Numerical methods for sound propagation in the atmosphere
Part I: BEM, FEM, Rays and Parabolic equations

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Outline

- Introduction: the large variety of sound propagation problems and the need for various numerical approaches
- A simple classification of the numerical approaches
- Full wave approaches (BEM, FEM)
- High frequency approximation and ray tracing
- Paraxial or parabolic equations
- Dealing with the randomness of the atmosphere

Propagation of acoustic waves in the atmosphere: a question of scales and of medium complexity

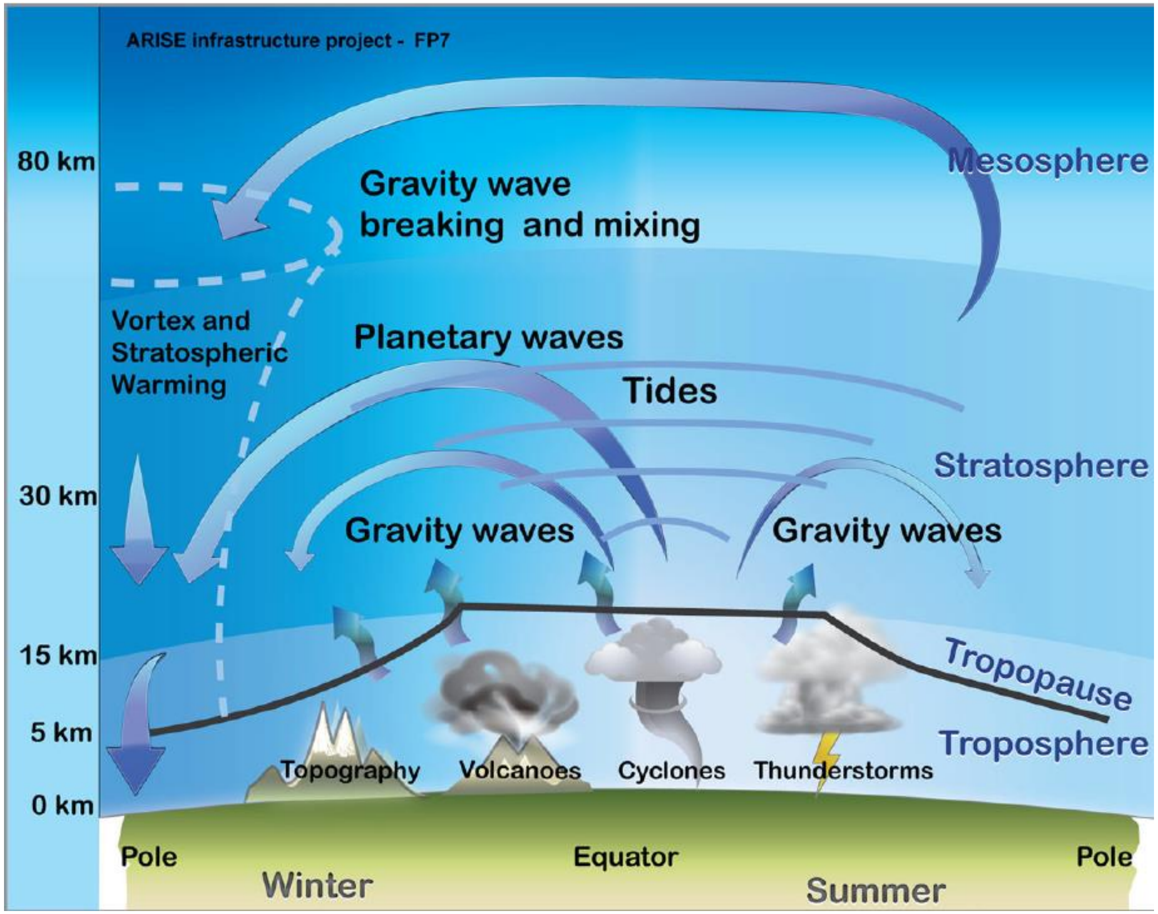
Propagation over short distances* in complex build environment (reflexions, diffraction, 3D complicated geometries). Temperature gradients and wind usually not important

*** distances have often to be compared to wavelength; short distance meaning then limited number of wavelengths***

acoustic source

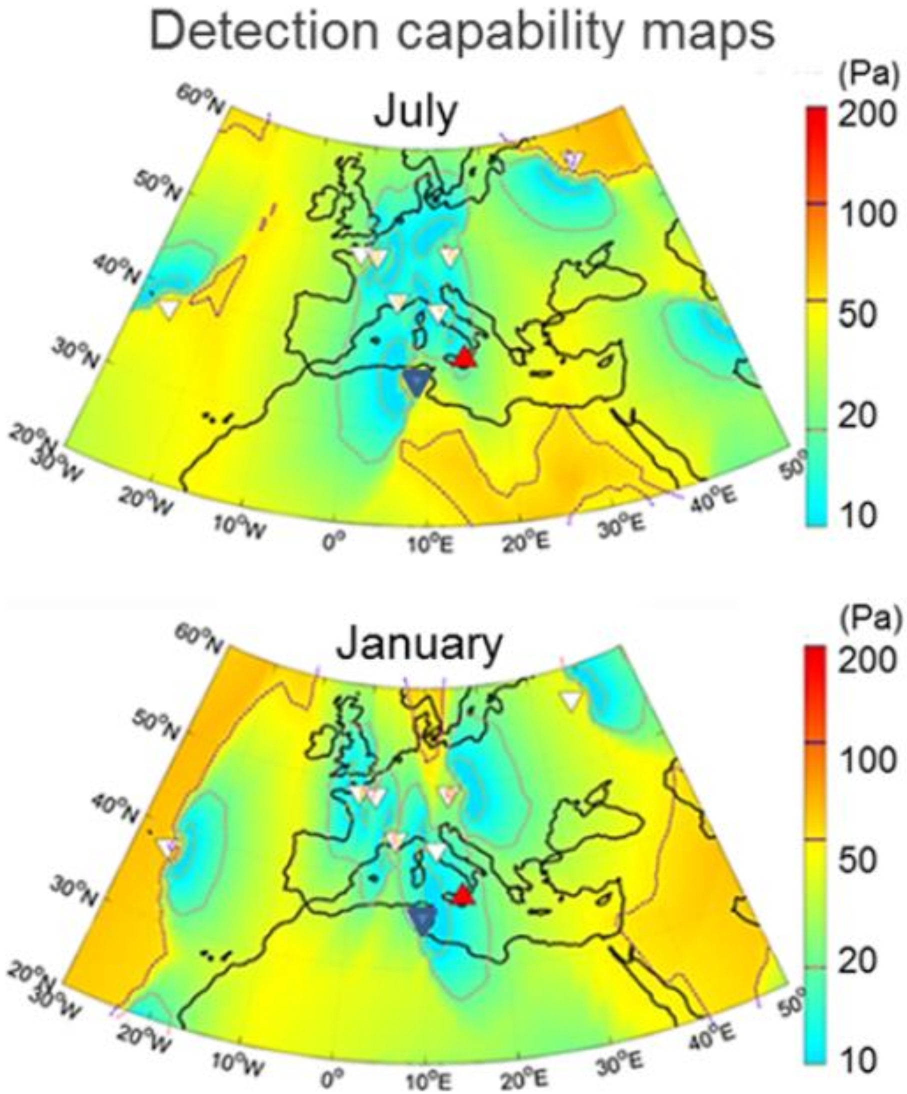
microphones

Propagation of acoustic waves in the atmosphere: a question of scales and of medium complexity



