# Statistics of peak overpressure and shock steepness for linear and nonlinear N-wave propagation in a kinematic turbulence

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(Received 19 May 2017; revised 5 October 2017; accepted 11 November 2017; published online 5 December 2017)

Linear and nonlinear propagation of high amplitude acoustic pulses through a turbulent layer in air is investigated using a two-dimensional KZK-type (Khokhlov–Zabolotskaya–Kuznetsov) equation. Initial waves are symmetrical *N*-waves with shock fronts of finite width. A modified von Kármán spectrum model is used to generate random wind velocity fluctuations associated with the turbulence. Physical parameters in simulations correspond to previous laboratory scale experiments where *N*-waves with 1.4 cm wavelength propagated through a turbulence layer with the outer scale of about 16 cm. Mean value and standard deviation of peak overpressure and shock steepness, as well as cumulative probabilities to observe amplified peak overpressure level in random foci for moderate initial amplitudes of *N*-waves thus increasing the probability to observe highly peaked waveforms. Saturation of the pressure level is observed for stronger nonlinear effects. It is shown that in the linear propagation regime, the turbulence mainly leads to the smearing of shock fronts, thus decreasing the probability to observe high values of steepness, whereas nonlinear effects dramatically increase the probability to observe steep shocks. © *2017 Acoustical Society of America*. https://doi.org/10.1121/1.5015991

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# I. INTRODUCTION

Nonlinear propagation of high amplitude acoustic waves through inhomogeneous media is an important problem for modern theoretical and applied acoustics.<sup>1</sup> Most research activity on high amplitude wave propagation in turbulent atmosphere was motivated by the sonic boom problem.<sup>2-6</sup> Sonic booms are generated during supersonic passage of an aircraft through the atmosphere.<sup>7</sup> The sonic booms heard on the ground are perceived by people as highly annoying noise.<sup>8–10</sup> This annoyance presents a major obstacle that prohibits overland supersonic flights. Perception studies suggest that both the perceived loudness and the annoyance of the sonic boom tend to increase with the increase of its peak overpressure and decrease of the shock rise time.<sup>11</sup> The rise time is classically defined as the time required for the acoustic pressure to increase from 10% to 90% of the peak overpressure [see Fig. 1(a)]. For N-shaped sonic booms, typical peak overpressure and rise time values at the ground are 50-100 Pa and 1-5 ms, respectively.<sup>6</sup>

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Although the sonic boom problem has been extensively explored since the early 1960s, recent efforts to develop small supersonic business jets (SSBJ) and low boom airplane concepts spurred interest in this research.<sup>12–14</sup> Due to advances in computational fluid dynamics, considerable progress has been achieved in predicting of the near-field and mid-field sonic booms.<sup>15</sup> However, certain effects related to the propagation of shock booms in unsteady atmosphere are still needed to be investigated. Note that SSBJ are designed to generate "low-boom" signatures, which are supposed to be quieter than classical *N*-waves.<sup>8</sup> However, it is still reasonable to use classical *N*-waves to investigate atmospheric propagation effects on acoustic shock waves since the results are simpler for interpretation and can be generalized to other cases with different wave signatures.

While propagating in the atmosphere from flight altitude (typically 10 km) toward the ground, a sonic boom wave is affected by numerous physical processes such as geometrical wavefront spreading, refraction, scattering, thermoviscous absorption, molecular relaxation, and nonlinear propagation effects. Appearance of classical *N*-shaped waves in a far-field is attributed to nonlinear effects.<sup>3</sup> In a quite atmosphere, the balance between molecular relaxation and nonlinear effects

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FIG. 1. (a) *N*-wave pressure signature with classical definition of shock rise time  $\tau_{sh}$  and (b) sketch of the propagation problem considered in the current paper.

results in sonic booms rise times on the order of one millisecond.<sup>16,17</sup> When an *N*-wave enters the lowest part of the atmosphere spanning the first 1–2 km above the ground [Planetary Boundary Layer (PBL)], its propagation can be affected by the turbulence. Variability of sonic boom signatures in terms of their shape, peak overpressure, and shock front rise time was reported since the earliest flight tests.<sup>2,18–20</sup> The rise times in the turbulent atmosphere were reported to be much larger than those predicted by the relaxation theory.<sup>21</sup>

It is now generally accepted that wind velocity turbulent fluctuations and sound speed thermal inhomogeneities in the PBL are the main factors which cause random distortions of the sonic boom signatures.<sup>5</sup> Several early analytical theories attempted to explain the large variability of the rise time and the peak overpressure in terms of basic physical principles, such as refraction, scattering, diffraction on turbulence inhomogeneities, formation of random caustics, and wavefront folding.<sup>22–25</sup> For example, Pierce and Maglieri<sup>26</sup> argued that acoustic wave refraction on the sound speed or wind inhomogeneities in a boundary layer produces the wavefront rippling with different spatial scales.<sup>26</sup> The rippling with large spatial scales leads to wavefront folding at caustics and the appearance of spikes via a "refraction-focusing-diffraction" mechanism.<sup>23</sup> At the same time, small scale ripples are responsible for wavefront folding which occurs repeatedly as the wavefront propagates through the turbulent layer. The resulting shock front thus can be considered as a tightly packed bundle of multiple separate microshocks, which effectively looks like a shock with increased rise time. On the other hand, nonlinear propagation effects tend to steepen the diffused shocks, thus acting in the opposite way as the turbulence does. A question remains: does the turbulence always result in shock front smearing or do sharper shocks occur due to nonlinear effects, particularly in random foci? Although several efforts have been made to clarify this question, no definitive answer has been given yet.<sup>27–31</sup>

In order to better understand different aspects of nonlinear pulse propagation in the turbulence, several laboratory scale experiments were performed.<sup>27,32-34</sup> The main advantage of such experiments is that both the source of shock waves and the turbulence can be well controlled. In these experiments, the pulse wavelength and characteristic scales of the turbulence were downscaled by a factor between 1000 and 10000. N-waves were usually produced by electrical sparks and turbulent field was generated either by jets<sup>27,32,33</sup> or by hot air convection.<sup>32,34</sup> Typical waveform distortions and peak overpressure statistics were shown to be similar to that observed for sonic booms. However, experimental results on the rise time statistics were not studied in detail. The limited bandwidth of the microphones (Brüel & Kjær 4138) did not permit the measurement of rise time values smaller than  $2.5 \,\mu s$ . However, using custom made microphones or optical methods, it was observed that the rise time can be significantly shorter, in the range of  $[0.2, 2] \mu s.^{27,35-37}$  Large rise times of distorted waveforms (up to tens of microseconds) thus were measured correctly, while the shortest rise times were overestimated.<sup>32–34</sup> Since the shock was not accurately measured in laboratory-scale experiments, the detailed analysis of turbulence effects on the shock structure must be done on the basis of numerical simulations.

The goal of this paper is to study statistics of the most important parameters of N-waves propagating through a turbulent layer using numerical simulations. A nonlinear parabolic two-dimensional (2D) KZK-type (Khokhlov-Zabolotskaya-Kuznetsov) evolution equation is used in a numerical model.<sup>38,39</sup> Homogeneous isotropic incompressible turbulence with modified von Kármán spectrum is chosen to set inhomogeneous distributions of the refraction index associated with wind velocity fluctuations. Physical parameters of the model are chosen to represent laboratory scale experiments.<sup>33,34</sup> An alternative method to analyze waveform signatures is used. Instead of the classical rise time definition (from 10% to 90% of the peak overpressure), the shock front steepness is defined as a more appropriate parameter to characterize the shock front structure of distorted waveforms.<sup>39</sup> The interplay between diffraction on random inhomogeneities and nonlinear propagation effects is investigated. Cumulative probabilities, mean value, and standard deviation of the peak overpressure and the shock front steepness are analyzed.

The paper is organized as follows. The theoretical model based on the KZK-type equation combined with the model of random velocity field is described in Sec. II. The definition of the shock steepness and the numerical algorithm used in simulations are presented in Sec. III. The effects of the intensity of turbulent fluctuations and nonlinearity on *N*-wave statistics are presented in Sec. IV. Concluding remarks are given in Sec. V.

#### **II. THEORETICAL MODEL**

#### A. Sound propagation

Numerical experiments based on different models were extensively used to study the problem of wave propagation in random media. Ray tracing methods predicted multiple focusing and formation of caustics at sufficiently long propa-gation distances.<sup>40–42</sup> The distance where the probability of the appearance of caustics reaches a maximum was determined in ray tracing simulations and calculated analytically.<sup>43</sup> However, ray tracing theories poorly predict the acoustic field near caustics since diffraction effects are not included. Other approaches based on a modified KZK equation or extensions of the Westervelt equation were more successful.<sup>28,38,39,44–48</sup> Among different physical effects, these models incorporate diffraction and are not expensive from the computational point of view if one compares them with full-wave propagation models.<sup>15,49,50</sup> The KZK-type equation was used to simulate the laboratory scale experiment with turbulence produced by a jet.<sup>39</sup> Simulations differ from experiments on several aspects: the model was implemented in 2D geometry and the initial N-wave had a plane wavefront, while in the experiment the spark source produces spherical N-waves which propagate through a three-dimensional turbulence. Despite these differences, simulations qualitatively and quantitatively agree with experimental results showing similar wave distortions and peak overpressure statistics.

