{n} = \mathbf{K}/|\mathbf{K}|$, form a closed system which can be integrated, starting from $t = \tau$, to obtain $\mathbf{X}(t)$ and $\mathbf{K}(t)$ for the ray.

The initial conditions

$$\mathbf{X}(\tau) = \mathbf{x}_{s}(\tau), \qquad \mathbf{K}(\tau) = \mathbf{n}_{s} \tag{2.14}$$

involve the source position, $\mathbf{x}_s(\tau)$, and wavefront unit normal, \mathbf{n}_s , at the emission time. Different rays are distinguished by emission time or \mathbf{n}_s . Note that the source is represented by a point, $\mathbf{x}_s(t)$. Given the assumed small size of the source compared with atmospheric scales, a different choice of representative point would modify the rays a little. It would also induce a compensating change in the acoustic waveform carried by the rays, the end result being asymptotically insensitive to the choice of \mathbf{x}_s .

In the case of an explosion (illustrated by Fig. 1(a)), the emission time, being that of the explosion, is the same for all rays, while the direction of the unit vector \mathbf{n}_s is unconstrained and determines the choice of particular ray. Rays can be parameterised using spherical polar angles, ϕ and ψ , for $\mathbf{n}_s(\phi, \psi)$ and the full set of rays, obtained by integration of the ray equations, expressed as $\mathbf{X}(t, \phi, \psi)$ and $\mathbf{K}(t, \phi, \psi)$. At any time *t*, the wavefront $\Phi = 0$ can be constructed from $\mathbf{X}(t, \phi, \psi)$ by varying the ray parameters ϕ and ψ .

A supersonic aircraft (see Fig. 1(b)) emits rays continuously and the wavefront $\Phi = 0$ has $\mathbf{x}_s(t)$ as its vertex. Thus, Φ has a constant (zero) value at $\mathbf{x} = \mathbf{x}_s(t)$, implying $\partial \Phi / \partial t + \mathbf{v}_s \cdot \nabla \Phi = 0$, where $\mathbf{v}_s = d\mathbf{x}_s/dt$ is the aircraft velocity. Subtracting (2.9), evaluated at $\mathbf{x} = \mathbf{x}_s(t)$, and recalling that $\mathbf{n} = \nabla \Phi / |\nabla \Phi|$, yields $(\mathbf{v}_s - \mathbf{v}_0(\mathbf{x}_s, t)) \cdot \mathbf{n} = c_0(\mathbf{x}_s, t)$. This equation defines a cone of possible vectors \mathbf{n} with axis in the direction $\mathbf{v}_s - \mathbf{v}_0$ and semi-angle $\arccos(1/M)$, where $M = |\mathbf{v}_s - \mathbf{v}_0|/c_0$ is the Mach number of the aircraft relative to the underlying flow. Since here \mathbf{n} is a normal vector to the wavefront at its apex, this implies the expected result that the wavefront asymptotes to a Mach cone with axis in the direction $\mathbf{v}_0 - \mathbf{v}_s$ and semi-angle $\arcsin(1/M)$. Rays emitted by the aircraft must have \mathbf{n}_s on the cone $(\mathbf{v}_s - \mathbf{v}_0) \cdot \mathbf{n}_s = c_0$, where $\mathbf{v}_s, \mathbf{v}_0$ and c_0 refer to the emission time of the ray. They can be parameterised by their emission time, τ , and a single polar angle, ψ , defining $\mathbf{n}_s(\tau, \psi)$ and leading to rays $\mathbf{X}(t, \tau, \psi)$ and $\mathbf{K}(t, \tau, \psi)$. As for an explosion, the wavefront surface $\Phi = 0$ can be determined from $\mathbf{X}(t, \tau, \psi)$ by varying the ray parameters, now τ and ψ .

Thus, in either case, the wavefront consists of a two-parameter family of rays. The precise parameters used are far from unique and depend on the application and on implementation choices. In what follows, β and γ denote a general set of ray parameters. We shall later require the partial derivatives of **X**(t, β , γ) and **K**(t, β , γ) with respect to each of the parameters. Equations governing the evolution of these partial derivatives following a ray are derived in Appendix A. These equations are added to the ray-tracing system (2.11) and (2.13).

3. Waveform propagation

3.1. Second-order analysis

Second-order analysis requires the determination of σ'_2 . This is described in Appendix B and involves use of the Newtonian constitutive law and entropy equation. For simplicity's sake, it is assumed that air is a thermodynamically simple fluid, *i.e.* its thermodynamic state is uniquely determined by just two independent state variables, which are chosen to be ρ and s, where s is entropy per unit mass. This implies, for instance, $p = p(\rho, s)$ for the pressure. Given the known importance of relaxation for atmospheric acoustic attenuation and the variation of atmospheric composition at high altitude, a full description requires more than just these two variables. We later (in Section 3.3 and Appendix D) describe the modifications which arise from allowing for relaxation and variable composition. Appendix B also makes the assumption that the coefficients of viscosity and heat conduction are $\mathcal{O}(\epsilon^2)$. This forms part of the formal asymptotic bookkeeping referred to earlier.

The results of Appendix B are $\sigma'_1 = -c_0^2 \rho'_1 \mathbf{I}$, assumed earlier, and that the components of σ'_2 are

$$\sigma_{2ij}' = \epsilon^{-2} \frac{c_0}{\rho_0} \left(\lambda_0 \delta_{ij} + 2\mu_0 n_i n_j \right) \left| \nabla \Phi \right| \frac{\partial \rho_1'}{\partial \eta} - \left(c_0^2 \left(\rho_2' + \frac{B}{2A\rho_0} {\rho_1'}^2 \right) + \Pi \right) \delta_{ij}, \tag{3.1}$$

where δ_{ij} is the Kronecker delta, λ_0 and μ_0 are the coefficients of viscosity of the underlying atmosphere, and

$$\frac{B}{2A} = \frac{\rho_0}{2c_0^2} \left. \frac{\partial^2 p}{\partial \rho^2} \right|_0 \qquad \Pi = \left. \frac{\partial p}{\partial s} \right|_0 s'_2 \tag{3.2}$$

are respectively a nonlinearity parameter and the pressure perturbation arising from entropy perturbations, represented by s'_2 . The latter is governed by (B.16), with S'_1 given by (B.15). We should perhaps explain the notation used in (3.2). Partial derivatives with a subscript 0 imply that the quantity being differentiated is a function of ρ and s. The derivative is taken with respect to the indicated variable, while holding the other one constant. The result is then evaluated at $\rho = \rho_0$, $s = s_0$. Introducing the expansions, (2.6), of **v**', ρ' and σ' into (2.4) and (2.5),

ntroducing the expansions, (2.6), of **v**, ρ and σ into (2.4) and (2.5)

$$\frac{\rho_0}{c_0} \left(\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi \right) \mathbf{n} \cdot \frac{\partial \mathbf{v}_2'}{\partial \eta} + c_0 \left| \nabla \Phi \right| \frac{\partial \rho_2'}{\partial \eta} = -f_1, \tag{3.3}$$

$$\left(\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi\right) \frac{\partial \rho_2'}{\partial \eta} + \rho_0 \left| \nabla \Phi \right| \mathbf{n} \cdot \frac{\partial \mathbf{v}_2'}{\partial \eta} = -f_2, \tag{3.4}$$

at order $\mathcal{O}(\epsilon^1)$, where the result of (2.4) has been scalar multiplied by $\rho_0 \mathbf{n}/c_0$ to obtain (3.3) and

$$f_{1} = \frac{\rho_{0}}{c_{0}} \left(\frac{\partial}{\partial t} + \mathbf{v}_{0} \cdot \nabla \right) \left(\frac{c_{0}\rho_{1}'}{\rho_{0}} \right) + \frac{1}{c_{0}} \mathbf{n} \cdot \nabla \left(c_{0}^{2}\rho_{1}' \right) + \frac{1}{\rho_{0}c_{0}} \mathbf{n} \cdot \left(\nabla \cdot \boldsymbol{\sigma}_{0} \right) \rho_{1}' + \mathbf{n} \cdot \left(\mathbf{n} \cdot \nabla \mathbf{v}_{0} \right) \rho_{1}' + \frac{c_{0}}{\rho_{0}} \frac{B}{A} \left| \nabla \boldsymbol{\Phi} \right| \rho_{1}' \frac{\partial \rho_{1}'}{\partial \eta} - \epsilon^{-2} \frac{\lambda_{0} + 2\mu_{0}}{\rho_{0}} \left| \nabla \boldsymbol{\Phi} \right|^{2} \frac{\partial^{2}\rho_{1}'}{\partial \eta^{2}} + \frac{1}{c_{0}} \left| \nabla \boldsymbol{\Phi} \right| \frac{\partial \Pi}{\partial \eta},$$
(3.5)

$$f_{2} = \frac{\partial \rho_{1}'}{\partial t} + \mathbf{v}_{0} \cdot \nabla \rho_{1}' + \rho_{0} \nabla \cdot \left(\frac{c_{0} \rho_{1}'}{\rho_{0}} \mathbf{n}\right) + \frac{c_{0} \rho_{1}'}{\rho_{0}} \mathbf{n} \cdot \nabla \rho_{0} + \rho_{1}' \nabla \cdot \mathbf{v}_{0} + \frac{2c_{0}}{\rho_{0}} \left| \nabla \Phi \right| \rho_{1}' \frac{\partial \rho_{1}'}{\partial \eta}.$$
(3.6)

Here $\mathbf{v}'_1 = c_0 \rho'_1 \mathbf{n} / \rho_0$, $\sigma'_1 = -c_0^2 \rho'_1 \mathbf{I}$ and (3.1) have been used to express \mathbf{v}'_1 , σ'_1 and σ'_2 . The space and time derivatives of $\mathbf{n} \cdot \mathbf{n} = 1$ have also been employed. Note that the Coriolis term in (2.4) does not contribute because its scalar product with \mathbf{n} is zero at the present order.