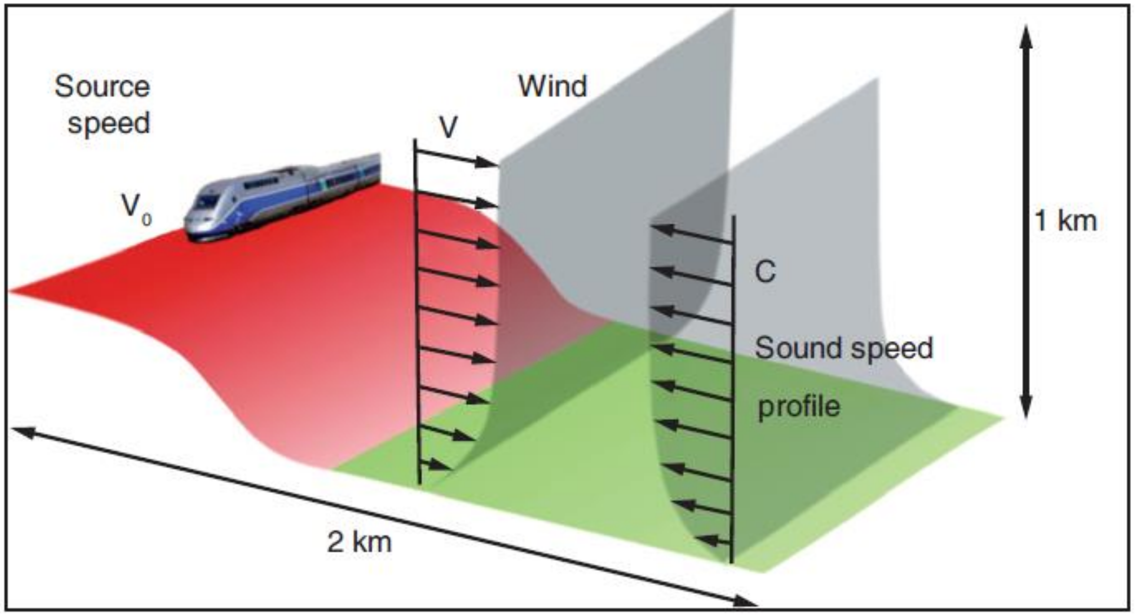
Propagation over very large distances with relatively smooth topography. Temperature gradients, wind effects very important. Random variability at smaller scales: it may be mandatory to introduce a statistical approach

Influence of seasonal variability of the atmosphere



Simulation of detection capability of explosive signals from Mt Etna (minimum amplitude detectable by an infrasound network)

Propagation of acoustic waves in the atmosphere: a question of scales and medium complexity



Sound propagation nearby a railway track.

Propagation over large distances with influence of topography near the source and of temperature and wind gradients.

Complexity of the source: rapidly moving (train, aircraft); flow generated sound (wind turbines)

To sum up

Multiple situations in terms of

- Propagation distance, from hundreds of meters to thousands of km
- Complexity of topography: from town areas to nearly flat terrain
- Complexity of physical effects to be taken into account: sound speed gradients, wind gradients, ground modelling, non linear effects for strong natural and artificial sources (volcanoes, thunder, explosions, sonic boom of supersonic aircraft …) and randomness (atmospheric turbulence)
- Complexity of noise sources: from point-like to complex aerodynamic and moving sources (trains, aircraft, meteorites for example)

Consequences

- No unique versatile numerical method to cover all these situations.
- Various model equations will be used from simple Helmholtz eq. to full Navier-Stokes eq.
- Often, for a given problem, at least 2 methods will have to be combined: a detailed, precise, method for describing the very near field of the source; and an “asymptotic ” method to propagate this near field up to distant observers.

Physics-based vs engineering (or heuristic) methods

Choosing which method to employ depends on the required fidelity, time constraints, and available computational resources. Short time constraints and reasonable fidelity requirements suggests the use of a heuristic model; these engineering models use different approximations to sound propagation and are typically used for noise mapping.
Sophisticated codes have been developed for such purposes, such as Nord2000 model or Harmonoise model.
Such approaches will be discussed in other lectures during this summer school.

In this talk we will focus on physics-based methods, which are by construction more precise but often computer- intensive and necessitate a detailed knowledge of the state of the atmosphere and of the boundary conditions.

Physics-based vs engineering (or heuristic) methods

In the following slides we will give a rapid description of what we think are the more appropriate methods to deal with "long range" sound propagation through the atmosphere.
In today's lecture we will consider only linear propagation and focus more on meteorological effects than terrain effects.
We will also focus on what can be now considered as "classical" approaches, mainly in the frequency domain.

Tomorrow Didier Dragna will discuss more deeply more recent approaches in the time-domain (FDTD) and the associated ground modelling.

A list (not complete!) of candidate numerical methods

1. **BEM: build environment; homogeneous medium at rest (or with uniform flow).**
2. **FEM for Helmholtz eq.: build environment; sound speed gradients possible.**
3. **Geometrical acoustics : sound speed and wind gradients OK; high frequency approximation; diffraction by buildings possible for simple geometries (GTD).**
4. **Paraxial eq. : sound speed gradients OK; wind possible with some approximations. Not applicable when back scattering important (town areas)**
5. **Finite Difference approximation of Euler/Navier-Stokes eq. (FDTD); all effects can be included but practical application limited to "short" distances (in terms of nb of wavelengths) due to computational cost. Often used as reference solution for benchmarking approximate methods.**

1-2: frequency domain mainly, but time domain possible. Linear acoustics.
3: freq. or time domain. Non linear extensions exist.
4: freq. domain mainly, but time domain possible. Non linear extensions exist.
5: mainly time domain (but freq. domain also possible). Linear to fully non linear.

Models and Equations

Homogeneous, non moving medium: Helmholtz equation

$$\boxed{\Delta p + k_0^2 p = 0} \; ; \; with \quad k_0 = \frac{\omega}{c_0}$$

(k_0 is the acoustic wave number)

Time dependence exp(+iωt)

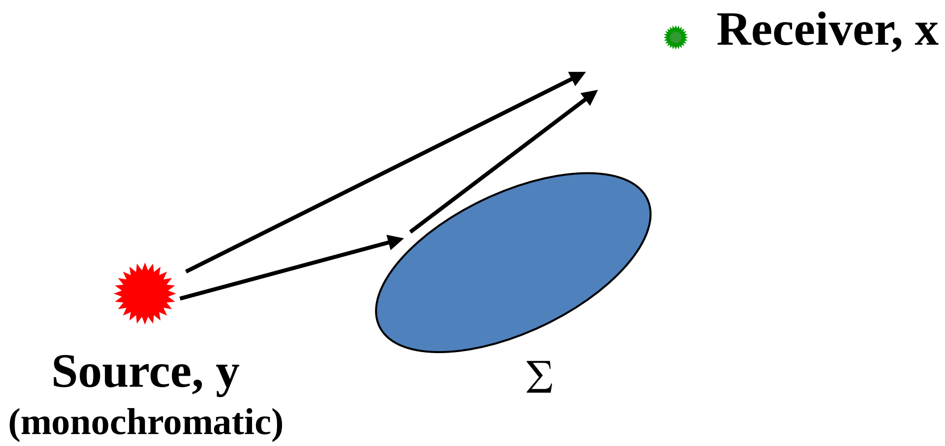
Boundary conditions (typically impedance ground; rigid bodies …)

Sommerfeld radiation condition (in 3D)

$$\boxed{\lim_{r\to\infty}\left[r\left(\frac{\partial p}{\partial r} + ik_0 p \right)\right] = 0}$$

Free field Green's function

$$\boxed{G(\vec{x}|\vec{y}) = \frac{e^{-ik_0 r}}{4\pi r} \; ; \; r = |\vec{x} - \vec{y}|}$$



Influence of temperature gradients (change of density and speed of sound)

$$\Delta p + k_0^2 p - \frac{1}{\rho_0} \nabla \rho_0 . \nabla p = 0, \text{ or}$$

$$\rho_0 \nabla \left(\frac{1}{\rho_0} \nabla p \right) + k_0^2 p = 0$$

*variation of density often negligible except for very low frequencies (infrasound typically); in all this lecture we will neglect also the effect of gravity.

No analytic form for the Green's function (in general)

Influence of flow

No exact equation for one variable only (acoustic pressure) in the more general case.

2nd order wave equations can be obtained as high frequency approximations (convected wave/Helmholtz equation; Pierce's equations - they are more general), or for a potential flow (which is not appropriate for propagation in the atmospheric boundary layer).

For a unidirectional shear flow, a 3rd order equation for the acoustic pressure can be obtained (Lilley/Goldstein equation).

For example with $\vec{V} = V_0(x_2)\vec{x}_1$

$$\boxed{\frac{D_0}{Dt}\left(\nabla\left(c_0^2\nabla p\right) - \frac{D_0^2 p'}{Dt^2}\right) - 2c_0^2\frac{dV_0}{dx_2}\frac{\partial^2 p'}{\partial x_1 \partial x_2} = 0}$$

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + V_0\frac{\partial}{\partial x_1}$$

This equation has been popular for some time in the aeroacoustics community but, due to its limited validity for arbitrary flows and the existence of unstable solutions induced by the coupling between acoustic perturbations and vorticity fluctuations, the main stream approach is now to use the linearized Euler equations (LEEs), a system of 3 coupled equations for pressure, velocity and entropy fluctuations.