As previously done in Ref. 39, *N*-wave propagation through the turbulence is simulated here using the 2D KZK-type parabolic equation

$$\frac{\partial^2 p}{\partial \tau \partial z} = \frac{c_0}{2} \Delta_\perp p + \frac{\beta}{2\rho_0 c_0^3} \frac{\partial^2 p^2}{\partial \tau^2} + \frac{\delta}{2c_0^3} \frac{\partial^3 p}{\partial \tau^3} + \frac{\mu}{c_0} \frac{\partial^2 p}{\partial \tau^2}.$$
 (1)

Here *p* is the acoustic pressure, *z* is the longitudinal spatial coordinate, *x* is the transversal spatial coordinate [Fig. 1(b)],  $\tau = t - z/c_0$  is the retarded time, *t* is time,  $\Delta_{\perp}p = \partial^2 p/\partial x^2$  is the transversal Laplacian in the case of 2D geometry;  $\rho_0$ ,  $c_0$ ,  $\beta$ , and  $\delta$  are the density, ambient sound speed, nonlinearity coefficient, and absorption coefficient of the medium, respectively. Values of the physical parameters in Eq. (1) are chosen to represent air conditions of earlier experiments<sup>33,34</sup> at 20 °C temperature with 40% humidity:  $\rho_0 = 1.18 \text{ kg/m}^3$ ,  $c_0 = 344 \text{ m/s}$ ,  $\beta = 1.2$ ,  $\delta = 38 \text{ mm}^2/\text{s}$ .

Equation (1) takes into account diffraction (the first term on the right hand side of the equation), nonlinearity (the second term), and thermoviscous absorption (the third term). The equation also includes refractive distortions of the wavefront produced by sound speed or wind velocity inhomogeneities (the fourth term). The refraction index that corresponds to these inhomogeneities is defined as

$$\mu = \frac{\Delta c + u_z}{c_0},\tag{2}$$

where the sound speed  $\Delta c(x, z)$  describes scalar-type inhomogeneities (for example, thermal), and the function  $u_z(x, z)$  is the z-component of the wind velocity vector field  $\mathbf{u} = (u_x, u_z)$ . The transversal component  $u_x(x, z)$  of the velocity vector is neglected since its effect on the acoustic field was shown to be weak in comparison with effect of the longitudinal component.<sup>39</sup> The model is applicable in the case of smooth velocity inhomogeneities with small Mach numbers  $u_z/c_0 \ll 1$ , which primarily results in scattering angles in the forward direction up to 20° off the axis. Note also that for realistic modeling of shock wave propagation in air, the vibrational relaxation of molecular nitrogen and oxygen must be taken into account since relaxation effects results in different structure of the shock front depending on its amplitude.<sup>51</sup> However, as this paper is mostly focused on the effects of diffraction on random inhomogeneities in combination with nonlinear propagation, relaxation effects are not added to the model.

# **B.** Turbulence

In this study, only wind velocity fluctuations are included in the model (kinematic turbulence), whereas sound speed inhomogeneities  $\Delta c$  are set to zero. Randomly inhomogeneous medium is considered as a static refraction index field ("frozen" turbulence). It is assumed therefore that the travel time of the acoustic wave through the inhomogeneity layer is much smaller than the characteristic evolution time of the turbulence. This assumption is valid since the sound speed in air is about 340 m/ s, while wind velocity in the atmospheric boundary layer and in laboratory experiments is on the order of 10 m/s.<sup>33,52</sup>

Random realizations of homogeneous incompressible isotropic turbulence are synthesized using a modified von Kármán spectrum.<sup>52</sup> This spectrum in the inertial region satisfies Kolmogorov's "five-thirds" power law<sup>53</sup> and allows the modelling of multi-scale effects of the turbulence on acoustic wave propagation. Although this model is an idealized representation of the real turbulence, it was widely used in many theoretical studies and numerical simulations of acoustic and electromagnetic wave propagation through turbulent media.<sup>28,43,53,54</sup> Note also that the von Kármán spectrum of the homogeneous isotropic turbulence model fits the turbulence spectra measured in the laboratory-scale experiments.<sup>33,34</sup> A summary of the modified von Kármán model is given below.

To describe the homogeneous isotropic turbulence, a spectral tensor  $\Phi_{ij}(K_1, K_2)$  is introduced.<sup>52</sup> Here the indices *i* and *j* take values of 1 or 2, which correspond to the velocity components  $u_x$  or  $u_z$ , respectively. The spectral tensor  $\Phi_{ij}(K_1, K_2)$  is a Fourier transform of the two point correlation function  $R_{ij}(\vec{r}) = \langle u_i(\vec{r}_0)u_j(\vec{r}_0 + \vec{r}) \rangle$  of the *i*th and *j*th components of the velocity vector fluctuations;  $\vec{r}_0$  and  $\vec{r}$  are arbitrary radius-vectors and brackets  $\langle \rangle$  indicate the ensemble average. For the incompressible flow, the spectral tensor can be expressed via the kinetic energy spectrum E(K) as<sup>52</sup>

$$\Phi_{ij}(K_1, K_2) = \frac{E(K)}{\pi K} \left[ \delta_{ij} - \frac{K_i K_j}{K^2} \right].$$
(3)

Here  $K_1$  and  $K_2$  are turbulence wavenumbers that correspond to the *x* and *z* coordinates,  $K = |\vec{K}| = \sqrt{K_1^2 + K_2^2}$ , and  $\delta_{ij}$  is the Kronecker symbol. The modified von Kármán energy spectrum of the turbulence is defined as<sup>55</sup>

$$E(K) = \frac{5}{3} \frac{11}{6} \frac{\langle \mu^2 \rangle}{L_0^{5/3}} \frac{K^3 \exp(-K^2/K_m^2)}{\left(K^2 + K_0^2\right)^{17/6}}.$$
(4)

Here the spectrum is written in terms of the refraction index  $\mu$ , in which turbulent fluctuations intensity is referred to as  $\mu_{\rm rms} = \sqrt{\langle \mu^2 \rangle} = \sqrt{\langle u_z^2 \rangle}/c_0$ . The spectrum in Eq. (4) has two

scaling parameters:  $K_0 = 1/L_0$  and  $K_m = 5.92/l_0$ . The parameter  $L_0$  is the "outer scale" that characterizes the size of the largest inhomogeneities; the parameter  $l_0$  is the "inner scale" that characterizes the smallest scales, where the energy of turbulent fluctuations dissipates due to viscous forces.

The values of the scaling parameters are set to match typical conditions of laboratory-scale experiments:  $L_0 = 160 \text{ mm}$  and  $l_0 = 5 \text{ mm}$ .<sup>27,32–34</sup> The refraction index fluctuation intensity varied between  $\mu_{\rm rms} = 0.25\%$  and  $\mu_{\rm rms} = 2\%$ , which also covers the typical range of this parameter measured in laboratory-scale experiments.

In addition to the energy spectrum E(K), one-dimensional spectra representing distribution of fluctuations intensity over spatial wavenumbers  $K_1$  or  $K_2$  separately were introduced.<sup>52</sup> The longitudinal spectrum of the  $u_z$  component as a function of the wavenumber  $K_2$ , and the transversal spectrum as a function of the wavenumber  $K_1$  are defined by Eqs. (5) and (6), respectively,

$$E_{22}^{(1)}(K_2) = \int_{-\infty}^{\infty} \frac{E(K)}{\pi K} \left[ 1 - \frac{K_2^2}{K^2} \right] dK_1,$$
(5)

$$E_{22}^{(2)}(K_1) = \int_{-\infty}^{\infty} \frac{E(K)}{\pi K} \left[ 1 - \frac{K_2^2}{K^2} \right] dK_2.$$
(6)

Note that even in the case of an isotropic homogenous turbulence, the spectral tensor  $\Phi_{ij}(K_1, K_2)$  of a single component, for example  $u_z$ , is anisotropic. Therefore, longitudinal [Eq. (5)] and transversal [Eq. (6)], one-dimensional spectra are different. The two points correlation function  $R_{ij}(\vec{r})$  is also anisotropic, and can be represented as a combination of longitudinal f(r) and transversal g(r) functions

$$R_{ij}(\vec{r}) = (f - g)\frac{r_i r_j}{r^2} + g\delta_{ij}.$$
(7)

The longitudinal f(r) and transversal g(r) correlation functions are not independent and are related by the following equation in 2D geometry:<sup>52</sup>

$$g(r) = \frac{\partial}{\partial r} (rf).$$
(8)

Following the method of random Fourier modes described in Ref. 28, the inhomogeneous velocity field is generated by summing 8000 randomly oriented spatial Fourier modes. Wavenumbers of Fourier modes are distributed between  $K_{\min} = 0.02 \text{ m}^{-1}$  and  $K_{\max} = 4000 \text{ m}^{-1}$  in logarithmical scale. The quality of generated turbulent fields is verified by calculating correlation functions and spatial spectra over large refraction index realizations (20 m by 20 m). There is a good agreement between theoretical correlation functions [Fig. 2(a), solid curves] and corresponding functions obtained from a random realization [Fig. 2(a), dashed curves]. It is seen [Fig. 2(a)] that the transversal correlation function is narrower than the longitudinal one. This means that inhomogeneities of the  $u_{\tau}$  velocity component are elongated along the z axis. Anisotropy of a single velocity component is explained by the fact that the fluid particles move



FIG. 2. (a) Longitudinal, f(z), and transversal, g(x), correlation functions of the turbulence for the modified von Kármán spectrum model: exact functions (solid curves), and computed by averaging over a  $20 \times 20$  m<sup>2</sup> synthesized refraction index field  $\mu(x, z)$  (dashed curves). Parameters of the spectrum are:  $L_0 = 160$  mm,  $l_0 = 5$  mm, and  $\mu_{\rm rms} = 1\%$ . (b) Corresponding longitudinal (dashed-dotted curves) and transversal (dotted curves) one-dimensional spectra.

more easily in the direction of a local disturbance than transverse to it. The same anisotropy is seen on the onedimensional spectra [Fig. 2(b)]. The transversal spectrum has higher amplitudes in the inertial wavenumbers range of  $K_0 < K_{1,2} < K_m$ , but lower amplitudes if  $K_{1,2} < K_0$  in comparison with the longitudinal spectrum.