Taking the sum of (3.3) and (3.4) and using (2.9) gives the solvability condition $f_1 + f_2 = 0$, leading to

$$\left(\frac{\rho_0}{c_0}\right)^{1/2} \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right) \left(\left(\frac{c_0}{\rho_0}\right)^{1/2} \rho_1' \right) = \frac{1}{2} \epsilon^{-2} \Delta |\nabla \Phi|^2 \frac{\partial^2 \rho_1'}{\partial \eta^2} - \left(1 + \frac{B}{2A}\right) \frac{c_0}{\rho_0} |\nabla \Phi| \rho_1' \frac{\partial \rho_1'}{\partial \eta} - \frac{1}{2} \left(\frac{1}{c_0} \nabla \cdot \left(c_0^2 \mathbf{n}\right) + \nabla \cdot \mathbf{v}_0 + \mathbf{n} \cdot \left(\mathbf{n} \cdot \nabla \mathbf{v}_0\right) \right) \rho_1',$$

$$(3.7)$$

where (B.16), (B.15) and the second of Eqs. (3.2) have been used to express $\partial \Pi / \partial \eta$ and

$$\Delta = \frac{1}{\rho_0} \left(\lambda_0 + 2\mu_0 + \kappa_0 \left(\frac{1}{c_{v0}} - \frac{1}{c_{p0}} \right) \right) \tag{3.8}$$

is the diffusivity of sound [14] due to viscosity and thermal conduction (κ is the thermal conductivity, while c_v , c_p are the specific heats at constant volume and pressure). Note that, in deriving (3.7) and (3.8), we have used the thermodynamic relation

$$\left(\frac{\partial\theta}{\partial\rho}\right)_{s}\left(\frac{\partial p}{\partial s}\right)_{\rho} = \theta c^{2} \left(\frac{1}{c_{v}} - \frac{1}{c_{p}}\right),\tag{3.9}$$

where θ is the absolute temperature, and the following term has been dropped from the right-hand side of (3.7):

$$-\frac{\rho_1'}{2\rho_0c_0}\mathbf{n}\cdot\left(\nabla\cdot\boldsymbol{\sigma}_0+c_0^2\nabla\rho_0+\left.\frac{\partial p}{\partial s}\right|_0\nabla s_0\right).$$
(3.10)

The justification for this is as follows. The sum in brackets in (3.10) can be rewritten as $\nabla \cdot (\sigma_0 + p_0 \mathbf{I}) = \nabla \cdot \tau_0$, where τ_0 is the underlying viscous stress tensor (not to be confused with the emission time τ). Given the assumption that the coefficients of viscosity are small, of $\mathcal{O}(\epsilon^2)$, $\nabla \cdot \tau_0$ is correspondingly small and is thus neglected.

3.2. Generalised Burgers equation

At this point, the second-order asymptotic analysis is complete, but there remains the task of simplifying and interpreting the result, (3.7).

The area of the wavefront consisting of rays with values of the ray parameters in the infinitesimal range $d\beta$, $d\gamma$ is $\nu d\beta d\gamma$, where $\nu = |\mathbf{X}_{\beta} \times \mathbf{X}_{\gamma}| = |\mathbf{n} \cdot (\mathbf{X}_{\beta} \times \mathbf{X}_{\gamma})|$ and \mathbf{X}_{β} , \mathbf{X}_{γ} denote the partial derivatives of **X** with respect to the ray parameters. As shown in Appendix C, following a ray,

$$\frac{2|\mathbf{K}|}{\nu^{1/2}}\frac{d}{dt}\left(\frac{\nu^{1/2}}{|\mathbf{K}|}\right) = \frac{1}{c_0}\nabla\cdot\left(c_0^2\mathbf{n}\right) + \nabla\cdot\mathbf{v}_0 + \mathbf{n}\cdot\left(\mathbf{n}\cdot\nabla\mathbf{v}_0\right),\tag{3.11}$$

of which the right-hand side will be recognised from (3.7).

For any given ray, $\mathbf{X}(t)$, $\mathbf{K}(t)$, $\nu(t)$, an associated waveform, $u(\xi, t)$, is defined from $\rho'_1(\eta, \mathbf{x}, t)$ by

$$u(\xi, t) = \frac{\epsilon}{|\mathbf{K}|} \left(\frac{\nu c_0}{\rho_0}\right)^{1/2} \rho_1'(\eta, \mathbf{X}(t), t), \qquad \xi = \frac{\epsilon \eta}{|\mathbf{k}_{\rm s}|},\tag{3.12}$$

with c_0/ρ_0 evaluated at the ray location, $\mathbf{x} = \mathbf{X}(t)$. Taking partial derivatives of the first of Eqs. (3.12) and using (2.11),

$$\frac{\partial u}{\partial \xi} = \frac{|\mathbf{k}_{\rm s}|}{|\mathbf{K}|} \left(\frac{\nu c_0}{\rho_0}\right)^{1/2} \frac{\partial \rho_1'}{\partial \eta},\tag{3.13}$$

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{|\mathbf{k}_{\rm s}|^2}{\epsilon |\mathbf{K}|} \left(\frac{\nu c_0}{\rho_0}\right)^{1/2} \frac{\partial^2 \rho_1'}{\partial \eta^2},\tag{3.14}$$

$$\frac{\partial u}{\partial t} = \epsilon \left(\frac{c_0}{\rho_0}\right)^{1/2} \rho_1' \frac{d}{dt} \left(\frac{\nu^{1/2}}{|\mathbf{K}|}\right) + \epsilon \frac{\nu^{1/2}}{|\mathbf{K}|} \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right) \left(\left(\frac{c_0}{\rho_0}\right)^{1/2} \rho_1'\right).$$
(3.15)

Finally, (3.7), (3.11) and $|\nabla \phi| = |\mathbf{k}_s| |\mathbf{K}|$ lead to

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta \left| \mathbf{K} \right|^2 \frac{\partial^2 u}{\partial \xi^2} - b \, u \frac{\partial u}{\partial \xi},\tag{3.16}$$

where

$$b(t) = |\mathbf{K}|^2 \left(1 + \frac{B}{2A}\right) \left(\frac{c_0}{\nu \rho_0}\right)^{1/2}.$$
(3.17)

(3.16) is a generalised Burgers equation which can be integrated following the ray to obtain $u(\xi, t)$ from initial conditions at the source. The two terms on the right-hand side represent thermoviscous attenuation and nonlinearity. (3.16), valid for a general atmosphere, is in agreement with previous results for special cases, *e.g.* [15,16].

The variables u and ξ , defined by (3.12), can be interpreted as follows. $\eta = \Phi/\epsilon$ and $|\nabla \Phi| = |\mathbf{k}_s| |\mathbf{K}|$ give $\Phi = |\nabla \Phi| \xi / |\mathbf{K}|$. Comparison with a Taylor's expansion of Φ about the wavefront $\Phi = 0$ shows that $\xi / |\mathbf{K}|$ is simply distance from the wavefront, measured positive in the direction of $\nabla \Phi$, *i.e.* the direction of acoustic propagation relative to the underlying atmosphere. Given the specialisation to $\mathbf{x} = \mathbf{X}(t)$ on the right-hand side of the first of the Eqs. (3.12), $u(\xi, t)$ refers to that point on the wavefront (the ray under consideration). At a particular time t, $u(\xi, t)$ describes, via the distance variable ξ , the acoustic perturbation along a line through $\mathbf{X}(t)$ perpendicular to the wavefront. The leading-order acoustic perturbation follows from $\rho' = p'/c_0^2$, $\mathbf{v}' = p'\mathbf{n}/\rho_0 c_0$ and

$$p' = |\mathbf{K}| \left(\frac{\rho_0 c_0^3}{\nu}\right)^{1/2} u \left(\mathbf{K} \cdot (\mathbf{x} - \mathbf{X}), t\right).$$
(3.18)

Another interpretation of $u(\xi, t)$ yields the acoustic perturbation at a fixed point, **x**, as a function of time. According to ray theories, such as the present one, the source is only heard if the wavefront, and hence a ray, passes through **x** at some

time t_a (otherwise **x** lies within a "shadow" zone). Taylor's expansion of $\Phi(\mathbf{x}, t)$ about $t = t_a$, using (2.9) to evaluate $\partial \Phi / \partial t$, leads to $\xi = \mathbf{w} \cdot \mathbf{K}(t_a - t)$. The pressure perturbation at **x** is thus

$$p' = |\mathbf{K}| \left(\frac{\rho_0 c_0^3}{\nu}\right)^{1/2} u \left(\mathbf{w} \cdot \mathbf{K}(t_a - t), t_a\right).$$
(3.19)

If there are several ray arrivals, their perturbations sum.

Initial conditions for (3.16) come from matching to the source. This requires numerical solution, analytical modelling (*e.g.* the Whitham F-function [17, section 9.3]) or measurements in the near-source region, where the atmosphere can be treated as locally steady and uniform. In particular, the acoustic perturbation is needed at distances from the source appropriate for matching. This implies distances large compared with the acoustic wavelength and source size, but small on the atmospheric scale. The distance must also be sufficiently large that the acoustic Mach number is small. For any given ray, a suitable matching point, $\mathbf{x} = \mathbf{X}(t_m)$, is chosen and $u(\xi, t)$ follows from (3.19) with $t_a = t_m$ and the pressure perturbation, $p'(\mathbf{x}, t)$, from the near-field problem.

The quantity v(t) goes to zero as $t \to \tau$, leading to an infinite singularity in b(t) at the emission time. It may be a good idea to remove the resulting singularity of the Burgers equation, prior to numerical integration, using a transformation of the time variable. Thus, a new time variable T = T(t) is used in place of t, where T(t) behaves like $(t - \tau)^{1/2}$ as $t \to \tau$ for the sonic-boom problem and like $\ln(t - \tau)$ for the explosive source.

3.3. Effects of relaxation and variable composition

As noted earlier, a more realistic description of atmospheric attenuation requires additional thermodynamic state variables. Details of the lengthy analysis this entails can be found in Appendix D. The result is simply that (3.16) becomes

$$\frac{\partial u}{\partial t} = -|\mathbf{K}| \frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} G(z,t) \, u(\xi + |\mathbf{K}|z,t) \, dz - b \, u \frac{\partial u}{\partial \xi},\tag{3.20}$$

i.e. the attenuation term in (3.16) becomes a more general linear operator. The function G(z) depends on the thermodynamic state (including composition) of the underlying atmosphere at time *t* and ray location $\mathbf{x} = \mathbf{X}(t)$, hence the time dependency, G(z, t), in (3.20). (3.16) can be recovered by taking $G(z) = \Delta \delta'(z)/2$, where δ' is the derivative of the Dirac function.