Linearized Euler Equations can be written as

$$\frac{D_0\rho'}{Dt} + v'.\nabla\rho_0 + \rho_0\nabla v' + \rho'\nabla V_0 = 0$$

$$\frac{D_0 v'}{Dt} + v'.\nabla V_0 + \frac{1}{\rho_0}\nabla p' - \frac{\rho'}{\rho_0^2}\nabla p_0 = 0$$

$$\frac{D_0\,\mathrm{s}'}{Dt} + v'\nabla s_0 = 0$$

With some reasonable assumptions for atmospheric propagation (eq. correct to first order in Mach number), Ostashev & Wilson reduced the problem to the solution of only 2 coupled eq. for acoustic pressure and velocity .

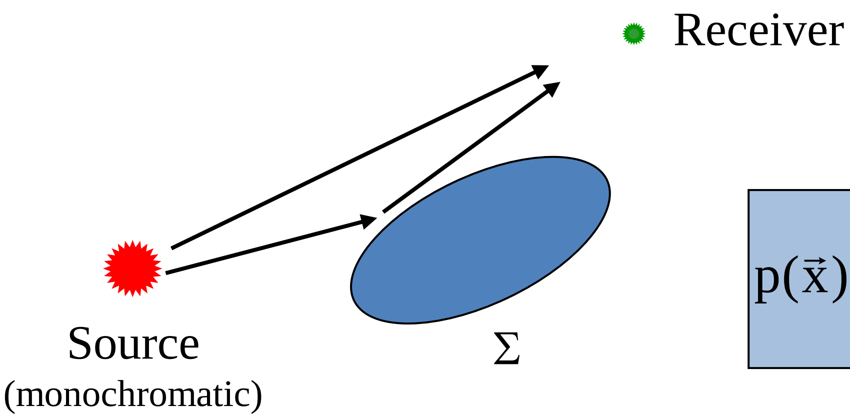
$$\frac{D_0\,\mathrm{p}'}{Dt} + +\rho_0 c_0^2\nabla v' = 0$$

$$\frac{D_0 v'}{Dt} + v'.\nabla V_0 + \frac{1}{\rho_0}\nabla p' = 0$$

These equations are often used in Finite Difference Time Domain simulations

Boundary Element Method (BEM)

Choice method for build environment when atmospheric effects can be neglected Based on the Kirchhoff-Helmholtz integral theorem; use of an analytic form of the Green's function of homogeneous Helmholtz equation (typically the free field Green's function); Sommerfeld radiation condition automatically taken into account.



Direct field

Diffracted field

$$p(\vec{x}) = \int_V f(\vec{y}) G(\vec{x}|\vec{y})\, d\vec{y} + \int_\Sigma \left(p(\vec{y}) \frac{\partial G}{\partial n_y} - G \frac{\partial p(\vec{y})}{\partial n_y} \right) d\vec{y}$$

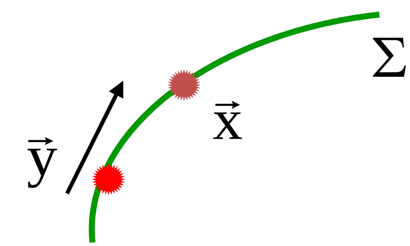
$$G(\vec{x}|\vec{y}) = \frac{e^{-ik_0 r}}{4\pi r} \; ; \; r = |\vec{x} - \vec{y}|$$

Other Green's functions can be used (a combination of similar functions for propagation above a reflective ground for example; a Hankel function for 2D problems).

The direct field is not a problem (the source is supposed to be known); it corresponds to the field which would be observed without the obstacle For the diffracted field, it is necessary to elaborate somewhat.

To determine the layer potentials (i.e. the unknown factors multiplying the Green's function and its derivative under the integral sign), it is necessary to put the observation point x on the surface Σ of the obstacle. (technically this is a bit subtle as the Green's function becomes singular, $G \sim 1/r$ and $G' \sim 1/r^2$). For a regular point (with a tangent; not a corner point for which a slightly different formula exists) the result is simply:

$$\boxed{p_D(\vec{x} \in \Sigma) = \frac{1}{2} \int_{\Sigma} \left(p(\vec{y}) \frac{\partial G}{\partial n_y} - G \frac{\partial p(\vec{y})}{\partial n_y} \right) d\vec{y}}$$



Warning: this is not a simple formula! But an integral equation, as the unknown (p) is present both inside and outside of the integral sign.

This equation is solved using a discretization of the integral by a collocation method or (usually) by a finite element approach (BEM for Boundary Element method). This step is the delicate part of the integral formulation, which necessitates the use of 2 different meshes:

1st step: Solve the integral equation which gives access to the layer potentials on the obstacles (element size typically smaller than $\lambda/6$).

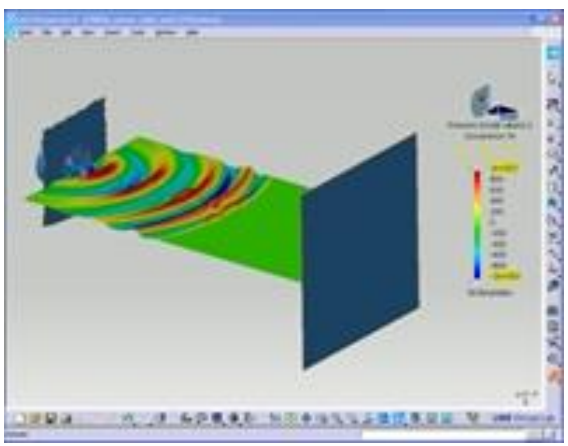
2nd step: Compute the acoustic field at various points (outside the obstacles) by the surface integration of a now perfectly known and non singular function.

A 3rd step consists simply in adding the direct field to get the final result

This approach is used in various commercial software, such as VirtualLab Acoustics (LMS/Siemens) or the Acoustic BEM module of VA ONE from ESI Group.

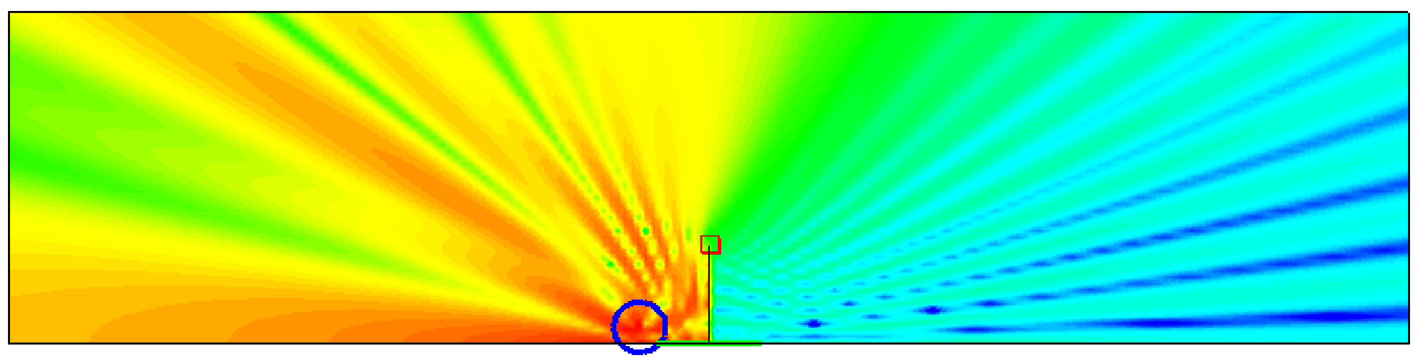
The boundary element method effectively reduces complex 3D geometry to 2D surface dimensions. Only the surface areas of the structural systems that are vibrating or scattering sound need to be modeled.

They are however some specific difficulties, such as the existence of “irregular frequencies”, or the fact that the matrices resulting from the FE discretization are full and complex. These questions are now well treated in commercial software, with efficient numerical techniques such as the Fast Multipole Method (FMM) implementation of BEM.
We refer the reader to the websites of the companies, and to the book by N. Atalla & F. Sgard, Finite Element and Boundary Methods in Structural Acoustics and Vibration, CRC Press, 2015.