# **III. NUMERICAL ALGORITHM**

# A. Numerical method

The KZK-type equation [Eq. (1)] is solved following the method of fractional steps with the operator splitting procedure of the second order.<sup>56</sup> Effects of diffraction, absorption, nonlinearity, and inhomogeneities are described by differential operators that are solved successively using different numerical schemes for each step along the propagation coordinate. Both time-domain and frequency-domain representations of the acoustic field are used to efficiently implement the method with these split operators. Transitions between the time and frequency domains are performed using fast Fourier transform (FFT) routines implemented in the FFTW library.<sup>57</sup>

The parabolic diffraction operator is calculated in the time domain using the Crank–Nikolson scheme.<sup>58</sup> The longitudinal transport of acoustic waveforms due to sound speed inhomogeneities is taken into account using the exact solution in the frequency domain in order to avoid dispersion and absorption associated with time-domain schemes for the transport equation. The absorption term is also calculated in the frequency domain using exact solutions for each harmonic. A conservative Godunov-type time domain algorithm is used to calculate the nonlinear term of the equation.<sup>59</sup> This allowed capturing accurately the evolution of nonlinear acoustic pulses using a small number of grid points at the shocks. All algorithms except the Crank–Nikolson scheme are implemented in parallel using OpenMP software technology.

The spatial grid steps ( $\Delta z = 1 \text{ mm}$  and  $\Delta x = 0.4 \text{ mm}$ ) are chosen according to the turbulence characteristic scales and the wavelength  $\lambda = 13.8 \text{ mm}$  of the initial *N*-wave measured as the distance between the front and rear shocks. The propagation distance is  $z_{\text{max}} = 6 \text{ m} (440 \lambda)$  in linear simulations and  $z_{\text{max}} = 4.5 \text{ m} (330 \text{ } \lambda)$  in nonlinear simulations. The transversal size of the computation domain is 19.2 m (1390  $\lambda$ ). The temporal grid step is 0.04  $\mu$ s, which is sufficient to model the fine structure of shocks. For example, in the case of an initial N-wave with peak overpressure  $p_0 = 400$  Pa, the rise time is  $\tau_{\rm sh} = 0.40 \,\mu \rm s$ . With the given temporal grid step there are at least ten grid points per shock front, and the number of grid points per initial N-wave is 1000. The time window is  $1280 \,\mu s$  long, comprising  $32\,000$ grid points, and the middle of the N-wave is located at  $t_0 = 300 \,\mu s$  after the beginning of the time window providing zero-padded intervals of  $280 \,\mu s$  and  $960 \,\mu s$  before and after the initial N-wave waveform, respectively. These large temporal margins are necessary since refraction on random inhomogeneities leads to large fluctuations of arrival time and increase of the waveform duration.

#### B. Initial pressure waveform and boundary condition

A plane wave with a symmetric *N*-wave pressure profile is set as a boundary condition to Eq. (1) (Fig. 1). The negative and positive peak pressures of the wave have the same magnitude, and the front and rear shocks have the same structure. The initial duration of  $40 \,\mu s$  corresponds to the laboratory-scale experiments.<sup>33</sup> Such waveform represents classical sonic booms in the far-field when entering the PBL<sup>3</sup> and is well reproducible in experiments with spark sources.<sup>35,36</sup> Note also that, despite the sonic boom community is now paying more attention to "low-boom" signatures,<sup>14</sup> symmetric *N*-waves are used here since experimental data are available, and because waveforms distorted by turbulence are "easier" to interpret. However, the effects of turbulence are expected to be similar for both types of the waveforms.

The rise time of finite amplitude waves is determined by a balance between thermoviscous absorption and nonlinear propagation effects.<sup>60</sup> For example, a stationary solution for a unipolar plane shock wave in a thermoviscous fluid is described by hyperbolic tangent function known as the Taylor shock. The classical 10%–90% rise time of the Taylor shock is inversely proportional to the peak overpressure

$$\bar{\tau}_{\rm sh} = \frac{4.4\delta\rho_0}{\beta p_{\rm max}} = \frac{C}{p_{\rm max}},\tag{9}$$

where the constant *C* for the chosen air parameters is equal to 166 Pa.  $\mu$ s. However, in the case of *N*-wave propagating in the quiescent air, better fit with simulation data is found with the value C = 150 Pa.  $\mu$ s. Thereby, the rise time of the initial *N*-wave is set according to the equation  $\bar{\tau}_{sh} = C/p_0$  where  $p_0$ is the initial peak overpressure, and C = 150 Pa.  $\mu$ s. In nonlinear simulations, the peak overpressure  $p_0$  is varied in the range between 50 and 400 Pa in order to investigate nonlinear effects of different strengths. In linear simulations, the initial waveform corresponds to the nonlinear propagation case with initial peak overpressure  $p_0 = 200$  Pa and rise time  $\bar{\tau}_{sh} = 0.75 \,\mu$ s. Rigid wall boundary conditions are set in the transverse direction. One-meter large buffer zones adjacent to the boundaries permit to minimize reflections from edges of the computational domain. In these buffer zones the refraction index is smoothly attenuated from its normal value to zero. Thus, at the edges of the computational domain, the wave is almost plane and it propagates parallel to the boundaries.

#### C. Statistical analysis method

For each particular combination of the model parameters, numerical simulations are run over four random realizations of the refraction index in order to increase statistical sampling. Buffer zones of each realization are discarded. Then four realizations with resulting transversal width of  $17.2 \text{ m} (1246\lambda)$  are combined to one realization with equivalent width of 68.8 m (4985 $\lambda$ ) in order to perform statistical analysis. The ergodicity hypothesis is assumed when statistical data along transversal coordinate at each propagation distance are collected. According to this hypothesis, the spatial averaging over one sufficiently long realization is equivalent to the averaging over many realizations at a single location (ensemble averaging). The ergodicity of the acoustic field in simulations originates from the ergodicity of the refraction index field and the fact that the wavefront of the incident wave is plane. Probability distributions of different waveform parameters are acquired at each propagation distance and used to calculate mean values, standard deviations, and cumulative probabilities. It is checked that in each particular case, the four different realizations mentioned above resulted in similar statistical distributions.

#### D. Definitions of the waveform parameters

The peak overpressure and the rise time are the main parameters used here to characterize wave signatures. The peak overpressure  $p_{max}$  is defined as the maximum peak positive overpressure of the waveform. The classical 10%–90% rise time is clearly defined and can be easily calculated for waveforms close to an *N*-wave (Fig. 1). On the contrary, it is much more intricate to evaluate in this way the rise time of the distorted waveforms. For example, such evaluation gives largely overestimated values in the case of significantly distorted waveform. An alternative method based on evaluating the time derivative of the pressure waveform is used here.<sup>33</sup> With this method, the steepest shock front of any waveform is automatically selected and a more meaningful rise time value is returned.

To illustrate various definitions of the rise time, an example of a waveform distorted during propagation through turbulence is given in Fig. 3(a). A segment of the waveform in the time interval between  $\tau = 253 \ \mu s$  and  $\tau = 273 \ \mu s$  is magnified in Fig. 3(b). In this particular example, two shocks are identified. The first shock is located at the time  $\tau = 255 \ \mu s$  and the second is around  $\tau = 270 \ \mu s$ . Following the classical definition of the rise time, times  $t_1$  and  $t_2$  defined by  $p(t_1) = 0.1 p_{max}$  and  $p(t_2) = 0.9 p_{max}$  are calculated and marked in Fig. 3(b) by triangle markers and solid vertical lines. Thus, the classical rise time in this particular case is equal to  $t_2 - t_1 = 15.4 \ \mu s$ ,



FIG. 3. (Color online) Rise time and shock steepness definitions based on analysis of the waveform derivative: (a) a waveform with several shocks; (b) zoom on the interval around the first two shocks; (c) waveform derivative over the zoomed interval.

which is very large in comparison with the apparent  $1 \mu s$  rise time of the first shock.

Following the alternative definition of the rise time based on the waveform derivative,<sup>33</sup> the first derivative of the waveform segment shown in Fig. 3(b) is calculated and is presented in Fig. 3(c). The two highest local maxima on the graph correspond to the two shock fronts. At the first step of the rise time evaluation, the highest (global) maximum of the first derivative  $d_{\text{max}}$  is found. This maximum corresponds to the steepest part of the waveform and thus automatically indicates the strongest shock, which is the first shock in the given example. Then the rise time is evaluated as the interval between the times on both sides of the maximum for which the level is  $0.386d_{max}$ . These two time points are marked in Fig. 3(b) and Fig. 3(c) as  $t_3$  and  $t_4$  (filled circle markers, dashed vertical lines). The threshold value  $0.386d_{max}$  is chosen such as the rise time estimated from the derivative fits the classical 10%-90% definition in the case of a classical Nwave in quiescent air.33 The resulting rise time in this example is 1.1  $\mu$ s, which is close to the apparent rise time.

As soon as the times  $t_3$  and  $t_4$  are found, the shock amplitude is defined as  $\Delta p = p(t_4) - p(t_3)$ . Then the steepness of the shock is defined as the ratio of the shock amplitude to the rise time  $\tau_{sh} = t_4 - t_3$ :

$$s_{\max} = \frac{\Delta p}{\tau_{\rm sh}}.$$
 (10)

Mathematically, the steepness is the first order finite difference, which approximates the first derivative near the steepest shock front. Note that since in quiescent air the rise time is inversely proportional to the peak overpressure [Eq. (9)], and  $\Delta p = 0.8 p_{\text{max}}$ , then the steepness is proportional to the overpressure square:

$$s_{\max} = 0.8 p_{\max}^2 / C.$$
 (11)

This means that the steepness is very sensitive to the changes in the peak overpressure and is expected to fluctuate more in turbulence than the peak overpressure.