Using the identity $u \partial u/\partial \xi = \partial (u^2/2)/\partial \xi$, the Fourier transform of (3.20) with respect to ξ gives

$$\frac{\partial \tilde{u}}{\partial t} = -\Gamma(|\mathbf{K}|q,t)\,\tilde{u} - \frac{1}{2}ibq\,\tilde{u}^2,\tag{3.21}$$

where

$$\tilde{u}(q,t) = \int_{-\infty}^{\infty} u(\xi,t) \, e^{-iq\xi} d\xi \tag{3.22}$$

is the Fourier transform of u, the (complex) temporal attenuation coefficient is

$$\Gamma(Q,t) = iQ \int_{-\infty}^{\infty} G(z,t) e^{iQz} dz$$
(3.23)

and $Q = |\mathbf{K}|q$ is the acoustic wavenumber.

Because they only depend on the local thermodynamic state of the underlying atmosphere, G(z) or $\Gamma(Q)$ can be obtained by specialising to the case of one-dimensional propagation according to linear acoustics in a uniform, time-independent atmosphere at rest. This allows use of existing models from the literature (*e.g.* [18]) to obtain Γ as a function of wavenumber, Q. Since, rather than $\Gamma(Q)$, most such models give the spatial attenuation rate, $\alpha(\omega)$, as a function of frequency, ω , it is perhaps worth noting that $\Gamma(Q) = c_0 \alpha(c_0 Q)$ relates the complex temporal and spatial attenuation coefficients in the case of small attenuation/dispersion per wavelength considered here. Observe that allowing for relaxation implies a small degree of choice in the definition of c_0 : it could be either the equilibrium or frozen sound speed. The expression used for $\Gamma(Q)$ should be consistent with this choice. Note also that, like G(z), $\Gamma(Q)$ is implicitly dependent on the thermodynamic state (including composition) of the underlying atmosphere at position $\mathbf{x} = \mathbf{X}(t)$ and time t, hence the time dependence, $\Gamma(Q, t)$, in (3.21).

3.4. Caustics

A caustic occurs when $\nu = 0$, leading to infinite amplitude according to ray theory, which does not apply in the neighbourhood of the caustic. Fold caustics are the simplest type, taking the form of curves on the wavefront at which it has a sharp pleat. Higher-order caustics may occur at particular points, but only a small fraction of rays are so affected and we restrict ourselves to fold caustics here.

Passage through a fold caustic implies a change of sign of $\mathbf{n} \cdot (\mathbf{X}_{\beta} \times \mathbf{X}_{\gamma})$, which can be detected when integrating the ray equations. The zero occurs at the caustic passage time t_c . Assuming linear acoustics holds in the caustic region, the effect on

the waveform of propagation through the caustic is well known. The generalised Burgers equation is integrated up to time t_c to give $u_{in}(\xi) = u(\xi, t_c)$, the waveform entering the caustic. The waveform leaving the caustic is the Hilbert transform of u_{in} , *i.e.*

$$\tilde{u}_{out}(q) = -i\operatorname{sgn}(q)\,\tilde{u}_{in}(q),\tag{3.24}$$

and integration of the Burgers equation can proceed for $t > t_c$, starting with $u(\xi, t_c) = u_{out}(\xi)$. Of course, neither u_{in} nor u_{out} apply within the caustic region itself. Calculation of the waveform in this (small) region, would require the solution of the Tricomi equation which describes that region (see [19]).

Perhaps the most questionable aspect of (3.24) is the implied neglect of nonlinearity. This is particularly true if the incident waveform contains thin shocks. The Hilbert transform of a discontinuity yields a logarithmic singularity, while a thin shock approaches this case, leading to a large peak in u_{out} and hence enhanced nonlinearity. [20], and references therein, investigated the consequences of nonlinearity on passage through the caustic. However, by far the simplest approach uses (3.24).

v(t) goes to zero as $t \to t_c$. As for the emission time, the resulting singularity of the Burgers equation can be removed using a transformed time variable, T(t), such that $T(t) - T(t_c)$ behaves like $sgn(t - t_c)|t - t_c|^{1/2}$ as $t \to t_c$.

4. Illustrative results

Here, the model derived in Sections 2 and 3 is illustrated by results obtained for a ground-level explosion: the welldocumented test case, known as Misty Picture (see *e.g.* [10,11,21,22]), which was conducted in 1987 by the US Defense Nuclear Agency. Data on atmospheric properties are needed as inputs to the calculations. Although the model is capable of treating a general atmosphere, it is difficult to obtain realistic three-dimensional, unsteady atmospheric data (though this is, in principle, possible using outputs from a real-time meteorological model, which would need to give accurate results at high altitude). Thus, we restrict attention to a particular steady, horizontally stratified atmosphere based on Misty-Picture observations. The full 3D, unsteady model is nonetheless used and can, in principle, treat the general case. Fig. 2 shows the sound speed and wind-velocity components as functions of altitude. The model also requires the atmospheric composition and acoustic attenuation. To this end, we use the composition/attenuation model of [18]. Note that the vertical component of wind velocity is neglected, being supposed small compared to the others.

4.1. Rays and wavefronts

As noted above, we consider the Misty-Picture explosion. Meteorological observations yield the profiles shown in Fig. 2, which are used to compute the rays. Results for rays which propagate eastwards and westwards at the initial instant are shown in Fig. 3(a) and (b). These figures show altitude as a function of distance from the source. Here, and in subsequent figures, "Distance" is that of the point obtained by vertical projection onto the ground. It is apparent that rays are reflected in either the stratosphere or thermosphere back towards the ground, thus yielding long-range arrivals (the explosive equivalent of secondary sonic boom). Cusps in the wavefront are symptomatic of caustics. The white zone without rays is the so-called shadow region, in which geometrical acoustics predicts no arrival. It goes without saying that the differences between Fig. 3(a) and (b) are due to wind.

During the Misty-Picture test, the pressure signature of the explosion was measured at several locations on the ground. One of these, on which we focus, is known as White River and lies 309 km west of the explosion. Simulation of the pressure signature requires the determination of the rays (known as eigenrays) which pass through the given ground location. This was done numerically for White River and we found four arrivals, illustrated in Fig. 4(a). Two of these rays are reflected in the thermosphere and are denoted *Ita* and *Itb*, while the other two undergo reflection in the stratosphere and are labelled *Isa* and *Isb*. Fig. 4(b) plots the ratio u/p', obtained from Eq. (3.18), of the waveform u to the acoustic pressure it represents. The ratio varies by six orders of magnitude along ray *Itb*, mainly due to the much smaller atmospheric density at high altitude, but also the effects of geometric spreading. Fig. 4(c) shows the nonlinear coefficient, b, which appears in the generalised Burgers equation and is given by Eq. (3.17). Geometric spreading initially causes b to decrease, but it then tends to increase with altitude due to the decreasing density, thus promoting nonlinearity. The spikes in Fig. 4(b) and (c) are due to caustics of *Ita* and *Isa*. Note that the caustics occur close to the maximum altitude for the given ray. Finally, Fig. 4(d) shows the attenuation coefficient $\Gamma_r(Q)$ (the subscript r denotes the real part) divided by Q^2 for the frequencies 0.1 Hz and 1 Hz. This ratio would have the same value at all frequencies if attenuation were purely thermoviscous. Significant differences between the two frequencies appear at altitudes around 40 km because the relaxation frequencies of O_2 and CO_2 lie between 0.1 Hz and 1 Hz. Elsewhere, the attenuation coefficient has the classical frequency dependence, like ω^2 , and tends to increase with altitude.

4.2. Waveform propagation

As discussed in Section 3.2, the waveform described by the generalised Burgers equation is initialised by matching to the source. Near the source, the overpressure due to the explosion is much greater than the atmospheric pressure and the Sedov (see e.g. [17]) blast-wave solution applies (note that the presence of the ground doubles the effective energy of the



Fig. 2. Atmospheric sound- and wind-velocity profiles. Profiles of both the zonal (west to east) and meridional (south to north) components of wind velocity are shown.



Fig. 3. Trajectories of rays launched towards: (a) the east, (b) the west. The corresponding wavefronts are also shown every 100s up to 800s and at 1100, 1400 and 1700s. Caustics are indicated by dashed curves.

explosion). This regime is very far from the weak nonlinearity assumed by the model and matching must be carried out sufficiently far from the source. According to the Sedov solution, the overpressure becomes comparable to the atmospheric



Fig. 4. (a) Eigenrays for White River, (b) plots of the ratio u/p' along the rays, (c) plots of the nonlinear coefficient, *b*, and (d) plots of $\Gamma_r(Q)/Q^2$ for frequencies 0.1 Hz and 1 Hz (the latter being represented by dotted lines).

pressure, p_0 , at distances of order $(E/p_0)^{1/3}$, often referred to as the blast radius, where *E* is the energy released by the explosion. The Sedov solution no longer holds at or beyond such distances. At distances large compared with the blast radius, the pressure perturbation due to the source becomes small compared to the atmospheric pressure, so matching can be applied. For the Misty-Picture explosion, the blast radius is of order 500 m and we used a matching distance of 2 km. At this distance, we estimate the pressure perturbation as 10⁴ Pa, about one tenth of the atmospheric pressure. Having decided on the distance for matching, an initial waveform is needed. We used an empirical expression for p'(t) suggested by [23].