Note that BEM can be formulated also in time domain (example from TDBEM in LMS Virtual Lab)

BEM computation of noise barrier efficiency



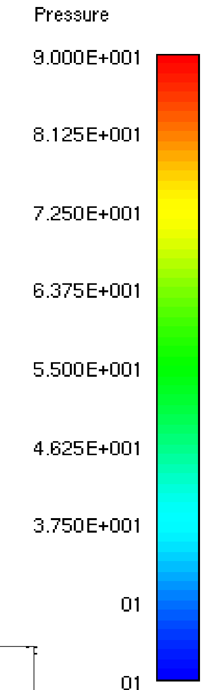
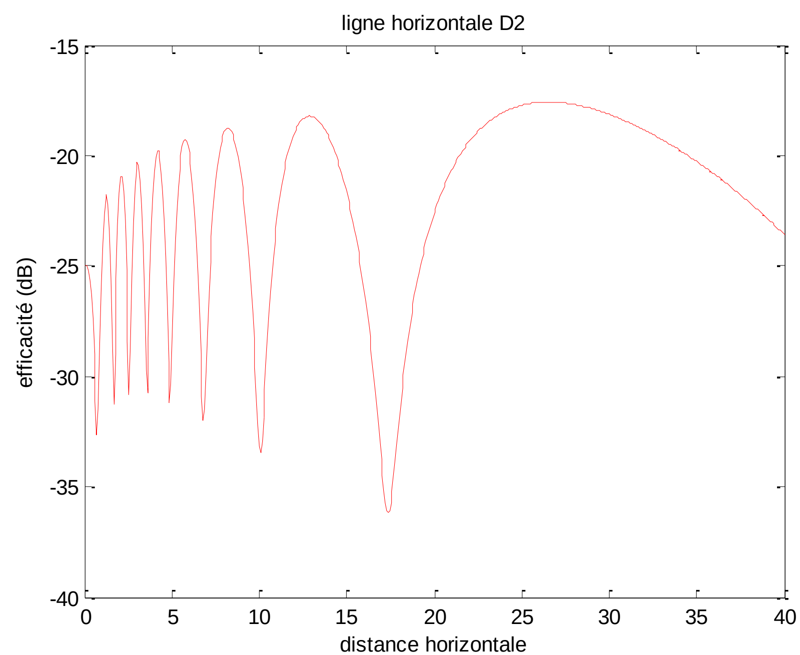


Illustration of the efficiency of the noise barrier: noise reduction in dB; difference between results obtained without and with the barrier (at mid height of the screen).

Note the existence of a strong interference pattern *(warning: map and efficiency plot do not correspond to the same frequency)*



FEM for non homogeneous Helmholtz equation

Main advantage relative to BEM: volume meshing allows for variation of density and speed of sound induced by temperature gradients. Very well adapted to encapsulated problems in which the areas to be meshed are clearly delimited.

Main drawbacks:

Due to volume meshing with element size smaller than $\lambda/6$ typically, the method is limited to low frequencies or short propagation range.

The presence of a numerical boundary at small distances from the source region may produce artificial reflexions (no way to reproduce simply the Sommerfeld radiation condition). In the past this was a real difficulty; a numerical trick sometimes used was the adjunction of a layer of so-called “infinite elements” around the FE domain. Nowadays the preferred approach is to add a rather limited region around the classical FE mesh in which a very rapid absorption of acoustic waves takes place with (nearly) no spurious reflexions, by achieving an impedance matching at the boundary of the FE domain: the Perfectly Matched Layer. PML zones are routinely used in commercial codes, whether specific to acoustic propagation (Virtual Lab for example) or generic FE codes, such as Comsol Multiphysics.

Perfectly Matched Layers

In this technique, initially developed for electromagnetic fields (Bérenger,1994), only finite elements are used. But in a region limiting the computational domain, the equations to be solved are changed in order to provide a very strong absorption of the incident waves.

For this an imaginary part of the wavenumber is introduced to provide an exponential decay of the acoustic waves inside the absorbing zone (this is similar to, but much stronger than the "real" absorption due to viscous and thermal effects in the atmosphere)
The exponential decay has to be sufficiently important so that the waves arriving at the exit boundary are of negligible amplitude and then the numerical BC at exit plays no role (we can impose a hard wall condition for example).
In practice numerical codes (such as Comsol Multiphysics) use a transformation of coordinates (into the complex plane) in the direction normal to the boundary (analytic continuation).

In Cartesian coordinates:

$$\hat{x}(x) = x - \frac{i}{\omega}\int_{x_0}^{x} \sigma(\xi)\,\mathrm{d}\,\xi \qquad \gamma(x) = 1 - \frac{i}{\omega}\sigma(x)$$

Solving the Helmholtz equation in the modified coordinate system is then equivalent to solve

$$\frac{d^2 p}{d\hat{x}^2} + k_0^2 p = 0$$

$$\frac{d^2 p}{dx^2} + \gamma^2 k_0^2 p = 0$$

(we have considered σ as constant to simplify)

which is a modified Helmholtz equation, with a complex wavenumber, whose imaginary part induces an exponential decay of the signal; for a right going wave:

$$k = k_0\left(1 - \frac{i\sigma}{\omega}\right);\ p(x) = \exp(-ik_0 x)\exp(-\frac{k_0\sigma}{\omega}x)$$

Denoting by Δ the depth of the PML, the incident amplitude is reduced by a factor:

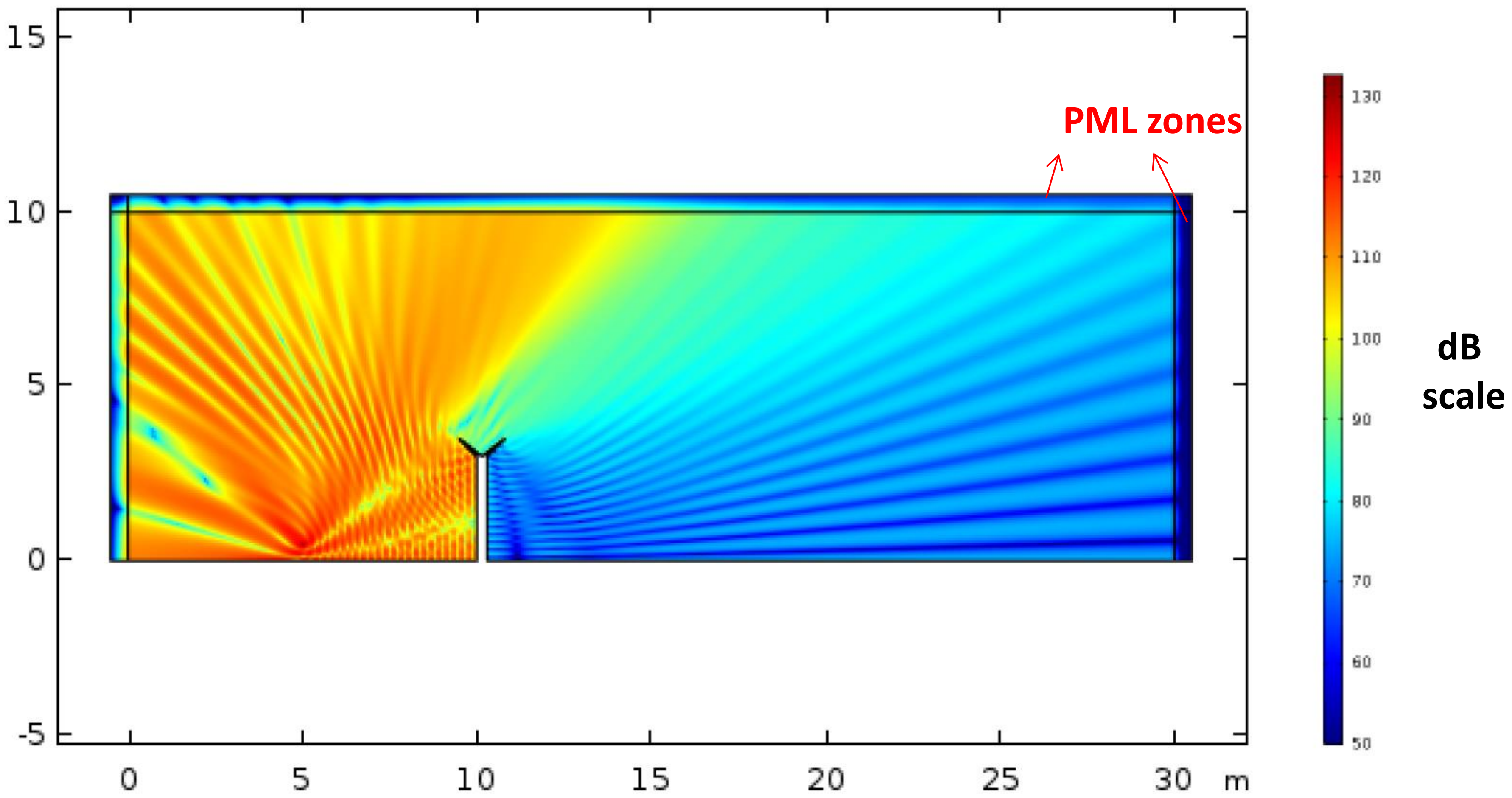
$$\exp(-\frac{k_0\sigma}{\omega}\Delta) = \exp(-\frac{\sigma}{c_0}\Delta)$$

At the exit of the PML zone, the residual amplitude can be made so small that the reflective wave will be negligible, whatever the imposed BC.