To highlight the effect of turbulence on the waveform parameters, the results are reported in normalized form in Secs. IV A and IV B. The peak overpressure  $p_{\text{max}}$  and the steepness  $s_{\text{max}}$  are normalized to their respective values in quiescent air  $p_{\text{max0}}$  and  $s_{\text{max0}}$  at the corresponding propagation distance

$$P_{\max}(x,z) = p_{\max}(x,z)/p_{\max}(z),$$
(12)

$$S_{\max}(x,z) = s_{\max}(x,z)/s_{\max}(z).$$
 (13)

# **IV. RESULTS AND DISCUSSION**

In the first part of this section, simulation results for linear wave propagation are presented and statistical properties of the acoustic field in the turbulence are discussed. In the second part, the role of nonlinear effects on random distortions of the acoustic field is investigated.

# A. Effect of turbulence intensity on N-wave statistics

The results of numerical simulations presented below demonstrate how the distance of formation of caustics and statistical characteristics of the acoustic field (probability distributions, mean value, standard deviation, and cumulative probabilities) change with the intensity of the turbulent fluctuations  $\mu_{\rm rms}$ . For this purpose, linear propagation of plane *N*-waves in turbulence with four different intensities  $\mu_{\rm rms}$  (0.25%, 0.5%, 1.0%, and 2.0%) is considered. In this first series of simulations, both the thermoviscous absorption and the nonlinear terms in Eq. (1) are disabled. In this case, only diffraction on turbulence inhomogeneities modifies the acoustic field during propagation.

Examples of spatial distributions of the refraction index  $\mu$  with  $\mu_{\rm rms} = 1\%$  and of the normalized peak overpressure  $P_{\rm max}$  are shown in Fig. 4. Examples of the distorted waveforms are presented in Fig. 5. Probability distributions of the normalized peak overpressure at four propagation distances are shown in Fig. 6. Mean value and standard deviation of the normalized peak overpressure are plotted in Fig. 7 as functions of the propagation distance.



FIG. 4. (a) Example of spatial realization of the refraction index  $\mu$  of kinematic turbulence with  $\mu_{\rm rms} = 1\%$ . (b) Corresponding normalized peak overpressure  $P_{\rm max}$  in the case of linear wave propagation. The vertical dashed line indicates the distance where the probability to observe strongly focused waveforms  $\Pr(P_{\rm max} > 2)$  has a maximum.

Characteristics of the random acoustic field are explained by the formation of caustics at a particular distance, which depend on the characteristics of turbulent inhomogeneities.<sup>26</sup> At the first stage of wave propagation, mean value and standard deviation of the normalized peak overpressure increase (Fig. 7), which indicates that the acoustic pressure fluctuates due to random wavefront rippling and



FIG. 5. Examples of randomly distorted waveforms at the propagation distance z = 1.1 m in the case of the turbulence intensity  $\mu_{\rm rms} = 1\%$  and linear propagation.



FIG. 6. Probability distributions of the normalized peak overpressure  $P_{\text{max}}$  at several propagation distances in the kinematic turbulence with  $\mu_{\text{rms}} = 1\%$ : (a) z = 0.28 m, (b) z = 0.55 m, (c) z = 1.1 m, and (d) z = 2.2 m. Class size of histograms is 0.01. Vertical dashed lines indicate the mean values.

following focusing and defocusing processes [Fig. 4(b)]. Probability distributions at this stage become wider as shown in Figs. 6(a), 6(b), and 6(c). A global maximum of the mean value and the standard deviation appears at a particular distance, where most of strong caustics are formed. These strong caustics are produced by the largest inhomogeneous structures of the multi-scale turbulent field. The amplitude gain in these large-scale caustics is greater than in smaller size caustics. The spatial distribution of the acoustic energy at this stage is the most inhomogeneous: amplitude fluctuations of the acoustic field attain their maximum due to formation of narrow high amplitude foci and large low amplitude defocusing zones [Fig. 4(b)].

Random focusing and defocusing processes continue further with the increase of the propagation distance in the turbulent layer. However, at this stage, the waveforms at



FIG. 7. (a) Mean value and (b) standard deviation of the normalized peak overpressure  $P_{\text{max}}$  as functions of the propagation distance z in the case of linear *N*-waves propagation in the kinematic turbulence with  $\mu_{\text{rms}} = 1\%$ . The gray line is the linear approximation for the initial rise of  $P_{\text{max}}$  standard deviation, given by equation  $0.5z/z_{\text{ref}}$ , where  $z_{\text{ref}}$  is the refraction length of the turbulent layer.

different spatial locations have random signatures with larger rise times, longer duration, and do not resemble the initial *N*-wave (Fig. 5). In other words, the acoustic field loses its coherence, the focusing process becomes less effective, and gradual decay of the mean value and the standard deviation is observed. However, the decrease of the standard deviation of  $P_{\text{max}}$  is rather slow in comparison with its initial rise. For example, probability distributions shown in Figs. 6(c) and 6(d) look very similar and almost equally wide, although the first of them is captured at the propagation distance where peak overpressure fluctuations have a maximum, and the second corresponds to the twice greater distance.

The mean value of the normalized peak overpressure  $P_{\text{max}}$  [Fig. 7(a)] has a maximum magnitude for the given simulation conditions, which is about 1.1. This magnitude of the maximum is determined by the focusing gain in caustics and depends on the spatial dimensions of the inhomogeneous structures and on the frequency content of the initial wave. For example, waveforms with shorter rise time have greater focusing gains than those with larger rise time. Note that in several publications where similar N-wave propagation simulations were performed, the mean value of the normalized peak overpressure was shown to be less than one.<sup>39,61</sup> It is explained by the fact that in these simulations, thermoviscous absorption was considered, so that the N-waves had larger rise times and correspondingly smaller focusing gains. The same behavior of the mean value of the normalized peak positive pressure was found in laboratory-scale experiments on N-wave propagation through turbulence.<sup>33,34</sup>

The standard deviation  $\delta P_{\text{max}}$  is a measure of the intensity of acoustic field fluctuations. It is seen in Fig. 7(b) that before reaching the global maximum, the standard deviation grows almost linearly with the propagation distance as shown by a solid line. Such linear increase of the standard deviation was also found in a model where the turbulent medium was represented by an infinitely thin random phase screen with a Gaussian correlation function and where the propagation of N-waves was simulated using the KZK nonlinear parabolic equation.<sup>61</sup> In the case of the phase screen model, analytical solutions for the first statistical moments of the N-wave amplitude were derived using geometrical acoustics approximation.<sup>42</sup> Linear increase of the standard deviation also was found:  $\delta P_{\text{max}} \approx 0.5 z/z_r$  for  $z \ll z_r$ , where  $z_r$  is the refraction length of the phase screen defined as a quantity inversely proportional to the standard deviation of rays convergence.<sup>62</sup> Mathematically, rays convergence of the phase screen is the second derivative of its phase. The refraction length indicates the distance where most of the first caustics occur.

In the present case of continuous inhomogeneous layer, one can define a refraction length similarly to the result of the geometrical acoustics obtained with the phase screen model. The initial linear increase of the standard deviation of the peak overpressure is fitted by the expression  $\delta P_{\text{max}}$ =  $0.5z/z_{\text{ref}}$ , where  $z_{\text{ref}}$  is defined as the refraction length of the turbulent layer. The refraction length approximately indicates the distance where the amplitude in random foci (caustics) is the highest. The refraction length is evaluated for the

0.25	0.5	1	2
2.8	1.5	0.8	0.43
3.4	1.9	1.1	0.60
18.8	18.7	19.5	19.3
3.9	4.4	4.9	5.4
	0.25 2.8 3.4 18.8 3.9	0.25         0.5           2.8         1.5           3.4         1.9           18.8         18.7           3.9         4.4	0.25         0.5         1           2.8         1.5         0.8           3.4         1.9         1.1           18.8         18.7         19.5           3.9         4.4         4.9

four values of  $\mu_{\rm rms}$  (Table I). It is found that the relation between the refraction length and the turbulence intensity can be written as follows:

$$z_{\rm ref} = z_{\rm ref0} (\mu_{\rm rms0} / \mu_{\rm rms})^{0.90},$$
 (14)

where  $z_{ref0} = 0.8$  m for the arbitrary chosen reference value of  $\mu_{rms0} = 1\%$ . The refraction length  $z_{ref}$  is then used to scale the propagation distance as

$$\bar{z} = z \frac{z_{\text{ref0}}}{z_{\text{ref}}}.$$
(15)

The mean value and the standard deviation of the normalized peak overpressure are plotted in Fig. 8 as functions of the scaled propagation distance. One can see that when the propagation distance is appropriately scaled, the propagation curves of the mean value and the standard deviation for different  $\mu_{\rm rms}$  are very similar. However, some small differences can be noticed. Despite the fact that nonlinear and thermoviscous absorption effects are disabled, the propagation conditions with low and high turbulence intensity are not fully equivalent. At lower  $\mu_{\rm rms}$  values (higher  $z_{\rm ref}$ ), the acoustic wave crosses more characteristic turbulent scales before arriving to the distance where strong caustics associated with larger turbulent structures form. Along this



FIG. 8. (Color online) Mean value (a) and standard deviation (b) of the normalized peak overpressure  $P_{\text{max}}$  as functions of the scaled propagation distance  $\bar{z}$  in the case of linear propagation with different levels of turbulence intensity  $\mu_{\text{rms}}$ : 0.25% (solid curve), 0.5% (dashed-dotted curve), 1% (dotted curve), and 2% (dashed curve). Note that in the case with  $\mu_{\text{rms}} = 1\%$  the normalized distance is equal to the physical distance in meters ( $\bar{z} = z$ ). The gray line is the linear approximation for the initial rise of  $P_{\text{max}}$  standard deviation, given by equation  $0.5z/z_{\text{ref}}$ , where  $z_{\text{ref}}$  is the refraction length of the turbulent layer.

propagation path the wave accumulates more phase distortions induced by small-scale turbulent structures. The most sensitive part of the acoustic pulse to these small-scale phase fluctuations is the shock front. Small-scale distortions lead to smoothening of the shock, thus reducing the peak overpressure in strong caustics. As a result, the standard deviation maximum is slightly lower for  $\mu_{\rm rms} = 0.25\%$  than for  $\mu_{\rm rms}$ = 2% [Fig. 8(b)].