Having integrated the generalised Burgers equation numerically along eigenrays, Fig. 5 shows some results. The maximum value of $|u|(\xi)$, denoted u_{\star} , is plotted in Fig. 5(a). A frequency characterising the dominant spectral content of the acoustic arrival can be defined as $f_{\star} = q_{\star} \mathbf{w} \cdot \mathbf{K}/2\pi$, where

$$q_{\star} = \frac{\int_{0}^{\infty} q |\tilde{u}(q)|^{4} dq}{\int_{0}^{\infty} |\tilde{u}(q)|^{4} dq}.$$
(4.1)

Thus, f_{\star}^{-1} is a measure of the duration of the arrival. Fig. 5(b) shows a plot of f_{\star} . Comparison of Fig. 5(a) and (b) indicates strong similarity. The decrease of f_{\star} represents a lengthening of the acoustic arrival with increasing propagation distance. This is a nonlinear effect that is reflected in the decrease of u_{\star} . The relative importance of nonlinearity and attenuation can be measured by the Gol'dberg number

$$R = \frac{bq_{\star}u_{\star}}{\Gamma_{r}(q_{\star}|\mathbf{K}|)},\tag{4.2}$$

which is plotted in Fig. 5(c). It is apparent that R > 1, indicating that nonlinearity is more important than attenuation for the present problem. The large values of R over most of the range suggest that nonlinear waveform steepening yields thin shocks in which a balance is struck between nonlinearity and attenuation, leading to dissipation of acoustic energy. Finally, Fig. 5(d) plots the maximum |p'| (denoted p'_{\star}) divided by the atmospheric pressure. The ratio is small, indicating weak nonlinearity (as required by the model), apart from the ray *Ita* near the caustic. This raises the question, posed earlier, concerning the validity of the linear relation, (3.24), used to describe passage through a caustic.

Fig. 6 shows plots of the acoustic arrival, represented by u and its energy spectral density, at different points along the ray *Ita*. Position on the ray is represented by the time, t_a , taken for $\mathbf{X}(t)$ to reach the given point, starting from the source at t = 0 (the time of the explosion). Fig. 6(a) and (b) show u and its energy spectral density at $t_a = 6.01$ s (the initialisation time), $t_a = 125$ s, $t_a = 500$ s and $t_a = 639.51$ s (the caustic passage time). It is apparent from Fig. 6(a) that, by $t_a = 125$ s, thin shocks have formed and the acoustic pulse has become an N-wave. The pulse lengthens and the maximum |u| decreases in accord with Fig. 5(a) and (b). Fig. 6(b) also shows the effects of pulse lengthening: the peak frequency moves to lower values. Pulse lengthening is mainly due to nonlinearity. Fig. 6(c) and (d) show $t_a = 639.51$ s just prior to passage through the caustic and just afterwards (the former being the same as the last of Fig. 5(a) and (b)). They also show $t_a = 685$ s and $t_a = 1258.9$ s (arrival at the ground). Recalling that the caustic occurs close to the maximum altitude, one can also think of Fig. 6(c) and (d) as corresponding to the descending phase of propagation. Caustic passage is modelled using the Hilbert transform, (3.24), a procedure which can be questioned, as discussed earlier. Comparing the post-caustic waveforms in Fig. 6(c), we see that





Fig. 6. Acoustic signature (*u* and its energy spectral density) at different propagation times, t_a , along the ray *lta*: (a) and (b): $t_a = 6.01$ s, $t_a = 125$ s, $t_a = 500$ s and $t_a = 639.51$ s, (c) and (d): $t_a = 639.51$ s, $t_a = 685$ s and $t_a = 1258.9$ s. Note the different scales used in (a) and (c). Apart from this, the largest t_a in figures (a) and (b) coincides with the smallest t_a in figures (c) and (d).

the initial evolution is quite rapid, but then slows down. Nonlinearity is relatively strong near the caustic, hence the rapid evolution, but weakens as the ray descends towards the ground. The energy spectral densities in Fig. 6(d) show little change during the downward propagation phase.

4.3. Ground arrivals and comparison with measurements

Fig. 7 shows measurements and model results. We use measured pressure signatures from three locations: Alpine, 232 km, White River, 309 km and Roosevelt, 416 km, west of the source.

Fig. 7(a) has reduced time on the vertical axis and distance from the source on the horizontal axis. By reduced time, we mean $t - d/c_{rep}$, where t = 0 is the explosion time, d is distance from the source, defined as before, and $c_{rep} = 340$ m/s is a representative sound speed. This reduces the dispersion of arrival times between different locations. Note that reduced time increases from the top of the figure to the bottom. Thus, for a given location, earlier arrivals appear towards the top and later ones towards the bottom. The wiggly lines in Fig. 7(a) show the observed pressure fluctuations at the three measurement



Fig. 7. Ground arrivals along a line through the source directed towards the west. (a) The vertical axis represents reduced time (defined in the text). The wiggly lines show the observed pressure fluctuations at three measurement stations. The curves represent the arrival times of rays according to the model. (b) Maximum |p'| as a function of distance. The circles, squares and triangles are measured values, whereas the curves are model results.

stations, whose locations are indicated by the labelling (AL, WR and RO) at the top of the figure. The curves represent the arrival times of rays according to the model.

AL lies in the shadow zone (see Fig. 3(b)), so ray theory predicts no arrivals. On the other hand, the measurements indicate that, in fact, two arrivals occur. This reflects the usual weakness of geometrical acoustics: that it cannot predict the acoustic signature inside shadow zones. However, as apparent in Fig. 3(b), the shadow boundary is located not far from AL. Thus, the observed arrivals can be interpreted as the result of "leak over" of rays into the shadow zone. The fact that the arrival times at AL are close to an extrapolation of the model results into the shadow zone lends credence to the idea of ray leakage. It also suggests an identification of the arrivals at AL, the first being stratospheric, the second thermospheric.

Turning attention to WR, as we saw earlier, the model indicates four arrivals, predicted to occur in the time-order *Is*, *Itb*, *Ita* according to Fig. 7(a) (note that *Isa* and *Isb* are predicted to arrive at almost the same time: here we use *Is* to refer to both of them). Of these, the measured and theoretical arrival times of *Is* and *Ita* are in reasonable accord, whereas *Itb* does not show up in the measurements. This can be attributed to its predicted relatively small amplitude at the ground (see Fig. 7(b)), which is due to it having propagated to higher altitude, where attenuation is greater.

At RO, the final arrival is in agreement with theory (*Ita*), but the two earlier ones raise questions. Although the first arrival is close to the predicted time for *Itb*, as noted above for WR we expect the ground amplitude of *Itb* to be so low that it would not be observed. The second arrival is also unexpected. It may be that these arrivals are due to "leakage" of stratospheric rays.

Fig. 7(b) compares the maximum |p'| according to measurements and theory. The curves are theoretical, while the symbols indicate experimental results (circles for stratospheric arrivals, squares for thermospheric ones and triangles for the first two arrivals at RO). For each arrival, the experimental p'_{\star} was obtained by maximising |p'| over a time window containing the arrival. These windows are indicated by the shaded rectangles in Fig. 7(a). To account for the presence of the ground (modelled as a hard wall), the pressure perturbation given by the model was doubled when calculating the theoretical values of p'_{\star} shown in Fig. 7(b). Once again, because the shadow zone is excluded by the model, there are no theoretical results for AL. Otherwise, the model results for *Ita* are in quite good agreement with the measurements, indicated by the squares. As before, *Itb* does not appear in the observations at WR. For RO, the two triangles near the *Itb* curve represent the unexpected arrivals discussed above. Finally, turning to *Isa* and *Isb* at WR, the model predicts a much higher amplitude for *Isa*, which we therefore expect to dominate. Comparing the theoretical result for *Isa* with experiment, the predicted p'_{\star} is seen to overestimate the experimental value. This may be due to scattering by small-scale atmospheric nonuniformities.

5. Conclusion

Beginning with the full equations of motion of a compressible fluid, we have used multiple-scale asymptotic analysis to derive a geometrical acoustics model for nonlinear propagation of impulsive waves in a general atmosphere (three dimensional, in-motion and time dependent). The approach used unifies the sonic boom and explosion problems and allows for arbitrary source motion in the case of a sonic boom. The ray equations appear as a solvability condition at first order, followed by a generalised Burgers equation at the next. To our knowledge this is the first time that the combination of ray theory and the Burgers equation has been derived using asymptotic analysis for such a general case.

In addition to deriving the model, results of its application to an explosive source have been given in Section 4. The model has also been employed to study sonic boom propagation, see *e.g.* [9,10,24–26].

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Appendix A. Partial derivatives of X and K

As seen in Section 2.4, rays, $\mathbf{X}(t)$, $\mathbf{K}(t)$, form a two-parameter family: $\mathbf{X}(t, \beta, \gamma)$, $\mathbf{K}(t, \beta, \gamma)$. (2.11) and (2.13) apply to any given ray, providing the partial derivatives of \mathbf{X} and \mathbf{K} with respect to t. Equations governing the partial derivatives with respect to the ray parameters are obtained using the method proposed by [7]. Taking the β -derivative of (2.11) and (2.13) and using $\mathbf{n} = \mathbf{K}/|\mathbf{K}|$ to express \mathbf{n} yields

$$\frac{dX_{\beta i}}{dt} = \left(\left.\frac{\partial v_{0i}}{\partial x_j}\right|_{\mathbf{x}=\mathbf{X}} + \left.\frac{\partial c_0}{\partial x_j}\right|_{\mathbf{x}=\mathbf{X}} n_i\right) X_{\beta j} + \frac{c_0}{|\mathbf{K}|} \left(K_{\beta i} - n_i n_j K_{\beta j}\right),\tag{A.1}$$

$$\frac{dK_{\beta i}}{dt} = -\left(\left.\frac{\partial v_{0j}}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}} + \left.\frac{\partial c_0}{\partial x_i}\right|_{\mathbf{x}=\mathbf{X}}n_j\right)K_{\beta j} - \left(\left.\frac{\partial^2 v_{0j}}{\partial x_i \partial x_k}\right|_{\mathbf{x}=\mathbf{X}} + \left.\frac{\partial^2 c_0}{\partial x_i \partial x_k}\right|_{\mathbf{x}=\mathbf{X}}n_j\right)K_j X_{\beta k},\tag{A.2}$$

where $\mathbf{X}_{\beta} = \partial \mathbf{X} / \partial \beta$, $\mathbf{K}_{\beta} = \partial \mathbf{K} / \partial \beta$. The same equations apply if β is replaced by γ .