There is also no reflexion at the entrance of the PML zone, as velocity and pressure are continuous functions at the interface (say in $x=0$), and this is true even if a very strong attenuation is selected; hence the name Perfectly Matched Layer: the acoustic impedance in the PML is matched to the medium value and therefore no reflexions can be generated.

The remarkable result is that this property of non reflexion at the interface, which is easy to obtain for normal incidence by using the simple BC $Z=Z_c=\rho c$), is also verified for oblique incidence, the attenuation being introduced only in the direction normal to the interface.

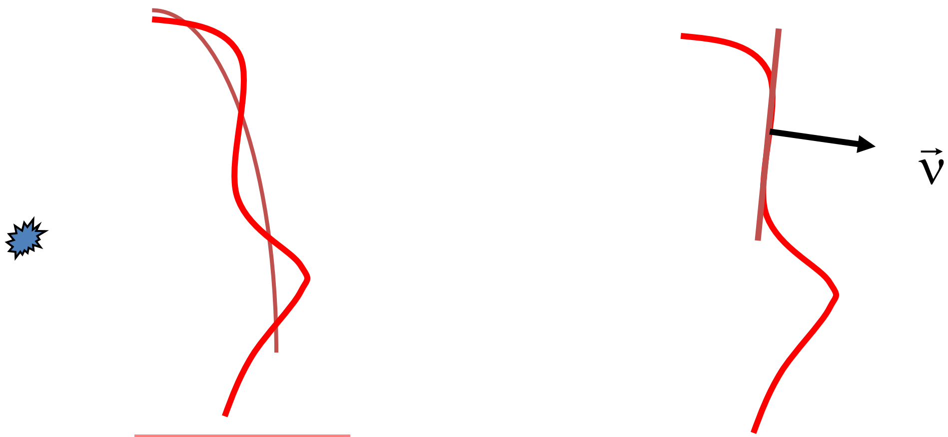
FEM computation of diffraction by a 2D noise barrier (f=800Hz)



Note the absence of visible reflections at the limits of the FE domain, and the extremely rapid decrease of pressure level inside the (quite small) PML zone

Ray tracing

The physical idea is to extend the concept of plane waves to non homogeneous and moving media. We will then define *locally plane waves*, associated to the tangent plane of the wavefront.



$$\tau(\vec{x}) = \text{cte}$$

(wavefront, i.e. position reached by acoustic perturbation τ seconds after emission at source position)

$$\boxed{\begin{aligned} p(\vec{x}) &= A(\vec{x}) \exp(-i\,\psi(\vec{x})) \\ \vec{K}(\vec{x}) &= \vec{\nabla}\psi = \omega \vec{\nabla}\tau \end{aligned}}$$

Homogeneous medium; Plane wave

$$p(\vec{x}) = A \exp(-i\vec{K}\,\vec{x});$$

$$\text{ie} \;\; \psi = \vec{K}\,\vec{x}\,;\vec{K} = \omega/c_0\,\vec{v}$$

The approximation will be better and better as the ratio between the wavelength and the characteristic length scale L of the non homogeneities decreases. Formally the equations of geometrical acoustics are obtained for the limit

$$\boxed{\lambda/L \to 0 \; (\Leftrightarrow f \to \infty)}$$

This physical idea of the existence of a small parameter λ/L, or equivalently a large one (angular frequency ω) can be formalized through a development in an asymptotic series. We illustrate below the mathematical mechanism on the simple Helmholtz equation (variable speed of sound, but density kept constant). The procedure can be extended to more complicated equations, including LEEs in the presence of an arbitrary flow.

The first step consists in introducing a "locally plane" form for the pressure fluctuation.

$$\Delta p + k^2(\vec{x})p = 0; k(\vec{x}) = \frac{\omega}{c(\vec{x})}$$

$$p = A(\vec{x})e^{-i\omega\tau(\vec{x})}$$

$$\frac{\partial p}{\partial x_i} = \left[\frac{\partial A}{\partial x_i} - i\omega\frac{\partial \tau}{\partial x_i}A\right]e^{-i\omega\tau}$$

K^2

$$\frac{\partial^2 p}{\partial x_i^2} = \left[\frac{\partial^2 A}{\partial x_i^2} - 2i\boxed{\omega\frac{\partial \tau}{\partial x_i}}\frac{\partial A}{\partial x_i} - i\omega\frac{\partial^2 \tau}{\partial x_i^2}A - \boxed{\omega^2\frac{\partial \tau}{\partial x_i}\frac{\partial \tau}{\partial x_i}}A\right]e^{-i\omega\tau}; \quad k^2 p = \frac{\omega^2}{c^2}$$

K$_i$

A/L^2 << $A/L\lambda$ << A/λ^2 A/λ^2

An asymptotic series is then introduced using the small parameter 1/ω ***(physically λ/L)***

$$A = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{\omega^n}$$

Helmholtz eq.:

$$\Delta A_0 + \sum_{1}^{\infty} \frac{1}{\omega^n} \Delta A_n - 2i\omega \vec{\nabla}\tau \vec{\nabla} A_0 - 2i \sum_{1}^{\infty} \frac{1}{\omega^{n-1}} \vec{\nabla}\tau \vec{\nabla} A_n - i\omega \Delta\tau A_0 - i\Delta\tau \sum_{1}^{\infty} \frac{1}{\omega^{n-1}} A_n$$

$$-\omega^2 \left(\vec{\nabla}\tau\right)^2 A_0 - \left(\vec{\nabla}\tau\right)^2 \sum_{1}^{\infty} \frac{1}{\omega^{n-2}} A_n + \frac{\omega^2}{c^2} A_0 + \frac{1}{c^2} \sum_{1}^{\infty} \frac{1}{\omega^{n-2}} A_n = 0$$

And by grouping terms in decreasing powers in ω:

$$\omega^2 \quad : \quad \left[\frac{1}{c^2} - \left(\vec{\nabla}\tau\right)^2 \right] A_0 = 0$$

$$\omega \quad : \quad -2i\vec{\nabla}\tau \vec{\nabla} A_0 - i\Delta\tau A_0 + \cancel{\left[\frac{1}{c^2} - \left(\vec{\nabla}\tau\right)^2 \right]} A_1 = 0$$

This series can be pursued, but in practice only the first two eq. are used

The first eq. is known as the Eikonal equation; it will allow to compute the paths followed by the acoustic energy (acoustic rays).
The second one will give access to the pressure amplitude along each ray.