Formation of high amplitude foci (caustics) can be investigated in details via calculation of the cumulative probabilities of the peak overpressure over the high amplitude tails of the probability distributions (Fig. 6). For this purpose, a cumulative probability  $Pr(P_{max} > \alpha)$  is defined here as the probability to observe the normalized peak overpressure which exceeds a given threshold  $\alpha$ . Inspection of the random realization presented in Fig. 4(b) indicates that in high amplitude foci, the peak overpressure is typically amplified by a factor between 2 and 4 in comparison with the reference values. The threshold value  $\alpha = 2$  is therefore chosen. The results for  $\alpha = 1.5$  are also shown, since the probability  $Pr(P_{max} > 1.5)$  is several times greater than  $Pr(P_{max} > 2)$  and is less susceptible to statistical sampling error.

In Fig. 9, the cumulative probabilities for two thresholds (a)  $\alpha = 1.5$  and (b)  $\alpha = 2.0$  are plotted using the scaled propagation distance  $\bar{z}$ . The probability curves are very similar for different  $\mu_{\rm rms}$  with small quantitative differences. They have a distinctive maximum, the position of which is marked in Fig. 9 by a vertical line and is denoted as  $z_{\text{caust}}$  in Table I. Appearance of this maximum corresponds to the formation of caustics and supports the general explanation of propagation curves of the mean value and the standard deviation given above. Since the position of the maximum is almost the same for different  $\mu_{\rm rms}$ , it means that the distance  $z_{\rm caust}$ depends on  $\mu_{\rm rms}$  almost similarly as  $z_{\rm ref}$ . Thus, approximately,  $z_{ref}$  and  $z_{caust}$  are proportional to each other:  $z_{\text{caust}} \approx 1.3 z_{\text{ref}}$  (see Table I for comparison). The maximum of the probability  $Pr(P_{max} > 2)$  is slightly lower for smaller values of the turbulence intensity  $\mu_{\rm rms}$  (Table I). This fact is explained similarly as previously done for the slightly different maxima of the standard deviation curves in Fig. 8. At the same time, maxima of the probability  $Pr(P_{max} > 1.5)$  have almost the same value for all turbulence intensities (about 19%). It means that small-scale ripples mainly affect the fine structure of the shock fronts, which is related to the focusing gain in random foci. From the present analysis where absorption and nonlinear propagation effects are not considered, one can conclude that random focusing leads to the appearance of significant number of waveforms with amplified peak overpressure [for example,  $Pr(P_{max} > 2) \approx 5\%$ ].

The theory of geometrical acoustics was used previously to analytically calculate the probability to observe caustics as a function of the propagation distance.<sup>43</sup> A typical probability curve of that kind has a maximum at a certain distance. The distance  $z_{caust}$  derived here from the position of the maximum of the probability  $Pr(P_{max} > 2)$  obtained with the KZK model can be compared to the distance of the most probable appearance of caustics in geometrical acoustics approximation. The geometrical acoustics analysis has led to a  $\mu_{rms}^{-2/3}$  scaling law for that distance, while  $z_{caust}$  is proportional to  $\mu_{rms}^{-0.90}$ .

The mean value and the standard deviation of the steepness as functions of the scaled propagation distance  $\bar{z}$  are shown in Fig. 10, and cumulative probabilities for the same threshold levels as for the peak overpressure ( $\alpha = 1.5$  and 2.0) are shown in Fig. 11. One can observe that the different scaled curves of the mean value and the standard deviation of the steepness are even more similar than the peak overpressure ones are. The evolution of the steepness statistics with the propagation distance indicates that, in the absence of absorption and nonlinearity, turbulence acts mostly as a destructive factor and leads to smoothening of sharp shock fronts. In contrast to the peak overpressure, the mean value of the steepness decreases monotonically and much more rapidly, and the maximum of the standard deviation is noticeably lower. For example, the maximum standard deviation of the normalized steepness is 0.33, whereas for the



FIG. 9. (Color online) Cumulative probabilities  $Pr(P_{max} > \alpha)$  to observe the normalized peak overpressure  $P_{max}$  greater than (a)  $\alpha = 1.5$  and (b)  $\alpha = 2.0$  as functions of the scaled propagation distance  $\bar{z}$  for different levels of  $\mu_{rms}$ : 0.25% (solid curve), 0.5% (dashed-dotted curve), 1% (dotted curve), and 2% (dashed curve). Vertical dashed lines indicate the distance where the probability  $Pr(P_{max} > 2)$  is maximum.



FIG. 10. (Color online) Mean value (a) and standard deviation (b) of the normalized steepness  $S_{\text{max}}$  as functions of the scaled propagation distance  $\bar{z}$  in the case of linear propagation of *N*-waves for different levels of  $\mu_{\text{rms}}$ : 0.25% (solid curve), 0.5% (dashed-dotted curve), 1% (dotted curve), and 2% (dashed curve). The gray line is the linear approximation for the initial rise of standard deviation of the normalized steepness.



FIG. 11. (Color online) Cumulative probabilities  $Pr(S_{max} > \alpha)$  to observe the normalized steepness greater than (a)  $\alpha = 1.5$  and (b)  $\alpha = 2.0$  as functions of the scaled propagation distance  $\bar{z}$  for  $\mu_{rms}$ : 0.25% (solid curve), 0.5% (dashed-dotted curve), 1% (dotted curve), and 2% (dashed curve). Vertical dashed lines indicate the distance where the probability  $Pr(P_{max} > 2)$  has a maximum.

normalized peak overpressure it is between 0.43 and 0.5. The destructive effect of turbulence on shock fronts is seen more clearly on cumulative probability curves: the maxima of the probabilities  $Pr(S_{max} > 2)$  and  $Pr(S_{max} > 1.5)$  are about 0.7% and 5.0%, respectively, which are considerably lower than maxima of the probabilities  $Pr(P_{max} > 2) = 5\%$ and  $Pr(P_{max} > 1.5) = 20\%$ . Note also that maxima of the cumulative probabilities  $Pr(S_{max} > 2)$  and  $Pr(S_{max} > 1.5)$ appear at half shorter distance compared to the maxima of the peak overpressure cumulative probabilities [Figs. 11(a) and 11(b)]. At the distance  $z_{\text{caust}}$ , where the peak overpressure cumulative probability  $Pr(P_{max} > 2)$  has its maximum, the steepness probability  $Pr(S_{max} > 2)$  is weak (0.1%). Thus, the shock front structure is significantly distorted by turbulence at much shorter propagation distances than the distance where strong caustics form.

These results support early analytical theories, which attempted to explain anomalously large and variable rise times measured in real atmosphere.<sup>26</sup> Here, the evidence of multiple shocks and smoothed shock fronts produced by small-scale rippling is demonstrated using several examples of waveforms at the propagation distance  $z = 1.1 \,\mathrm{m}$  for  $\mu_{\rm rms} = 1\%$  (Fig. 5). Note that an amplified waveform [Fig. 5(a)] with the peak overpressure of about two times higher than the incident wave has a very smooth shock (rise time about  $6 \mu s$ ) composed presumably of three separate shocks with slightly different arrival times. On the contrary, strongly distorted waveforms shown in Fig. 5(b) have much shorter rise times of about 1.5  $\mu$ s. These results indicate that, at least in the linear propagation model, high values of the peak overpressure are not necessarily associated to steep shocks.

In the presence of absorption and nonlinear effects, greater discrepancies between scaled propagation curves of the mean value, standard deviation, and cumulative probabilities are expected. For example, without inhomogeneities, absorption leads to an increase in the rise time with the propagation distance. In turbulence, on average, this process also takes place. Since the distance of formation of strong caustics is almost inversely proportional to the turbulence intensity [see Eq. (14)] and the *N*-wave waveforms have, on average, larger rise times at larger distances, smaller amplitudes are expected in random caustics for smaller turbulence intensities. As a result, thee standard deviation and the cumulative probability  $Pr(P_{max} > 2)$  are expected to be smaller as well.

Linear *N*-wave propagation investigated in this section can be considered as a reference case to compare with more general situations. Despite the complex interplay of different physical effects, overall evolution of the mean value, standard deviation, and cumulative probabilities presented for the linear case should be more or less the same. In the following section, nonlinear and absorption effects are considered in the propagation model and statistical data of the peak overpressure and steepness are analyzed with fixed intensity of the kinematic turbulent field (reference case  $\mu_{\rm rms} = 1\%$ ).

# B. Effect of nonlinear propagation on *N*-wave statistics

In this section, nonlinear propagation effects are considered. The nonlinear and thermoviscous absorption terms in Eq. (1) are enabled together with the inhomogeneous term. A balance between thermoviscous absorption and nonlinear steepening of the waveform provides finite rise time of the shock front. Four cases with different amplitudes of the initial *N*-wave varying from 50 to 400 Pa are simulated to investigate nonlinear effects of different strength. The intensity of the turbulent field is fixed at  $\mu_{\rm rms} = 1\%$ , so the normalized distance is identical to the physical distance  $\bar{z} = z$ .