Initial conditions for the partial derivatives can be obtained for the explosion and boom problems using (2.14). In the case of the explosion, the partial derivative of (2.14) with respect to γ yields

$$\mathbf{X}_{\gamma}(\tau) = 0, \qquad \mathbf{K}_{\gamma}(\tau) = \frac{\partial \mathbf{n}_{s}}{\partial \gamma}$$
(A.3)

and the same equations hold if γ is replaced by β .(A.3) also applies to the boom problem if the ray parameters are chosen as $\beta = \tau$ and γ , γ being any parameterisation of \mathbf{n}_s which respects the Mach cone. (2.11), (2.13), (2.14) yield the coefficients of Taylor's expansions up to linear terms of $\mathbf{X}(t)$ and $\mathbf{K}(t)$ about $t = \tau$. Taking partial derivatives of these expansions with respect to τ and setting $t = \tau$:

$$\mathbf{X}_{\tau}(\tau) = \mathbf{v}_{\mathrm{s}}(\tau) - \mathbf{v}_{\mathrm{0}}(\mathbf{x}_{\mathrm{s}},\tau) - c_{\mathrm{0}}(\mathbf{x}_{\mathrm{s}},\tau)\mathbf{n}_{\mathrm{s}},\tag{A.4}$$

$$K_{\tau i}(\tau) = \frac{\partial n_{si}}{\partial \tau} + \frac{\partial v_{0j}}{\partial x_i} n_{sj} + \frac{\partial c_0}{\partial x_i}$$
(A.5)

provide initial conditions for \mathbf{X}_{β} and \mathbf{K}_{β} (since we have here specialised to $\beta = \tau$). Note that the derivatives of v_{0j} and c_0 in Eq. (A.5) are to be evaluated at position $\mathbf{x} = \mathbf{x}_s(\tau)$ and time $t = \tau$.

Appendix B. Stress-tensor evaluation to second order

Closure of Eqs. (2.1) and (2.2) requires the Newtonian constitutive law

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \tag{B.1}$$

and the entropy equation

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = S,\tag{B.2}$$

where τ is the viscous stress tensor (not to be confused with the emission time, τ), given by

$$\tau_{ij} = \lambda D_{kk} \delta_{ij} + 2\mu D_{ij}, \quad D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \tag{B.3}$$

and

$$S = \frac{1}{\rho\theta} \left(\mathbf{D} : \boldsymbol{\tau} + \boldsymbol{\nabla} \cdot (\kappa \, \boldsymbol{\nabla} \theta) + \boldsymbol{q}_{\star} \right). \tag{B.4}$$

The pressure *p* and viscosity coefficients λ , μ appearing in (B.1), (B.3) are functions of the thermodynamic state of the air. As discussed in the main text, we suppose air to be thermodynamically simple, so that $p = p(\rho, s)$, with similar expressions for λ and μ . In (B.4), $\theta = \theta(\rho, s)$ is absolute temperature, $\kappa = \kappa(\rho, s)$ is thermal conductivity, and q_* represents volumetric heating by absorption of radiation. Subtracting the equation of the underlying atmosphere, (B.2) gives

$$\frac{\partial s'}{\partial t} + \mathbf{v}_0 \cdot \nabla s' = S' - \mathbf{v}' \cdot \nabla s_0 - \mathbf{v}' \cdot \nabla s'.$$
(B.5)

The expansions (2.6) are supplemented by similar ones for the pressure, entropy and temperature perturbations, p', s' and θ' . (B.3) gives

$$\tau_{ij} = \lambda_0 D_{kk} \delta_{ij} + 2\mu_0 D_{ij},\tag{B.6}$$

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_{0i}}{\partial x_i} + \frac{\partial v_{0j}}{\partial x_i} + \frac{\partial \Phi}{\partial x_j} \frac{\partial v'_{1i}}{\partial \eta} + \frac{\partial \Phi}{\partial x_i} \frac{\partial v'_{1j}}{\partial \eta} \right)$$
(B.7)

at leading order, while the heat conduction term in (B.4) has the leading-order expression

$$\nabla \cdot (\kappa \nabla \theta) = \epsilon^{-1} \kappa_0 |\nabla \Phi|^2 \frac{\partial^2 \theta_1'}{\partial \eta^2}.$$
(B.8)

As discussed in the main text, it is assumed that λ_0 , μ_0 and κ_0 are of $\mathcal{O}(\epsilon^2)$. It follows from (B.6)–(B.8) that the viscous and heat conduction terms in (B.4) are respectively $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon)$. Heating by radiation absorption is important in determining the state of the underlying atmosphere, but is here assumed negligible as regards acoustic propagation. Thus, S' has the asymptotic expansion

$$S' = \epsilon S'_1 + \cdots, \tag{B.9}$$

where

$$S_1' = \epsilon^{-2} \frac{\kappa_0}{\rho_0 \theta_0} |\nabla \Phi|^2 \frac{\partial^2 \theta_1'}{\partial \eta^2}.$$
(B.10)

Using (B.9), (B.5) yields

$$\left(\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi\right) \frac{\partial s_1'}{\partial \eta} = 0 \tag{B.11}$$

at $\mathcal{O}(\epsilon^0)$. One solution of (B.11) is

$$\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi = 0, \tag{B.12}$$

which contradicts (2.9), the eikonal equation of acoustic waves. (B.12) is, in fact, the eikonal equation of the two non-acoustic modes supported by the Navier-Stokes equations, namely entropy and vorticity "waves" (quotes are used here because these modes are carried by the flow, rather than propagating with respect to the fluid). Such modes are not the subject of this paper and we specialise to the acoustic mode from here on by adopting the other solution of (B.11), namely $\partial s'_1 / \partial \eta = 0$. Arguing, as following (2.10), that the perturbation is zero at $\eta \to +\infty$ implies $s'_1 = 0$.

Subtracting the equation of the underlying atmosphere, (B.1) gives

$$\sigma' = -p'\mathbf{I} + \tau',\tag{B.13}$$

while (B.6) and (B.7) imply

$$\tau_{ij}' = \lambda_0 \frac{\partial \Phi}{\partial x_i} \frac{\partial v_{1i}'}{\partial \eta} \delta_{ij} + \mu_0 \left(\frac{\partial \Phi}{\partial x_j} \frac{\partial v_{1i}'}{\partial \eta} + \frac{\partial \Phi}{\partial x_i} \frac{\partial v_{1j}'}{\partial \eta} \right)$$
(B.14)

at leading order. Because the viscosity coefficients are $\mathcal{O}(\epsilon^2)$, so is the viscous term in (B.13). Thus, (B.13) gives $\sigma'_1 = -p'_1 \mathbf{I}$. Since $s'_1 = 0$, Taylor's expansion of $p(\rho, s)$ about ρ_0, s_0 implies $p'_1 = c_0^2 \rho'_1$. This justifies the usual linear acoustics relation $\sigma'_1 = -c_0^2 \rho'_1 \mathbf{I}$, introduced following Eqs (2.7), (2.8). Employing Taylor's expansion of $\theta(\rho, s)$ about ρ_0, s_0 and $s'_1 = 0$, (B.10) leads to

$$S_1' = \epsilon^{-2} \frac{\kappa_0}{\rho_0 \theta_0} \left. \frac{\partial \theta}{\partial \rho} \right|_0 |\nabla \Phi|^2 \left. \frac{\partial^2 \rho_1'}{\partial \eta^2} \right|_0 (B.15)$$

Using (2.9), $s'_1 = 0$, $\mathbf{v}'_1 = c_0 \rho'_1 \mathbf{n} / \rho_0$ and $\mathbf{n} = \nabla \Phi / |\nabla \Phi|$, (B.5) gives

$$|\nabla \Phi| \frac{\partial s_2'}{\partial \eta} = (\mathbf{n} \cdot \nabla s_0) \frac{\rho_1'}{\rho_0} - \frac{S_1'}{c_0}$$
(B.16)

at $\mathcal{O}(\epsilon^1)$. Taylor's expansion of $p(\rho, s)$ about ρ_0, s_0 implies

$$p_{2}' = c_{0}^{2} \left(\rho_{2}' + \frac{B}{2A\rho_{0}} {\rho_{1}'}^{2} \right) + \Pi,$$
(B.17)

while $\mathbf{v}'_1 = c_0 \rho'_1 \mathbf{n} / \rho_0$ and (B.14) give

$$\tau_{ij}' = \frac{c_0}{\rho_0} \left(\lambda_0 \delta_{ij} + 2\mu_0 n_i n_j \right) |\nabla \Phi| \frac{\partial \rho_1'}{\partial \eta}$$
(B.18)

at $\mathcal{O}(\epsilon^2)$. Finally, using (B.17) and (B.18) in (B.13) yields (3.1).

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Appendix C. Derivation of equation (3.11)

This appendix is devoted to a derivation of (3.11). To this end, we temporarily lift the restriction to rays coming from the source, imposed from just after (2.12) onwards.

Consider a volume, V(t), consisting at all times of the same rays. According to (2.11), rays move with velocity $\mathbf{w}(\mathbf{x}, t)$, thus the volume of V(t) evolves according to

$$\frac{d}{dt} \int_{V(t)} dv = \int_{V(t)} \nabla \cdot \mathbf{w} \, dv. \tag{C.1}$$

This relation follows from Reynolds' theorem and is well known in the context of fluid mechanics, where the analogue of $\mathbf{w}(\mathbf{x}, t)$ is the fluid velocity.

Let S(t) be part of the wavefront surface $\Phi = 0$ consisting always of the same rays. Define *V* at some initial time by infinitesimally thickening *S* on both sides of $\Phi = 0$ out to $\Phi = \pm \delta/2$ (thus $\delta > 0$ is infinitesimal), then let S(t) and V(t) move with the rays. Since Φ is constant following a ray, *V* remains an infinitesimally thickened version of *S*, bounded by $\Phi = \pm \delta/2$. Taylor's expansion of Φ shows that *V* has thickness $\delta/|\mathbf{k}|$, hence

$$\int_{V(t)} dv = \delta \int_{\Sigma} \frac{v}{|\mathbf{k}|} \, d\beta d\gamma, \tag{C.2}$$

where β and γ are ray parameters, we have used the fact that the area element on the wavefront is $dS = \nu d\beta d\gamma$ and the integral is over the part, Σ , of the $\beta - \gamma$ plane corresponding to S(t). Likewise, the integral on the right-hand side of (C.1) is

$$\int_{V(t)} \nabla \cdot \mathbf{w} \, dv = \delta \int_{\Sigma} \frac{\nu \nabla \cdot \mathbf{w}}{|\mathbf{k}|} \, d\beta d\gamma.$$
(C.3)

From here on, there is no need to consider rays other than those coming from the source and we again restrict attention to such rays.