The eikonal equation is a first order non linear equation and it can be solved by the method of characteristic lines

$$\left[\frac{1}{c^2}-\left(\vec{\nabla}\tau\right)^2\right]=0$$

or

$$\left|\vec{K}\left(\vec{x}\right)\right|^2=\frac{\omega^2}{c\left(\vec{x}\right)^2}$$

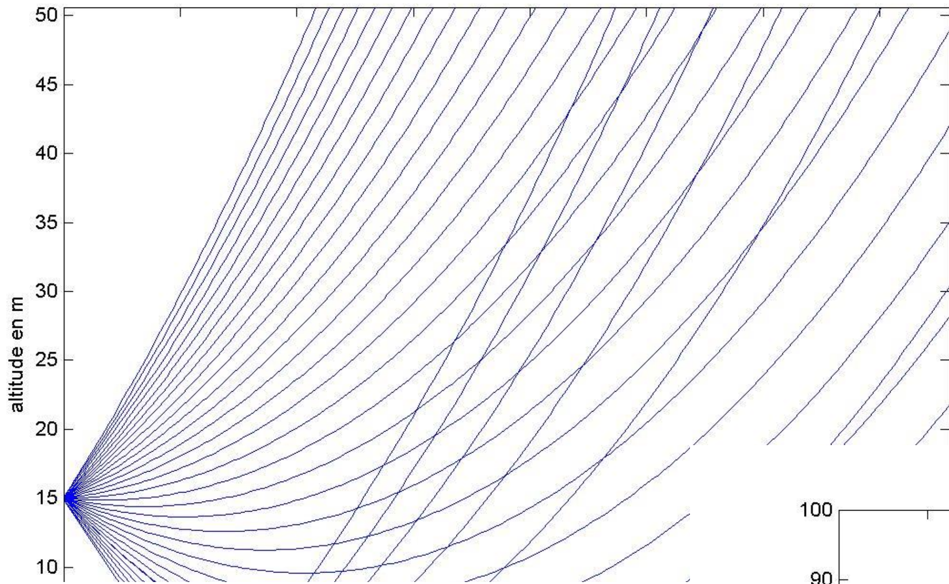
This results in a system of coupled first order ODEs which are easily solved using for example a Runge-Kutta method.

$$\frac{dx_i}{dt}=c\,\frac{K_i}{K}$$

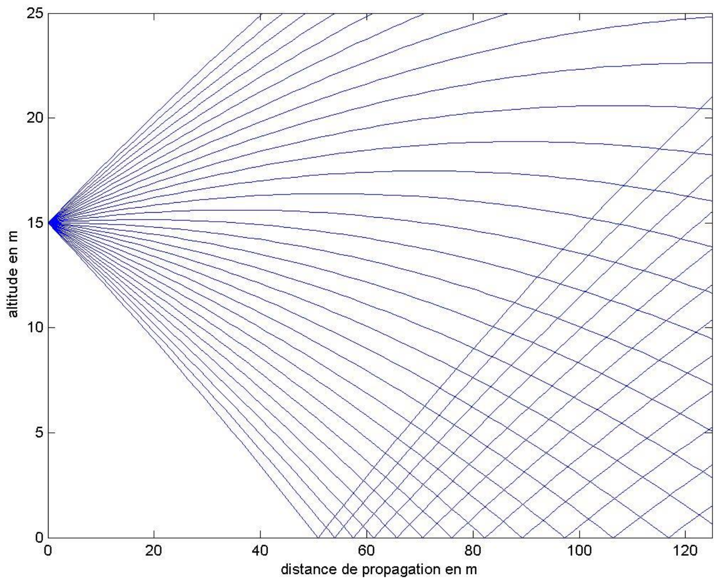
$$\frac{dK_i}{dt}=-K\,\frac{\partial c}{\partial x_i}$$

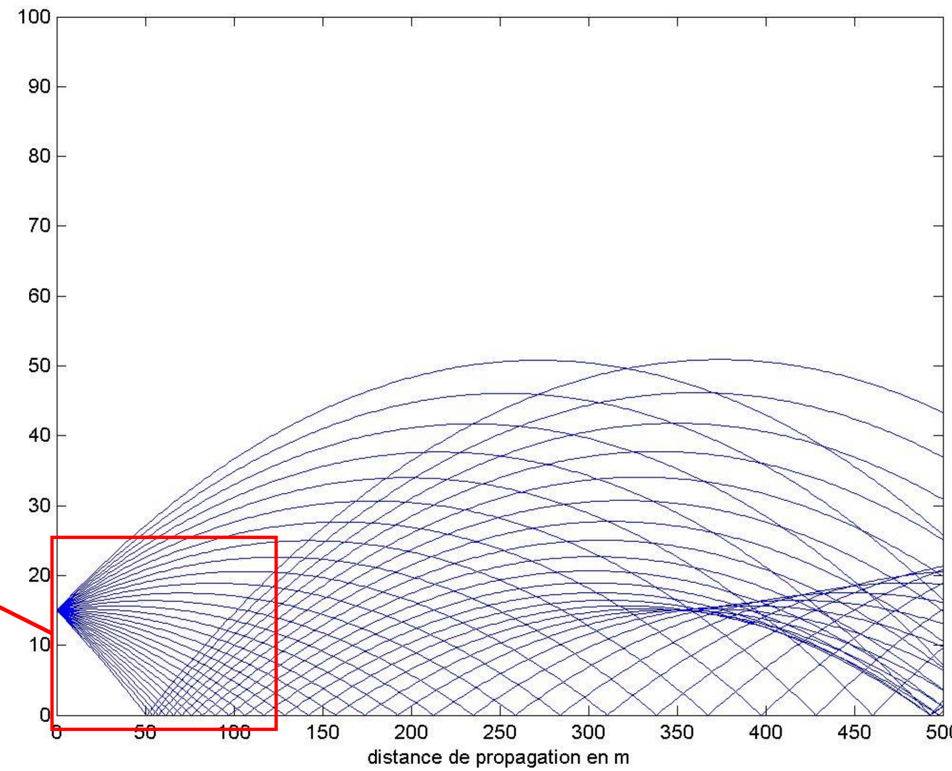
The first system describes the convection of acoustic waves at the local speed of sound.
The second system is associated to the refraction of the wave by the sound speed gradient .
(x is the courant position along the ray, and t is the travel time along the ray).

Sound speed decreasing with height



Sound speed increasing with height





Specular reflection on the ground surface

Ray equations in the presence of an arbitrary flow: a very similar form

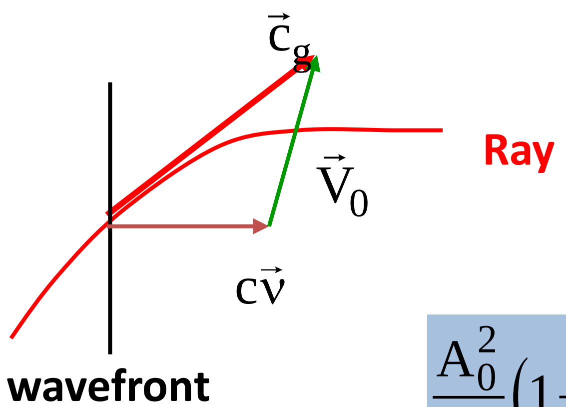
$$\frac{dx_i}{dt} = c\frac{K_i}{K} \boxed{+ V_{0_i}}$$

$$\frac{dK_i}{dt} = -K\frac{\partial c}{\partial x_i} \boxed{- K_j \frac{\partial V_{0_j}}{\partial x_i}}$$

Convection of acoustic wave by the local flow velocity. As a consequence rays are no longer orthogonal to wave fronts!