Nonlinear effects greatly complicate the analysis of *N*-wave propagation in a randomly inhomogeneous medium. The most important manifestation of nonlinear effects is their tendency to steepen the shock front in the case of amplified shock amplitude, while this effect is partly balanced by thermoviscous absorption. Higher amplitudes thus necessarily result in sharper shocks. As a result, nonlinear propagation effects tend to counterbalance turbulence effects that lead to increased rise time. This process is known as "healing" of the shock front.<sup>17</sup> Since the focusing gain in the caustics is higher and the rise time is shorter, nonlinear effects are expected to increase both the fluctuations of the peak overpressure and the steepness. In particular, the probability to observe peaked waveforms with high amplitude is expected to be higher in the case of stronger nonlinearities. However, when the pulse amplitude is sufficiently high, nonlinear effects lead to the saturation of the focusing gain in random foci due to effective absorption of wave energy at the shock and to a nonlinear refraction effect. The nonlinear refraction mechanism limits the focusing gain via flattening the concave wavefront of focused wave since wavefront segments with higher amplitudes propagate faster than wavefront parts with lower amplitudes.<sup>63,64</sup>

The mean value and the standard deviation of the normalized peak overpressure are plotted as functions of the propagation distance z in Fig. 12. Qualitatively, their evolutions with the distance are similar for all initial amplitudes and agree with the results obtained with the linear propagation model (Sec. IV A). However, the evolution of the mean value with the distance has no pronounced maximum in the nonlinear case [Fig. 12(a)]. Instead, at propagation distances shorter than the refraction length, the mean value of the normalized peak pressure is almost constant and differs from unity by few percents only. At distances greater than the refraction length, the mean value decreases. Note that the same behavior of the peak overpressure mean value was observed in laboratory scale experiments.<sup>27,33,34</sup> Also, the rate of decrease of the mean value depends on the initial *N*-wave amplitude and is the lowest for the highest amplitudes.

The linear growth rate of the standard deviation is different at different amplitudes and also deviates from the rate obtained in Sec. IV A with the linear propagation model. This result is expected since *N*-waves with different initial amplitudes have different initial rise times. Higher values of the standard deviation for small and moderate amplitudes (50, 100, and 200 Pa) indicate the increase of intensity of fluctuations at distances closer than the refraction length. At the highest initial pressure level (400 Pa), the focusing gain saturation in random foci tends to decrease the standard deviation. However, at larger distances, higher standard deviation levels are observed in the case with strongest nonlinearity.

The cumulative probability curves of the peak overpressure shown in Fig. 13 clarify the role of nonlinear effects in the process of random focusing. The probability to observe waveforms with the peak overpressure amplified at least by a factor of two  $[Pr(P_{max} > 2), Fig. 13(b)]$  nearly doubles (from 1.9% to 3.4%) when the initial N-wave amplitude  $p_0$ increases from 50 to 100 Pa (Table II). The maximum of the probability  $Pr(P_{max} > 2)$  is about 4% for  $p_0 = 200$  Pa, then it decreases for further increase of  $p_0$ . This behavior of cumulative probabilities results from the nonlinear saturation of the focusing gain in random foci as described above. At the same time, the maximum of the probability  $Pr(P_{max} > 1.5)$ does not vary as much as the probability  $Pr(P_{max} > 2)$ . It means that the nonlinear amplification of the peak overpressure mainly appears in high amplitude foci. Another important observation is that, around their maximum, the



FIG. 12. (Color online) Mean value (a) and standard deviation (b) of the normalized peak overpressure  $P_{\rm max}$  in the case of nonlinear propagation with different amplitudes of the initial *N*-wave: 50 Pa (solid curve), 100 Pa (dashed-dotted curve), 200 Pa (dotted curve), 400 Pa (dashed curve). Gray curves correspond to linear wave propagation. Vertical dashed line indicates the distance where the probability  $Pr(P_{\rm max} > 2)$  is maximum in the linear case.



FIG. 13. (Color online) Cumulative probabilities  $Pr(P_{max} > \alpha)$  to observe the normalized peak overpressure  $P_{max}$  greater than the thresholds (a)  $\alpha$ = 1.5 and (b)  $\alpha$  = 2.0 as functions of the propagation distance z in the case of nonlinear propagation with different amplitudes of the initial *N*-wave: 50 Pa (solid curve), 100 Pa (dashed-dotted curve), 200 Pa (dotted curve), 400 Pa (dashed curve). The gray curve corresponds to linear wave propagation. Vertical dashed lines indicate the distance where the probability  $Pr(P_{max} > 2)$  is maximum in the linear case.

probability curves  $Pr(P_{max} > 1.5)$  and  $Pr(P_{max} > 2)$  broaden with the increase of the nonlinearity strength, and the probability remains high beyond the maxima located approximately at the distance  $z_{caust}$  (vertical dashed line in Fig. 13). Nonlinear effects tend to maintain the effectiveness of random focusing due to a nonlinear shock front "healing" mechanism even in the case of a partially diffused acoustic field.

Since the probability curves of  $Pr(P_{max} > 2)$  are very wide and the statistical sampling error can shift the position of their maximum  $z_{max}$ , it is more accurate to analyze the propagation distance where this probability reaches 90% of its maximal value. This distance is denoted as  $z_{90max}$  and compared with  $z_{max}$  in Table II. It is seen that the distance  $z_{90max}$  is 10%–20% shorter than  $z_{max}$  and is equal approximately to one meter.

The most significant change of the statistics due to nonlinear effects is observed for the shock front steepness. While the mean value does not greatly change with the increase of the initial *N*-wave amplitude [Fig. 14(a)], the standard deviation indicates higher variability [Fig. 14(b)]. For example, at  $p_0 = 50$  Pa, the maximum of the standard deviation is  $\delta S_{\text{max}} = 0.45$ , whereas at  $p_0 = 400$  Pa, it is more than twice higher ( $\delta S_{\text{max}} = 1.05$ ). Note the difference with the results obtained for the standard deviation of the peak overpressure, where its values are in a range between

TABLE II. Cumulative probabilities to observe waveforms with normalized peak overpressure greater than  $\alpha = 2$  and  $\alpha = 1.5$  for different amplitudes of the initial *N*-wave: maximum values of  $Pr(P_{max} > 2)$  and  $Pr(P_{max} > 1.5)$ ; distance  $z_{max}$  where the probability  $Pr(P_{max} > 2)$  is maximum (in meters); and distance  $z_{90max}$  where  $Pr(P_{max} > 2)$  attains 90% of its maximum value.

<i>p</i> <sub>0</sub> , Pa	50	100	200	400
$\Pr(P_{\max} > 2), \max.$	1.9	3.4	4.0	3.2
$Pr(P_{max} > 1.5), max.$	9.8	12.5	13.0	11.3
$\Pr(P_{\max} > 2), z_{\max}, m$	1.22	1.17	1.18	1.44
$Pr(P_{max} > 2), z_{90max}, m$	1.03	1.06	1.03	1.10



FIG. 14. (Color online) Mean value (a) and standard deviation (b) of the normalized steepness  $S_{\text{max}}$  in the case of nonlinear propagation with different amplitudes of the initial *N*-wave: 50 Pa (solid curve), 100 Pa (dasheddotted curve), 200 Pa (dotted curve), 400 Pa (dashed curve). Gray curves correspond to linear wave propagation.

0.4 and 0.5 for all considered initial wave amplitudes [Fig. 12(b)]. The initial linear growth rate of the steepness standard deviation  $\delta S_{\text{max}}$  also varies with the pressure amplitude. It can be higher or lower than the rate observed with the linear propagation case [gray line in Fig. 14(b)].

Cumulative probability curves indicate that the maximum of the probability  $Pr(S_{max} > 2)$  is almost proportional to the initial wave amplitude [Fig. 15(b)]. However, for  $p_0 = 400$  Pa, the saturation of this probability is observed. Also, as it is found for peak overpressure probabilities  $Pr(P_{max} > 1.5)$  and  $Pr(P_{max} > 2)$ , nonlinear effects broaden the steepness probability curves  $Pr(S_{max} > 1.5)$  and  $Pr(S_{max} > 2)$ , the probability remains high beyond the caustics formation distance  $z_{\text{caust}}$ . The maximum value of the steepness probability  $Pr(S_{max} > 2)$  is found at a distance about 20%-30% shorter than it is for the maximum of the peak overpressure probability  $Pr(P_{max} > 2)$ . The same difference is observed in z<sub>90max</sub> distances calculated for  $Pr(S_{max} > 2)$  and  $Pr(P_{max} > 2)$ . This demonstrates that random focusing effects can be observed for the shock front steepness at shorter distances than for the peak overpressure.

High sensitivity of the shock steepness to nonlinear effects is explained by the fact that the steepness is proportional to the square of the shock amplitude [Eq. (11)]. Thus, the rise of the peak overpressure in random foci leads to a fast increase of the steepness, which in turn enhances subsequent focusing. However, there are several limiting factors such as diffraction effects, effective absorption on shocks, and nonlinear refraction, which do not allow the amplitude and the steepness to grow infinitely. Note also how fast the



FIG. 15. (Color online) The cumulative probabilities  $Pr(S_{max} > \alpha)$  to observe the normalized steepness  $S_{max}$  greater than (a)  $\alpha = 1.5$  and (b)  $\alpha = 2.0$  as functions of the propagation distance *z* in the case of nonlinear propagation with different amplitudes of the initial *N*-wave: 50 Pa (solid curve), 100 Pa (dashed-dotted curve), 200 Pa (dotted curve), 400 Pa (dashed curve). Vertical dashed lines indicate the distance where the probability  $Pr(P_{max} > 2)$  is maximum in the linear case. Gray curves correspond to linear wave propagation.

absolute values of the steepness grow with the initial *N*-wave amplitude (Table III): an 8 times increase of  $p_0$  results in a 35 times increase of the steepness at the propagation distance  $z_{\text{caust}}$ . This observation could be important in the context of sonic boom if the same nonlinear enhancement of the shock front steepness is observed in sonic booms.