Since rays have constant values of β and γ and S(t) is made up of rays, Σ is independent of t. Using (C.2) and (C.3) in (C.1), the time derivative can be taken inside the integral on the left-hand side, where it becomes a partial derivative at constant β and γ . This is equivalent to the derivative following a ray, hence

$$\int_{\Sigma} \left(\frac{d}{dt} \left(\frac{\nu}{|\mathbf{k}|} \right) - \frac{\nu \nabla \cdot \mathbf{w}}{|\mathbf{k}|} \right) d\beta d\gamma = 0$$
(C.4)

for any choice of Σ . Taking the limit in which Σ shrinks down to a point,

$$\frac{d}{dt}\left(\frac{\nu}{|\mathbf{k}|}\right) = \frac{\nu\nabla\cdot\mathbf{w}}{|\mathbf{k}|}.$$
(C.5)

Given $|\mathbf{k}| = |\mathbf{k}_s| |\mathbf{K}|$ and constancy of $|\mathbf{k}_s|$ for a ray,

$$\frac{d}{dt}\left(\frac{\nu}{|\mathbf{K}|}\right) = \frac{\nu \nabla \cdot \mathbf{w}}{|\mathbf{K}|}.$$
(C.6)

Using $\mathbf{w} = \mathbf{v}_0 + c_0 \mathbf{n}$, we conclude that

$$\frac{|\mathbf{K}|}{\nu}\frac{d}{dt}\left(\frac{\nu}{|\mathbf{K}|}\right) = \nabla \cdot \mathbf{v}_0 + \nabla \cdot (c_0 \mathbf{n}).$$
(C.7)

Multiplying (2.13) by K_i gives

$$\frac{1}{2}\frac{d}{dt}\left(\left|\mathbf{K}\right|^{2}\right) = -\mathbf{K}\cdot\left(\mathbf{K}\cdot\nabla\mathbf{v}_{0}\right) - \left|\mathbf{K}\right|\mathbf{K}\cdot\nabla c_{0}.$$
(C.8)

Dividing by $|\mathbf{K}|^2$,

$$\frac{1}{2|\mathbf{K}|^2}\frac{d|\mathbf{K}|^2}{dt} = -\mathbf{n}\cdot(\mathbf{n}\cdot\nabla\mathbf{v}_0) - \mathbf{n}\cdot\nabla c_0.$$
(C.9)

Finally, subtracting (C.9) from (C.7) and using the identities

$$\frac{|\mathbf{K}|}{\nu}\frac{d}{dt}\left(\frac{\nu}{|\mathbf{K}|}\right) - \frac{1}{2|\mathbf{K}|^2}\frac{d|\mathbf{K}|^2}{dt} = \frac{2|\mathbf{K}|}{\nu^{1/2}}\frac{d}{dt}\left(\frac{\nu^{1/2}}{|\mathbf{K}|}\right)$$
(C.10)

and

$$\nabla \cdot (c_0 \mathbf{n}) + \mathbf{n} \cdot \nabla c_0 = \frac{1}{c_0} \nabla \cdot (c_0^2 \mathbf{n})$$
(C.11)

yields (3.11).

Appendix D. Inclusion of relaxation and variable composition

Section 2 is unaffected by relaxation and variable composition, but Section 3 and Appendix B are significantly modified, as described below.

D.1. Stress-tensor evaluation to second order

The fluid is a mixture of different molecular species, counting excited states as separate species, and the mass fraction of species α is denoted Y_{α} . These additional thermodynamic variables have associated evolution equations

$$\frac{\partial Y_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla Y_{\alpha} = Z_{\alpha},\tag{D.1}$$

where

$$Z_{\alpha} = R_{\alpha} - \frac{1}{\rho} \nabla \cdot \mathbf{J}_{\alpha}, \tag{D.2}$$

 R_{α} represents production of species α by relaxation processes and \mathbf{J}_{α} its diffusive mass-flux vector. Note that, since the sum of the Y_{α} is 1, one of them is redundant and is dropped in what follows. Given that unexcited molecular nitrogen is the dominant species in the atmosphere, it is the natural choice for elimination. The thermodynamic state variables are ρ , s, Y_{α} , all other thermodynamic quantities (*e.g.* pressure and temperature), the coefficients of viscosity and R_{α} being functions of these variables. Subtracting the equation of the underlying atmosphere, (D.1) gives

$$\frac{\partial Y'_{\alpha}}{\partial t} + \mathbf{v}_0 \cdot \nabla Y'_{\alpha} = Z'_{\alpha} - \mathbf{v}' \cdot \nabla Y_{\alpha 0} - \mathbf{v}' \cdot \nabla Y'_{\alpha}. \tag{D.3}$$

The entropy equation still has the form (B.2), but (B.4) is replaced by

$$S = \frac{1}{\rho\theta} \left(\mathbf{D} : \boldsymbol{\tau} - \boldsymbol{\nabla} \cdot \mathbf{q} + q_{\star} \right) - \sum_{\alpha} \chi_{\alpha} Z_{\alpha}, \tag{D.4}$$

where **q** is the heat-flux vector and χ_{α} is the difference of chemical potentials (per unit mass) between species α and the species eliminated earlier, divided by the absolute temperature, θ . Subtraction of the entropy equation of the underlying atmosphere gives

$$\frac{\partial s'}{\partial t} + \mathbf{v}_0 \cdot \nabla s' = S' - \mathbf{v}' \cdot \nabla s_0 - \mathbf{v}' \cdot \nabla s'$$
(D.5)

as before.

The generalised Fourier-Fick laws:

$$\mathbf{q} = -\kappa \nabla \theta - \kappa_s \nabla s - \sum_{\alpha} \kappa_{\alpha} \nabla Y_{\alpha}, \tag{D.6}$$

$$\mathbf{J}_{\alpha} = -\zeta_{\alpha} \nabla \rho - \zeta_{\alpha}^{s} \nabla s - \sum_{\beta} \zeta_{\alpha\beta} \nabla Y_{\beta}$$
(D.7)

describe the heat- and mass-flux vectors appearing in (D.2) and (D.4). It perhaps goes without saying that χ_{α} in (D.4), and the coefficients in (D.6) and (D.7), are functions of the thermodynamic state variables. In keeping with the assumptions concerning the coefficients of viscosity and heat conduction made in Appendix B, we suppose the coefficients in (D.6) and (D.7) of $\mathcal{O}(\epsilon^2)$.

The expansions (2.6) are supplemented by similar ones for s', θ' and Y'_{α} . (D.2), (D.7) and Taylor's expansion of $R_{\alpha}(\rho, s, Y_{\beta})$ yield

$$Z'_{\alpha} = \left. \frac{\partial R_{\alpha}}{\partial \rho} \right|_{0} \rho' + \left. \frac{\partial R_{\alpha}}{\partial s} \right|_{0} s' + \sum_{\beta} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_{0} Y'_{\beta} + \epsilon^{-2} \frac{1}{\rho_{0}} \left| \nabla \Phi \right|^{2} \frac{\partial^{2}}{\partial \eta^{2}} \left(\zeta_{\alpha 0} \rho' + \zeta_{\alpha 0}^{s} s' + \sum_{\beta} \zeta_{\alpha \beta 0} Y'_{\beta} \right)$$
(D.8)

at leading order, where the derivatives with subscript 0 imply that the quantity being differentiated is a function of ρ , s, Y_{α} . The derivative is taken with respect to the indicated variable, while holding all others constant. As indicated by the subscript 0, the result is then evaluated at $\rho = \rho_0$, $s = s_0$ and $Y_{\alpha} = Y_{\alpha 0}$. (D.8) is $\mathcal{O}(\epsilon)$, hence (D.3) implies

$$\left(\frac{\partial \Phi}{\partial t} + \mathbf{v}_0 \cdot \nabla \Phi\right) \frac{\partial Y'_{\alpha 1}}{\partial \eta} = \mathbf{0}.$$
(D.9)

Arguing as we did following (B.11), we conclude that $Y'_{\alpha 1} = 0$. As we saw in Appendix B, $s'_1 = 0$ for a thermodynamically simple fluid and this remains the case here. Taylor's expansion of $p(\rho, s, Y_{\alpha})$ implies that $\sigma'_1 = -c_0^2 \rho'_1 \mathbf{I}$ as before. Note that $c_0^2 = \partial p/\partial \rho|_0$, where the derivative is taken at constant *s* and Y_{α} , *i.e.* c_0 is the frozen sound speed.

Since $Y'_{\beta 1} = 0$, the term $\sum_{\beta} \partial R_{\alpha} / \partial Y_{\beta} |_{0} Y'_{\beta}$ in (D.8) appears to be of $\mathcal{O}(\epsilon^{2})$, and thus negligible at $\mathcal{O}(\epsilon)$. However, this term is essential for relaxation. To ensure its contribution to the Burgers equation, we take $\partial R_{\alpha} / \partial Y_{\beta} |_{0} = \mathcal{O}(\epsilon^{-1})$ in what follows. This additional bookkeeping assumption is needed so that relaxation is effective during the short passage time, $\mathcal{O}(\epsilon)$, of the acoustic perturbation.