Refraction by velocity gradients

Group velocity $\vec{c}_g = c\vec{\nu} + \vec{V}_0$

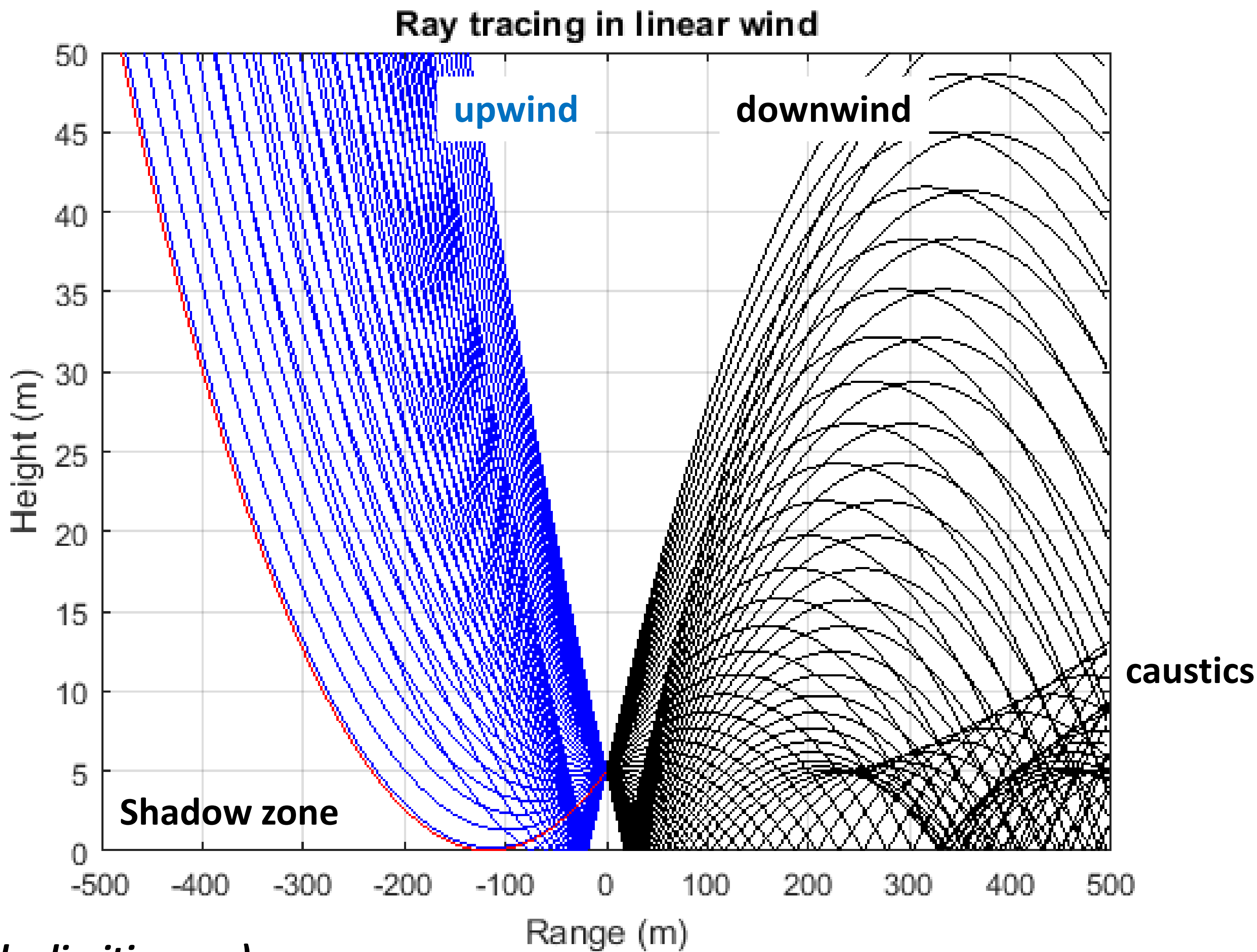


Conservation of acoustic energy flux through a ray tube (consequence of the 2nd eq. in the asymptotic series) gives access to the pressure amplitude along each ray.

$$\frac{A_0^2}{\rho c}\left(1+\vec{M}.\vec{\nu}\right)\left|\vec{\nu}+\vec{M}\right| d\Sigma = cte$$

M is the local Mach number V/c
dΣ the area of an infinitesimal ray tube and ν the unit normal to the wavefront.

*** *In practice Σ will be estimated by solving an other set of ODEs for the geodesic elements of the ray system***



(in red the limiting ray)

Launch angles $\pm 15°$

Ray tracing pros and cons

Ray tracing is very simple to implement; it gives a clear physically appealing, qualitative, view of sound propagation in a complex environment.

But some difficulties appear when willing to obtain quantitative information (sound levels):

Shadow zones: no rays and so no other information than “low sound level”.

Caustics: neighbouring rays cross and then amplitude is theoretically infinite; in practice the prediction of level will be unprecise.

These 2 difficulties are a consequence of having neglected all diffraction effects, by taking a limit of “infinite” frequency.
“Corrections” have been proposed but in practice, when it is necessary to estimate precisely the sound pressure level in these regions, it is better to turn to a wave approach such as paraxial approximations.

Ray tracing pros and cons

Outside these problematic zones, computing sound level is a bit complicated as ray tracing is fundamentally a Lagrangian approach: we trace acoustic trajectories, not knowing in advance where they will arrive.

To compute the sound level at a given point, one has to determine all the rays joining the source to the receiver (the so called eigenrays) and add the different contributions in amplitude and phase. This can be tedious especially in 3D.

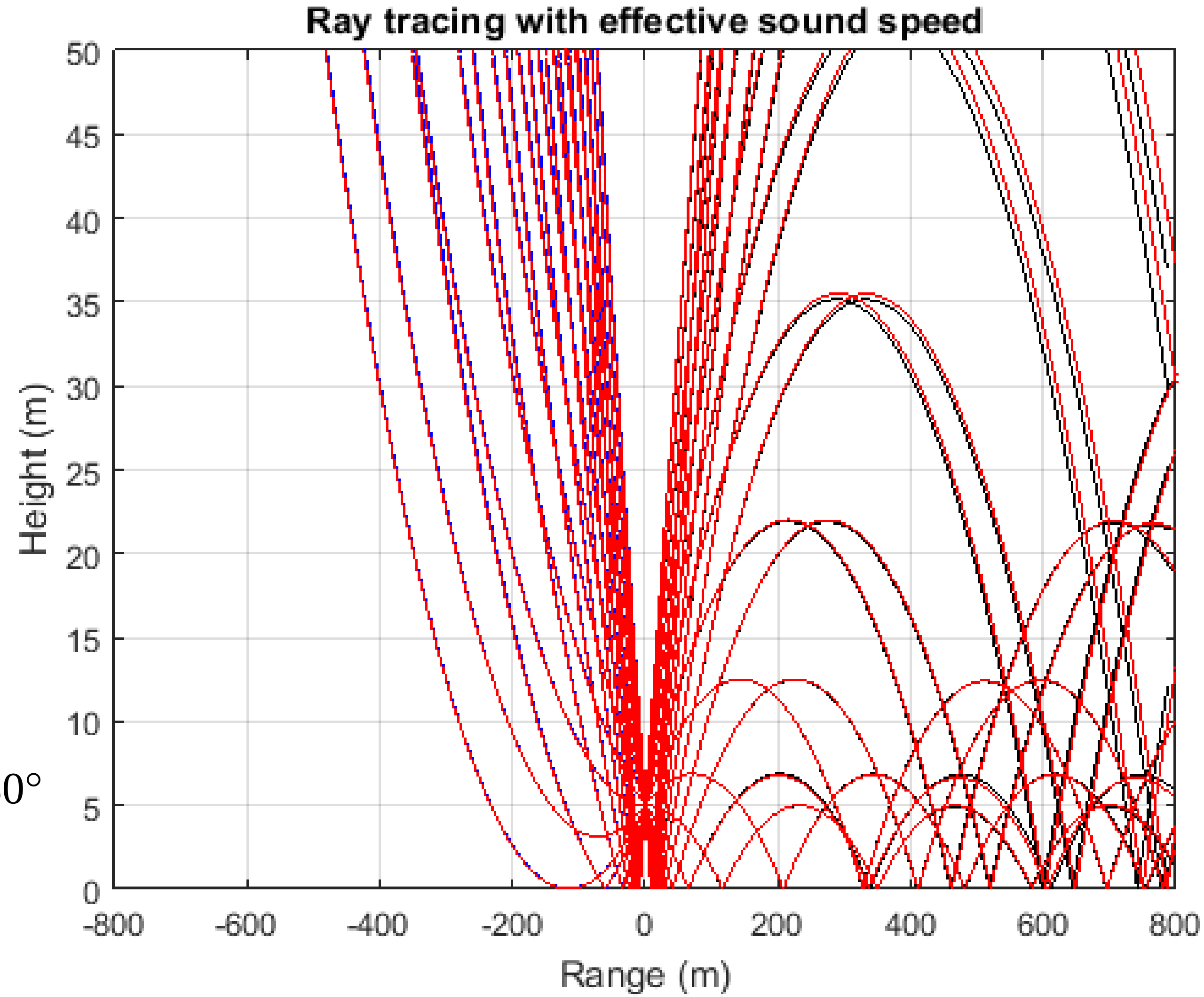
To finish, it is important to note that an eigenray corresponds to an extremum value of the travel time between source and receiver (Fermat's principle); the stability of the travel time relative to small variations of the ray path (induced by say local temperature variations) is central to the development of travel time acoustic tomography, an inversion technique used to infer the structure of thermal and velocity fluctuations in the atmosphere from acoustic measurements.

Effective sound speed

As shown above, the presence of flow changes dramatically the equations of sound propagation. In the high frequency limit we have seen that the propagation becomes anisotropic due to the vector nature of the velocity; as a consequence rays are no more normal to wavefronts.
However when the velocity is small, as is the case for sound propagation near the ground, it is possible to introduce an approximate method to take into account the effect of wind via an "effective sound speed"; this trick permits the use of wave methods developed for a non homogeneous medium at rest, for example the parabolic equation method.
The wind is supposed to be nearly horizontal with component V, and the vertical propagation angle is also supposed to be small.
The effective sound speed is given by

$$c_{eff} = c + V\cos\psi$$

ψ **being the azimuthal angle (between the vertical plane containing source and receiver and the wind velocity);** ***note that out-of-plane effects are thus neglected.***
The following slide offers a comparison of ray traces obtained with the exact equations and the "scalar" ones obtained with the effective sound speed (no temperature gradient; linear evolution of wind with height).