# V. SUMMARY AND CONCLUSIONS

Linear and nonlinear propagation of short *N*-waves in turbulence with modified von Kármán spectrum is considered using numerical simulations based on the KZK-type 2D parabolic equation. The influence of the turbulence intensity and the nonlinear propagation effects on the acoustic field are studied. To investigate the structure of the shock front, a steepness parameter is defined as a ratio of the shock amplitude to the corresponding rise time. The shock front amplitude and the rise time are calculated using the waveform derivative. Mean value, standard deviation, and cumulative probabilities of the peak overpressure and shock front steepness are analyzed as functions of the propagation distance. Cumulative probabilities to observe waveforms with peak

TABLE III. Cumulative probabilities to observe waveforms with normalized shock front steepness greater than  $\alpha = 2$  and  $\alpha = 1.5$  for different amplitudes of the initial *N*-wave: maximum values of  $Pr(S_{max} > 2)$  and  $Pr(S_{max} > 1.5)$ ; distance  $z_{max}$  where the probability  $Pr(S_{max} > 2)$  is maximum (in meters); and distance  $z_{90max}$  where  $Pr(S_{max} > 2)$  attains 90% of its maximum value. The last line shows the absolute values of the shock front steepness in homogeneous air ( $s_{max0}$ ) at a propagation distance z = 1.1 m.

<i>p</i> <sub>0</sub> , Pa	50	100	200	400
$\Pr(S_{\max} > 2)$ , max.	2.2	4.5	7.3	8.5
$\Pr(S_{\max} > 1.5), \max.$	8.0	10.6	12.7	13.6
$\Pr(S_{\max} > 2), z_{\max}, m$	0.90	0.87	0.96	1.18
$Pr(S_{max} > 2), z_{90max}, m$	0.74	0.72	0.76	0.82
$s_{\text{max0}}$ at $z = z_{\text{caust}} = 1.1 \text{ m}$ , Pa/ $\mu$ s	15	50	170	520

overpressure and steepness amplified by a factor of two or higher are used to detect the appearance of highly peaked waveforms and sharp shock fronts.

Results of numerical experiments within a linear propagation model corroborated the conclusions of analytical theories describing the shock front interaction with turbulence. It is shown that multiple wavefront folding produced by small-scale inhomogeneities mainly leads to smoothening of shock fronts. For example, the highest probability to observe shock fronts with doubled or higher steepness is relatively low and rated below 0.7%. At the same time, large-scale inhomogeneities are still able to produce high peak overpressures in random foci. The cumulative probability to observe waveforms with peak overpressure amplified by a factor of two or higher is about 5%.

Variation of the turbulence intensity  $\mu_{\rm rms}$  in the absence of absorption and nonlinear effects results in longitudinal scaling of the acoustic field statistics according to a power law  $\mu_{\rm rms}^{-0.9}$ . This result means that in stronger turbulence, random foci appear at a closer propagation distance, and this distance is almost inversely proportional to the turbulence intensity.

Nonlinear *N*-wave propagation in the turbulence results in important quantitative differences from the linear propagation case. The cumulative probability to observe waveforms with high peak overpressures is shown to be sensitive to the strength of nonlinear effects. For moderate pressure amplitude of the initial *N*-wave, the cumulative probability increases with the initial wave amplitude. However, for higher initial wave amplitude, the saturation of the focusing gain in random foci leads to saturation of the cumulative probability curves. The variation range of the cumulative probability of  $P_{\text{max}}$  is relatively small. For example, the maximum of the probability of twofold gain in  $P_{\text{max}}$  is rated between 2% and 4% for eightfold increase of the initial amplitude between  $p_0 = 50$  Pa and  $p_0 = 400$  Pa.

Statistics of the shock steepness is found to be more sensitive to the strength of the nonlinear effects in comparison to statistics of peak overpressure. The maximum of the cumulative probability to observe doubled and higher steepness is rated between 2% and 8% for the same range of initial *N*-wave amplitudes (between 50 and 400 Pa), which is almost one order of magnitude higher in comparison with the linear propagation case. This result proves that even if nonlinear effects are weak, the shock front steepening they produce is sufficient to compensate shock front smoothening due to diffraction on turbulent inhomogeneities. Also, waveforms with high values of the shock steepness appear at 20%–30% smaller propagation distances than waveforms with high peak overpressures.

In this paper, the problem of linear and nonlinear acoustic pulse propagation through a turbulent medium is analyzed for characteristic parameters close to previous laboratory scale experiments. However, similar results can be expected in the case of the real atmosphere and sonic boom scales since physical mechanisms involved in acoustic wave propagation are the same. Application of the presented numerical model to investigate statistics of sonic booms is the subject of the future work.

#### ACKNOWLEDGMENTS

This work was supported by RSF-17-72-10277 and the LabEx Centre Lyonnais d'Acoustique of Université de Lyon, operated by the French National Research Agency (ANR-10-LABX-0060/ANR-11-IDEX-0007). Numerical simulations were performed on the SKIF "Chebyshev" and "Lomonosov" clusters of the Moscow State University supercomputer center.

<sup>1</sup>V. Krasilnikov, "Linear and nonlinear sound propagation in turbulent and inhomogeneous media," Acoust. Phys. **44**, 559–569 (1998), available at http://www.akzh.ru/pdf/1998\_4\_559-569.pdf.

- <sup>2</sup>D. J. Maglieri, "Some effects of airplane operations and the atmosphere on sonic-boom signatures," J. Acoust. Soc. Am. **39**, S36–S42 (1966).
- <sup>3</sup>H. Carlson and D. Maglieri, "Review of sonic boom generation theory and prediction methods," J. Acoust. Soc. Am. **51**, 675–685 (1972).
- <sup>4</sup>K. Plotkin and D. Maglieri, "Sonic boom research: History and future," in *Proceedings of the 33rd AIAA Fluid Dynamics Conference and Exhibit*, Orlando, FL (June 23–26, 2003), AIAA2003-3575.
- <sup>5</sup>K. J. Plotkin, "State of the art of sonic boom modeling," J. Acoust. Soc. Am. **111**, 530–536 (2002).
- <sup>6</sup>D. Maglieri, P. Bobbitt, K. Plotkin, K. Shepherd, P. Coen, and D. Richwine, "Sonic boom: Six decades of research," Technical Report No. NASA/SP-2014-622, L-20381, NF1676L-18333, NASA, Washington, DC (2014).
- <sup>7</sup>R. Seebass, "Sonic boom theory," J. Aircr. 6, 177–184 (1969).
- <sup>8</sup>J. Leatherwood, B. Sullivan, K. Shepherd, D. McCurdy, and S. Brown, "Summary of recent NASA studies of human response to sonic booms," J. Acoust. Soc. Am. **111**, 586–598 (2002).
- <sup>9</sup>A. Niedzwiecki and H. Ribner, "Subjective loudness of N-wave sonic booms," J. Acoust. Soc. Am. 64, 1617–1621 (1978).
- <sup>10</sup>B. Sullivan, J. Klos, R. Buerhle, D. McCurdy, and E. Hearing, "Human response to low-intensity sonic booms heard indoors and outdoors," Technical Report No. TM-2010-216685, NASA, Washington, DC (2010).
- <sup>11</sup>S. Fidell, L. Silvati, and K. Pearsons, "Relative rates of growth of annoyance of impulsive and non-impulsive noises," J. Acoust. Soc. Am. 111, 576–585 (2002).
- <sup>12</sup>F. Simmons III and D. Freund, "Morphing concept for quiet supersonic jet boom mitigation," in *Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit*, Reno, NV (January 10–13, 2005), AIAA2005-1015.
- <sup>13</sup>D. C. Howe, "Improved sonic boom minimization with extendable nose spike," in *Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit*, Reno, NV (January 10–13, 2005), AIAA2005-1014.
- <sup>14</sup>J. Pawlowski, D. Graham, C. Boccadoro, P. Coen, and D. Maglieri, "Origins and overview of the shaped sonic boom demonstration program," in *Proceedings of the 43rd AIAA Aerospace Sciences Meeting and Exhibit*, Reno, NV (January 10–13, 2005), AIAA2005-5.
- <sup>15</sup>F. Alauzet and A. Loseille, "High-order sonic boom modeling based on adaptive methods," J. Comput. Phys. 229, 561–593 (2010).
- <sup>16</sup>A. Pierce and J. Kang, "Molecular relaxation effects on sonic boom waveforms," in *Frontiers of nonlinear acoustics: Proceedings of the 12th ISNA*, edited by M. F. Hamilton and D. T. Blackstock (Elsevier Applied Science, London, 1990).
- <sup>17</sup>H. Bass, R. Raspet, J. Chambers, and M. Kelly, "Modification of sonic boom waveforms during propagation from the source to the ground," J. Acoust. Soc. Am. **111**, 481–487 (2002).
- <sup>18</sup>D. Maglieri, V. Sotchcott, and T. Keefer, "A summary of XB-70 sonic boom signature data for flights during march 1965 through may 1966," NASA Contracor Report No. 189630, NASA, Washington, DC (1992).
- <sup>19</sup>G. Herbert, W. Hass, and J. Angell, "A preliminary study of atmospheric effects on the sonic boom," J. Appl. Meteor. **8**, 618–626 (1969).
- <sup>20</sup>R. Blumrich, F. Coulouvrat, and D. Heimann, "Meteorologically induced variability of sonic-boom characteristics of supersonic aircraft in cruising flight," J. Acoust. Soc. Am. 118, 707–722 (2005).
- <sup>21</sup>W. L. Willshire, Jr. and D. Devilbiss, "Preliminary results from the white sands missile range sonic boom propagation experiment," in *Proceedings* of the High-Speed Research Workshop on Sonic Boom, edited by C. M. Darden (NASA Langley Research Center, Hampton, VA, 1992), Vol. 1, pp. 137–149.
- <sup>22</sup>A. Pierce, "Statistical theory of atmospheric turbulence effects on sonicboom rise times," J. Acoust. Soc. Am. 49, 906–924 (1971).