 $\partial R_{\alpha}/\partial Y_{\beta}|_{0} = \mathcal{O}(\epsilon^{-1})$ means that relaxation is rapid on the atmospheric time scale, hence the underlying atmosphere is always close to relaxational equilibrium. To make this statement more precise, consider a fluid particle, $\mathbf{x}(t)$, which follows the underlying flow and is located at \mathbf{x}_{0} at time t_{0} , *i.e.* $d\mathbf{x}/dt = \mathbf{v}_{0}$, $\mathbf{x}(t_{0}) = \mathbf{x}_{0}$. Applying (D.1) and (D.2) to the underlying atmosphere,

$$\frac{dY_{\alpha 0}}{dt} = R_{\alpha}(\rho_0(\mathbf{x}, t), s_0(\mathbf{x}, t), Y_{\beta 0}) - \frac{1}{\rho_0} \nabla \cdot \mathbf{J}_{\alpha 0}.$$
(D.10)

Given the time scale, $\mathcal{O}(\epsilon)$, for relaxation, we consider times $t = t_0 + \mathcal{O}(\epsilon)$. Taylor's expansion gives

$$R_{\alpha}(\rho_{0}(\mathbf{x},t),s_{0}(\mathbf{x},t),Y_{\beta0}) = R_{\alpha}(\rho_{0}(\mathbf{x}_{0},t_{0}),s_{0}(\mathbf{x}_{0},t_{0}),Y_{\beta0}) + \frac{\partial R_{\alpha}}{\partial \rho} \Big|_{0} \frac{d\rho_{0}}{dt}(t-t_{0}) + \frac{\partial R_{\alpha}}{\partial s} \Big|_{0} \frac{ds_{0}}{dt}(t-t_{0}),$$
(D.11)

in which the final two terms are $\mathcal{O}(\epsilon)$. According to (D.7), the diffusive term in (D.10) is $\mathcal{O}(\epsilon^2)$. If $R_{\alpha}(\rho_0(\mathbf{x}_0, t_0), s_0(\mathbf{x}_0, t_0), Y_{\beta 0})$ were asymptotically larger than $\mathcal{O}(\epsilon)$, it would dominate and (D.10) would give

$$\frac{dY_{\alpha 0}}{dt} = R_{\alpha}(\rho_0(\mathbf{x}_0, t_0), s_0(\mathbf{x}_0, t_0), Y_{\beta 0})$$
(D.12)

at leading order. This equation implies relaxation of $Y_{\alpha 0}$ to equilibrium on a time scale $\mathcal{O}(\epsilon)$. Thus, if it exceeded $\mathcal{O}(\epsilon)$ at $t = t_0$, $R_\alpha(\rho_0(\mathbf{x}_0, t_0), s_0(\mathbf{x}_0, t_0), Y_{\beta 0})$ would decrease rapidly until it reached $\mathcal{O}(\epsilon)$ and the remaining terms in (D.11) kick in. Since the underlying atmosphere has already had sufficient time to reach this state, we conclude that $R_{\alpha 0} = \mathcal{O}(\epsilon)$, an expression of closeness to equilibrium.

Relaxation processes can be considered as a special type of chemical reaction. R_{α} can be expressed as a sum over such reactions:

$$R_{\alpha} = \sum_{l} \omega^{l} R_{\alpha}^{l}, \tag{D.13}$$

where ω^l is the rate of reaction *l*, which depends on the thermodynamic state variables, and R^l_{α} are constant coefficients expressing stoichiometry. According to chemical thermodynamics, equilibrium leads to zero entropy production, *i.e.*

$$\sum_{\alpha} \chi_{\alpha} R_{\alpha}^{l} = 0 \tag{D.14}$$

for each reaction. Since the underlying atmosphere is close to equilibrium,

$$\sum_{\alpha} \chi_{\alpha 0} R_{\alpha}^{l} = o(1). \tag{D.15}$$

Note that $\partial R_{\alpha}/\partial Y_{\beta}|_{0} = \mathcal{O}(\epsilon^{-1})$ arises from $\partial \omega_{l}/\partial Y_{\beta}|_{0} = \mathcal{O}(\epsilon^{-1})$.

Using $s'_{1} = Y'_{\alpha 1} = 0$, (D.8) implies

$$Z_{\alpha} = \epsilon Z_{\alpha 1} + \cdots, \tag{D.16}$$

where

$$Z'_{\alpha 1} = \left. \frac{\partial R_{\alpha}}{\partial \rho} \right|_{0} \rho'_{1} + \epsilon \sum_{\beta} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_{0} Y'_{\beta 2} + \epsilon^{-2} \frac{\zeta_{\alpha 0}}{\rho_{0}} \left| \nabla \Phi \right|^{2} \frac{\partial^{2} \rho'_{1}}{\partial \eta^{2}}. \tag{D.17}$$

Employing (2.9) and $\mathbf{v}'_1 = c_0 \rho'_1 \mathbf{n} / \rho_0$, (D.3) gives

$$|\nabla \Phi| \frac{\partial Y'_{\alpha 2}}{\partial \eta} = (\mathbf{n} \cdot \nabla Y_{\alpha 0}) \frac{\rho'_1}{\rho_0} - \frac{Z'_{\alpha 1}}{c_0}$$
(D.18)

as equation for $Y'_{\alpha 2}$.

The final term in (D.4) contributes

$$-\left(\sum_{\alpha}\chi_{\alpha}Z_{\alpha}\right)' = -\sum_{\alpha}\chi_{\alpha0}Z_{\alpha}' - \sum_{\alpha}\chi_{\alpha}'Z_{\alpha0} - \sum_{\alpha}\chi_{\alpha}'Z_{\alpha}'$$
(D.19)

to S'. Using (D.17),

$$\sum_{\alpha} \chi_{\alpha 0} Z'_{\alpha} = \epsilon \sum_{\alpha} \chi_{\alpha 0} \left(\frac{\partial R_{\alpha}}{\partial \rho} \bigg|_{0} \rho'_{1} + \epsilon \sum_{\beta} \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \bigg|_{0} Y'_{\beta 2} + \epsilon^{-2} \frac{\zeta_{\alpha 0}}{\rho_{0}} |\nabla \Phi|^{2} \frac{\partial^{2} \rho'_{1}}{\partial \eta^{2}} \right)$$
(D.20)

at leading order. Differentiating (D.13),

$$\frac{\partial R_{\alpha}}{\partial \rho}\Big|_{0} = \sum_{l} \frac{\partial \omega^{l}}{\partial \rho}\Big|_{0} R_{\alpha}^{l}, \qquad \epsilon \frac{\partial R_{\alpha}}{\partial Y_{\beta}}\Big|_{0} = \epsilon \sum_{l} \frac{\partial \omega^{l}}{\partial Y_{\beta}}\Big|_{0} R_{\alpha}^{l}$$
(D.21)

and (D.15) implies that the corresponding terms in (D.20) are negligible once the sum over α is taken. It follows that the first term in (D.19) has the expansion

$$-\sum_{\alpha} \chi_{\alpha 0} Z'_{\alpha} = -\epsilon^{-1} \frac{1}{\rho_0} |\nabla \Phi|^2 \sum_{\alpha} \chi_{\alpha 0} \zeta_{\alpha 0} \frac{\partial^2 \rho'_1}{\partial \eta^2} + \cdots$$
(D.22)

and hence is $\mathcal{O}(\epsilon)$.

Applying (D.2) to the underlying atmosphere, (D.7) and $R_{\alpha 0} = \mathcal{O}(\epsilon)$ imply $Z_{\alpha 0} = \mathcal{O}(\epsilon)$, while Taylor's expansion of $\chi_{\alpha}(\rho, s, Y_{\beta})$ gives $\chi'_{\alpha} = \mathcal{O}(\epsilon)$. Thus, the second term in (D.19) is $\mathcal{O}(\epsilon^2)$, negligible compared with (D.22). The final term in (D.19) is also $\mathcal{O}(\epsilon^2)$, hence negligible, because $\chi'_{\alpha} = \mathcal{O}(\epsilon)$ and $Z'_{\alpha} = \mathcal{O}(\epsilon)$. Thus,

$$-\left(\sum_{\alpha}\chi_{\alpha}Z_{\alpha}\right)' = -\epsilon^{-1}\frac{1}{\rho_{0}}|\nabla\Phi|^{2}\sum_{\alpha}\chi_{\alpha0}\zeta_{\alpha0}\frac{\partial^{2}\rho_{1}'}{\partial\eta^{2}} + \cdots$$
(D.23)

(D.6) and $s'_1 = Y'_{\alpha 1} = 0$ lead to

$$-\frac{1}{\rho\theta}\nabla\cdot\mathbf{q} = \epsilon^{-1}\frac{\kappa_0}{\rho_0\theta_0}|\nabla\Phi|^2\frac{\partial^2\theta_1'}{\partial\eta^2} + \cdots$$
(D.24)

for the heat-conduction term in (D.4). According to (B.6) and (B.7), the viscous term in (D.4) is $\mathcal{O}(\epsilon^2)$, small compared with (D.23) and (D.24). Neglecting the volumetric heating term as before, (D.4), (D.23), (D.24) and Taylor's expansion of $\theta(\rho, s, Y_{\alpha})$ lead to

$$S_{1}^{\prime} = \epsilon^{-2} \frac{1}{\rho_{0}} |\nabla \Phi|^{2} \left(\frac{\kappa_{0}}{\theta_{0}} \left. \frac{\partial \theta}{\partial \rho} \right|_{0} - \sum_{\alpha} \chi_{\alpha 0} \zeta_{\alpha 0} \right) \frac{\partial^{2} \rho_{1}^{\prime}}{\partial \eta^{2}}.$$
(D.25)

Using (2.9) and $\mathbf{v}'_{1} = c_{0}\rho'_{1}\mathbf{n}/\rho_{0}$, (D.5) gives

$$|\nabla \Phi| \frac{\partial s_2'}{\partial \eta} = (\mathbf{n} \cdot \nabla s_0) \frac{\rho_1'}{\rho_0} - \frac{S_1'}{c_0}.$$
 (D.26)

(D.17), (D.18), (D.25) and (D.26) provide governing equations for $Y'_{\alpha 2}$ and s'_2 . Finally, (3.1) remains valid, but the second of the Eqs. (3.2) becomes

$$\Pi = \frac{\partial p}{\partial s} \bigg|_{0}^{s_{2}} + \sum_{\alpha} \left. \frac{\partial p}{\partial Y_{\alpha}} \right|_{0}^{s_{2}} Y_{\alpha 2}^{\prime}. \tag{D.27}$$

D.2. Completion of the second-order analysis

Defining

$$\hat{Y}_{\alpha} = Y'_{\alpha 2} + \epsilon^{-2} \frac{\zeta_{\alpha 0}}{\rho_0 c_0} \left| \nabla \Phi \right| \frac{\partial \rho'_1}{\partial \eta} - a_{\alpha} \rho'_1 + \Xi \mathbf{n} \cdot \nabla Y_{\alpha 0}, \tag{D.28}$$

where

$$a_{\alpha} = \epsilon^{-2} \frac{1}{\rho_0 c_0} \sum_{\beta} r_{\alpha\beta} \zeta_{\beta 0}, \qquad r_{\alpha\beta} = \frac{\epsilon}{c_0} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_0, \tag{D.29}$$

$$\Xi = \frac{1}{\rho_0 |\nabla \Phi|} \int_{\eta}^{\infty} \rho_1' d\eta, \tag{D.30}$$