- <sup>23</sup>A. Pierce, "Spikes on sonic-boom pressure waveforms," J. Acoust. Soc. Am. 44, 1052–1061 (1968).
- <sup>24</sup>S. Crow, "Distortion of sonic bangs by atmospheric turbulence," J. Fluid Mech. 37, 529–563 (1969).
- <sup>25</sup>A. George and K. Plotkin, "Propagation of sonic booms and other weak nonlinear waves through turbulence," Phys. Fluids 14, 548–554 (1971).
- <sup>26</sup>A. Pierce and D. Maglieri, "Effects of atmospheric irregularities on sonicboom propagation," J. Acoust. Soc. Am. **51**, 702–924 (1972).
- <sup>27</sup>B. Lipkens and D. Blackstock, "Model experiment to study sonic boom propagation through turbulence. Part I: Model experiment and general results," J. Acoust. Soc. Am. 103, 148–158 (1998).
- <sup>28</sup>P. Blanc-Benon, B. Lipkens, L. Dallois, M. F. Hamilton, and D. T. Blackstock, "Propagation of finite amplitude sound through turbulence: Modeling with geometrical acoustics and the parabolic approximation," J. Acoust. Soc. Am. 111, 487–498 (2002).
- <sup>29</sup>P. Boulanger, R. Raspet, and H. Bass, "Sonic boom propagation through a realistic turbulent atmosphere," J. Acoust. Soc. Am. **98**, 3412–3417 (1995).
- <sup>30</sup>R. Raspet, H. Bass, L. Yao, P. Boulanger, and W. McBride, "Statistical and numerical study of the relationship between turbulence and sonic boom characteristics," J. Acoust. Soc. Am. 96, 3621–3626 (1994).
- <sup>31</sup>M. Kelly, R. Raspet, and H. Bas, "Scattering of sonic booms by anisotropic turbulence in the atmosphere," J. Acoust. Soc. Am. **107**, 3059–3064 (2000).
- <sup>32</sup>S. Ollivier and P. Blanc-Benon, "Model experiment to study acoustic *N*-wave propagation through turbulence," in *Proceedings of the 10th AIAA/ CEAS Aeroacoustics Conference*, Manchester, UK (May 10–12, 2004), AIAA2004-2921.
- <sup>33</sup>M. Averiyanov, S. Ollivier, V. Khokhlova, and P. Blanc-Benon, "Random focusing of nonlinear acoustic *N*-waves in fully developed turbulence: Laboratory scale experiment," J. Acoust. Soc. Am. **130**, 3595–3607 (2011).
- <sup>34</sup>E. Salze, P. Yuldashev, S. Ollivier, V. Khokhlova, and P. Blanc-Benon, "Laboratory-scale experiment to study nonlinear *N*-wave distortion by thermal turbulence," J. Acoust. Soc. Am. **136**, 556–566 (2014).
- <sup>35</sup>W. Wright, "Propagation in air of *N*-waves produced by sparks," J. Acoust. Soc. Am. **73**, 1948–1955 (1983).
- <sup>36</sup>P. Yuldashev, S. Ollivier, M. Averiyanov, O. Sapozhnikov, V. Khokhlova, and P. Blanc-Benon, "Nonlinear propagation of spark-generated *N*-waves in air: Modeling and measurements using acoustical and optical methods," J. Acoust. Soc. Am. **128**, 3321–3333 (2010).
- <sup>37</sup>P. Yuldashev, M. Karzova, V. Khokhlova, S. Ollivier, and P. Blanc-Benon, "Mach-Zehnder interferometry method for acoustic shock wave measurements in air and broadband calibration of microphones," J. Acoust. Soc. Am. 137, 3314–3324 (2015).
- <sup>38</sup>M. Averiyanov, V. Khokhlova, O. Sapozhnikov, P. Blanc-Benon, and R. Cleveland, "Parabolic equation for nonlinear acoustic wave propagation in inhomogeneous moving media," Acoust. Phys. **52**, 623–632 (2006).
- <sup>39</sup>M. Averiyanov, P. Blanc-Benon, R. Cleveland, and V. Khokhlova, "Nonlinear and diffraction effects in propagation of N-waves in randomly inhomogeneous moving media," J. Acoust. Soc. Am. **129**, 1760–1772 (2011).
- <sup>40</sup>P. Blanc-Benon, D. Juvé, and G. Comte-Bellot, "Occurrence of caustics for high-frequency acoustic waves propagating through turbulent fields," Theor. Comput. Fluid Dyn. 2, 271–278 (1991).
- <sup>41</sup>R. Cleveland, J. Chambers, H. Bass, R. Raspet, D. T. Blackstock, and M. F. Hamilton, "Comparison of computer codes for the propagation of sonic boom waveforms through isothermal atmospheres," J. Acoust. Soc. Am. **100**, 3017–3027 (1996).
- <sup>42</sup>O. Rudenko and B. Enflo, "Nonlinear N-wave propagation through a one-dimensional phase screen," Acta Acust. united Acust. **86**, 229–238 (2000), available at http://www.ingentaconnect.com/search/article?option1= title&value1=one-dimensional+phase+screen&pageSize=10&index=1.

- <sup>43</sup>P. Blanc-Benon, D. Juvé, V. Ostashev, and R. Wandelt, "On appearance of caustics for plane sound-wave propagation in moving random media," Waves Random Media 5, 183–199 (1995).
- <sup>44</sup>O. Godin, "Wide-angle parabolic equations for sound in a 3D inhomogeneous moving medium," Dokl. Phys. 47, 643–646 (2002).
- <sup>45</sup>F. Coulouvrat, "Parabolic approximation in ray coordinates for high-frequency nonlinear waves in a inhomogeneous and high speed moving fluid," Wave Motion **45**, 804–820 (2008).
- <sup>46</sup>F. Dagrau, M. Rénier, R. Marchiano, and F. Coulouvrat, "Acoustic shock wave propagation in a heterogeneous medium: A numerical simulation beyond the parabolic approximation," J. Acoust. Soc. Am. **130**, 20–32 (2011).
- <sup>47</sup>F. Coulouvrat, "New equations for nonlinear acoustics in a low mach number and weakly heterogeneous atmosphere," Wave Motion 49, 50–63 (2012).
- <sup>48</sup>L.-J. Gallin, M. Rénier, E. Gaudard, T. Farges, R. Marchiano, and F. Coulouvrat, "One-way approximation for the simulation of weak shock wave propagation in atmospheric flows," J. Acoust. Soc. Am. 135, 2559–2570 (2014).
- <sup>49</sup>O. Marsden, C. Bogey, and C. Bailly, "A study of infrasound propagation based on high-order finite difference solutions of the Navier–Stokes equations," J. Acoust. Soc. Am. **135**, 1083–1095 (2014).
- <sup>50</sup>V. Sparrow and R. Raspet, "A numerical method for general finite amplitude wave propagation in two dimensions and its application to spark pulses," J. Acoust. Soc. Am. **90**, 2683–2691 (1991).
- <sup>51</sup>L. Orenstein, "The rise time of *N*-waves produced by sparks," Technical Report No. ARL-TR-82-51, Applied Research Laboratories, University of Texas at Austin, Austin, TX (1982).
- <sup>52</sup>G. Comte-Bellot and C. Bailly, *Turbulence* (CNRS, Paris, France, 2003), pp. 132–136.
- <sup>53</sup>V. I. Tatarskii, *The Effects of the Turbulent Atmosphere on Wave Propagation* (Israel Program for Scientific Translations, Jerusalem, 1971), pp. 51–59.
- pp. 51–59.
  <sup>54</sup>J. Martin and S. M. Flatte, "Intensity images and statistics from numerical simulation of wave propagation in 3-D random media," Appl. Opt. 27, 2111–2126 (1988).
- <sup>55</sup>V. Ostashev, "Sound propagation and scattering in media with random inhomogeneities of sound speed, density and medium velocity," Waves Random Media 4, 403–428 (1994).
- <sup>56</sup>J. Tavakkoli, D. Cathignol, R. Souchon, and O. Sapozhnikov, "Modeling of pulsed finite-amplitude focused sound beams in time domain," J. Acoust. Soc. Am. **104**, 2061–2072 (1998).
- <sup>57</sup>M. Frigo and S. G. Johnson, "The design and implementation of FFTW3," Proc. IEEE 93, 216–231 (2005).
- <sup>58</sup>W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in FORTRAN. The Art of Scientific Computing*, 2nd ed. (Cambridge University Press, Cambridge, UK, 1992), p. 840.
- <sup>59</sup>K. Kurganov and E. Tandmor, "New high-resolution central schemes for nonlinear conservation laws and convention-diffusion equations," J. Comp. Phys. **160**, 241–282 (2000).
- <sup>60</sup>O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Consultants Bureau, New York, 1977), Chap. 2.
- <sup>61</sup>P. Yuldashev, N. Bryseva, M. Averiyanov, P. Blanc-Benon, and V. Khokhlova, "Statistical properties of nonlinear diffracting *N*-wave behind a random phase screen," Acoust. Phys. **56**, 158–167 (2010).
- <sup>62</sup>A. Dubrovskii, V. Khokhlova, and O. Rudenko, "Fluctuation characteristics of a sonic boom after passing through randomly inhomogeneous layer," Acoust. Phys. **42**, 550–554 (1996), available at http:// www.akzh.ru/pdf/1996\_5\_623-628.pdf.
- <sup>63</sup>M. Karzova, M. Averiyanov, O. Sapozhnikov, and V. Khokhlova, "Mechanisms for saturation of nonlinear pulsed and periodic signals in focused acoustic beams," Acoust. Phys. **58**, 81–89 (2012).
- <sup>64</sup>O. Rudenko and O. Sapozhnikov, "Self-action effects for wave beams containing shock fronts," Phys. Usp. 47, 907–922 (2004).