(D.17) and (D.18) give

$$|\nabla \Phi| \frac{\partial \hat{Y}_{\alpha}}{\partial \eta} + \sum_{\beta} r_{\alpha\beta} \hat{Y}_{\beta} = -d_{\alpha} \rho_1' + \frac{\epsilon}{c_0} \Xi \mathbf{n} \cdot \sum_{\beta} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_0 \nabla Y_{\beta 0}, \tag{D.31}$$

where

$$d_{\alpha} = \frac{1}{c_0} \left. \frac{\partial R_{\alpha}}{\partial \rho} \right|_0 + \sum_{\beta} r_{\alpha\beta} a_{\beta}.$$
(D.32)

As we saw earlier,

$$R_{\alpha}(\rho_0, s_0, Y_{\beta 0}) = \mathcal{O}(\epsilon), \tag{D.33}$$

whose gradient yields

$$\frac{\partial R_{\alpha}}{\partial \rho} \bigg|_{0} \nabla \rho_{0} + \frac{\partial R_{\alpha}}{\partial s} \bigg|_{0} \nabla s_{0} + \sum_{\beta} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_{0} \nabla Y_{\beta 0} = \mathcal{O}(\epsilon).$$
(D.34)

The first two terms in (D.34) are of $\mathcal{O}(1)$, hence

$$\sum_{\beta} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_{0} \nabla Y_{\beta 0} = \mathcal{O}(1). \tag{D.35}$$

Recalling the bookkeeping assumption $\partial R_{\alpha}/\partial Y_{\beta}|_{0} = \mathcal{O}(\epsilon^{-1})$, terms in the sum over β in (D.35) are individually of $\mathcal{O}(\epsilon^{-1})$, but this equation shows that they are nearly cancelling once the sum is taken. Using (D.35), (D.31) becomes

$$|\nabla \Phi| \frac{\partial \dot{Y}_{\alpha}}{\partial \eta} + \sum_{\beta} r_{\alpha\beta} \hat{Y}_{\beta} = -d_{\alpha} \rho_{1}^{\prime}.$$
(D.36)

Eq. (D.36) and the boundary condition $\hat{Y}_{\alpha} \to 0$ as $\eta \to +\infty$ determine \hat{Y}_{α} . Using (D.25)–(D.28), we find

$$|\nabla \Phi| \frac{\partial \Pi}{\partial \eta} = 2c_0 |\nabla \Phi| \frac{\partial}{\partial \eta} \left(\Lambda \rho_1' + \sum_{\alpha} e_{\alpha} \hat{Y}_{\alpha} \right)$$

$$- \epsilon^{-2} \frac{1}{\rho_0 c_0} |\nabla \Phi|^2 \left(\frac{\kappa_0}{\theta_0} \frac{\partial \theta}{\partial \rho} \Big|_0 \frac{\partial p}{\partial s} \Big|_0 + \sum_{\alpha} \zeta_{\alpha 0} \left(\frac{\partial p}{\partial Y_{\alpha}} \Big|_0 - \chi_{\alpha 0} \frac{\partial p}{\partial s} \Big|_0 \right) \right) \frac{\partial^2 \rho_1'}{\partial \eta^2}$$

$$+ \mathbf{n} \cdot \left(\frac{\partial p}{\partial s} \Big|_0 \nabla s_0 + \sum_{\alpha} \frac{\partial p}{\partial Y_{\alpha}} \Big|_0 \nabla Y_{\alpha 0} \right) \frac{\rho_1'}{\rho_0},$$
 (D.37)

where

$$\Lambda = \sum_{\alpha} e_{\alpha} a_{\alpha}, \quad e_{\alpha} = \frac{1}{2c_0} \left. \frac{\partial p}{\partial Y_{\alpha}} \right|_0. \tag{D.38}$$

Eqs. (3.3)–(3.6) hold as before, but with (D.37) for $\partial \Pi / \partial \eta$. The solvability condition, $f_1 + f_2 = 0$, gives

$$\left(\frac{\rho_0}{c_0}\right)^{1/2} \left(\frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla\right) \left(\left(\frac{c_0}{\rho_0}\right)^{1/2} \rho_1' \right) = \frac{1}{2} \epsilon^{-2} \Delta |\nabla \Phi|^2 \frac{\partial^2 \rho_1'}{\partial \eta^2} - \left(1 + \frac{B}{2A}\right) \frac{c_0}{\rho_0} |\nabla \Phi| \rho_1' \frac{\partial \rho_1'}{\partial \eta} - |\nabla \Phi| \frac{\partial}{\partial \eta} \left(\Delta \rho_1' + \sum_{\alpha} e_{\alpha} \hat{Y}_{\alpha} \right) - \frac{1}{2} \left(\frac{1}{c_0} \nabla \cdot (c_0^2 \mathbf{n}) + \nabla \cdot \mathbf{v}_0 + \mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{v}_0) \right) \rho_1',$$
(D.39)

where

$$\Delta = \frac{1}{\rho_0} \left\{ \lambda_0 + 2\mu_0 + \frac{1}{c_0^2} \left(\frac{\kappa_0}{\theta_0} \frac{\partial \theta}{\partial \rho} \Big|_0 \frac{\partial p}{\partial s} \Big|_0 + \sum_{\alpha} \zeta_{\alpha 0} \left(\frac{\partial p}{\partial Y_{\alpha}} \Big|_0 - \chi_{\alpha 0} \frac{\partial p}{\partial s} \Big|_0 \right) \right) \right\}$$
(D.40)

is the diffusivity of sound. Note that, in deriving (D.39), the following term has been dropped from the right-hand side:

$$-\frac{\rho_1'}{2\rho_0c_0}\mathbf{n}\cdot\left(\nabla\cdot\boldsymbol{\sigma}_0+c_0^2\nabla\rho_0+\left.\frac{\partial p}{\partial s}\right|_0\nabla s_0+\sum_{\alpha}\left.\frac{\partial p}{\partial Y_{\alpha}}\right|_0\nabla Y_{\alpha 0}\right).\tag{D.41}$$

As before, this is because the sum in brackets can be rewritten as $\nabla \cdot \tau_0$, which is $\mathcal{O}(\epsilon^2)$ and thus negligible. The second-order asymptotic analysis is now complete, the results being (D.36) and (D.39).

D.3. Generalised Burgers equation

In keeping with (3.12), we introduce

$$y_{\alpha}(\xi,t) = \frac{\epsilon^2}{|\mathbf{K}|} \left(\frac{\nu c_0}{\rho_0}\right)^{1/2} \hat{Y}_{\alpha}(\eta,\mathbf{X}(t),t).$$
(D.42)

Eqs. (D.36) and (D.39) yield

$$|\mathbf{K}|\frac{\partial y_{\alpha}}{\partial \xi} + \sum_{\beta} \bar{r}_{\alpha\beta} y_{\beta} = -d_{\alpha} u \tag{D.43}$$

and

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta |\mathbf{K}|^2 \frac{\partial^2 u}{\partial \xi^2} - b \, u \frac{\partial u}{\partial \xi} - |\mathbf{K}| \frac{\partial}{\partial \xi} \left(\bar{\Lambda} u + \sum_{\alpha} e_{\alpha} y_{\alpha} \right), \tag{D.44}$$

where

$$\bar{r}_{\alpha\beta} = \epsilon^{-1} r_{\alpha\beta} = \frac{1}{c_0} \left. \frac{\partial R_{\alpha}}{\partial Y_{\beta}} \right|_0, \tag{D.45}$$

$$\bar{\Lambda} = \epsilon \Lambda = \sum_{\alpha} e_{\alpha} \bar{a}_{\alpha}, \tag{D.46}$$

$$\bar{a}_{\alpha} = \epsilon a_{\alpha} = \frac{1}{\rho_0 c_0} \sum_{\beta} \bar{r}_{\alpha\beta} \zeta_{\beta 0} \tag{D.47}$$

remove the ϵ -dependency.

The solution of (D.43) can be expressed using a Green's function, $g_{\alpha}(z)$ being the solution of

$$\frac{dg_{\alpha}}{dz} - \sum_{\beta} \bar{r}_{\alpha\beta} g_{\beta} = d_{\alpha} \delta(z), \tag{D.48}$$

where $g_{\alpha}(z) = 0$ for z < 0. The solution of (D.43) which goes to zero as $\xi \to +\infty$ is

$$y_{\alpha}(\xi) = \int_{-\infty}^{+\infty} g_{\alpha}(z)u(\xi + |\mathbf{K}|z)dz.$$
(D.49)

This result is used in (D.44) to obtain (3.20), where

$$G(z) = \frac{1}{2}\Delta\delta'(z) + \bar{\Lambda}\delta(z) + \sum_{\alpha} e_{\alpha}g_{\alpha}(z)$$
(D.50)

depends only on the thermodynamic state of the underlying atmosphere at position $\mathbf{x} = \mathbf{X}(t)$ and time *t*.

We would not, of course, advocate actually calculating G(z) using the above procedure, it being much simpler to adopt existing models from the literature, as described in Section 3.3. The important point is not the detailed expression for G(z), but that: (a) the attenuation term has the form given in (3.20), and (b) G(z) depends only on the thermodynamic state of the underlying atmosphere at $\mathbf{x} = \mathbf{X}(t)$ and time t. These results are not *a priori* obvious, at least not to us. In particular, nonuniformity of the underlying atmosphere appears in the analysis via $\nabla \cdot \sigma_0$, $\nabla \rho_0$, ∇s_0 and $\nabla Y_{\alpha 0}$, but does not contribute to the Burgers equation, thanks to (D.35) and smallness of (D.41).

Finally, adding a multiple of $\delta(z)$ to G(z) (as does the term containing \overline{A} in (D.50)) is equivalent to an $\mathcal{O}(\epsilon)$ change in the sound speed. This allows a limited degree of choice in c_0 . Although the frozen sound speed is the natural one for asymptotic analysis, \overline{A} can be incorporated into the definition of c_0 . The equilibrium sound speed (with or without incorporation of \overline{A}) can also be used. For each such choice of c_0 , a consistent expression for G(z) or $\Gamma(Q)$ should be employed in the Burgers equation or its Fourier transform.

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