Launch angles $\pm 30°$

Exact rays (black) vs effective sound speed rays (red): differences are noticeable only for large values of emission angle

Parabolic equations

Parabolic (or paraxial)approximations are trying to combine numerical efficiency (compared to FE solution of Helmholtz eq.) with partial inclusion of diffraction effects.

The key idea is to construct one-way equations, that is equations describing only the forward propagation of waves (from source to receiver) in contrast to the classical Helmholtz equation (a two-way eq.) supporting both forward propagating and backward propagating waves.

This can be seen as a generalization of the decomposition of the 1D wave eq. with constant speed of sound:

$$\frac{d^2p}{dx_1^2} + k_0^2 p = 0 \Leftrightarrow \left(\frac{d}{dx_1} + ik_0\right)\left(\frac{d}{dx_1} - ik_0\right)p = 0 \Leftrightarrow$$

$$p = p^+ + p^-$$

$$\left(\frac{d}{dx_1} + ik_0\right)p^+ = 0$$ **Forward propagating wave (with exp(+iωt) time dependence)**

$$\left(\frac{d}{dx_1} - ik_0\right)p^- = 0$$ **Backward propagating wave;**
coupling between the 2 waves can occur only through a reflecting BC.

For a stratified medium, in which the refraction index does not depend on x_1 (in what follows we limit ourselves to 2D propagation in a vertical plane), we can write formally:

$$\frac{\partial^2 p}{\partial x_1^2}+\frac{\partial^2 p}{\partial x_2^2}+k_0^2 n^2(x_2)p=0 \Leftrightarrow \left(\frac{\partial}{\partial x_1}+ik_0 Q\right)\left(\frac{\partial}{\partial x_1}-ik_0 Q\right)p=0$$

$$Q^2=n^2\left(x_2\right)+\frac{1}{k_0^2}\frac{\partial^2}{\partial x_2^2} \qquad n(x_2)=\frac{c_0}{c_{eff}(x_2)}$$

Where c_0 is a "reference value" of the speed of sound, for example the value at the source position, or an average over some vertical distance.
This factorization is exact as Q does not depend on x_1 and then commutes with the x_1 derivative. In this case a one-way equation can be formally written for the forward propagating wave as

$$\left(\frac{\partial}{\partial x_1}+ik_0 Q\right)p^+=0$$

This eq. is or parabolic type as only a first order derivative along x_1 appears. Efficient "marching" algorithms in x_1 direction are available for such a parabolic eq., which is similar to the time-dependent heat equation (x_1 playing here the role of time).

How to obtain an explicit form to Q?

The physical idea is to consider Q as a slightly perturbed operator; if we write Q as

$$Q=\left(n^2+\frac{1}{k_0^2}\frac{\partial^2}{\partial x_2^2}\right)^{1/2}=\left(1+\underbrace{\left(n^2-1\right)}_{\varepsilon<<1}+\underbrace{\frac{1}{k_0^2}\frac{\partial^2}{\partial x_2^2}}_{\mu=?}\right)^{1/2}$$

It is clear that ε is a small parameter for sound propagation in the atmosphere, typically smaller than 0.1; to interpret μ let us consider a plane wave inclined by an angle θ relative to the horizontal direction.

$$p\sim e^{-ik_0\cos\theta x_1}e^{-ik_0\sin\theta x_2}$$

$$\mu p=\frac{1}{k_0^2}\frac{\partial^2 p}{\partial x_2^2}=-\sin^2\theta\, p;\text{ie } \mu\sim\sin^2\theta$$

It is now clear that μ can also be considered as a small parameter for waves propagating at small angles relative to the horizontal direction .

A Taylor expansion is thus quite natural and we have to first order

$$Q \approx 1 + \frac{\varepsilon}{2} + \frac{\mu}{2} = 1 + \frac{n^2 - 1}{2} + \frac{1}{2k_0^2}\frac{\partial^2}{\partial x_2^2}$$

and the equation for p is then

$$\frac{\partial p}{\partial x_1} + ik_0 p + i\frac{n^2 - 1}{2}k_0 p + \frac{i}{2k_0}\frac{\partial^2 p}{\partial x_2^2} = 0$$

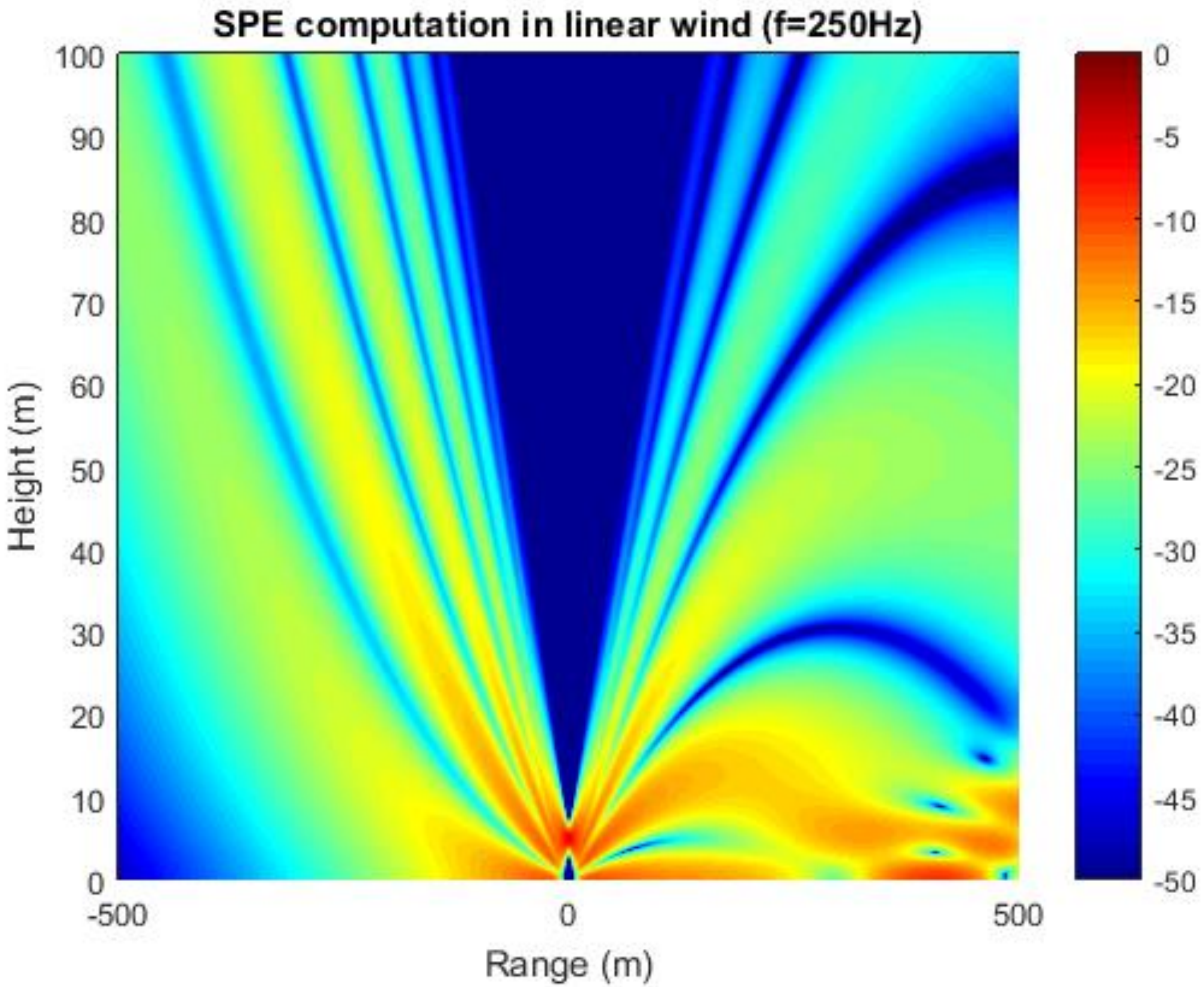
which is known as the Standard Parabolic Equation SPE, developed in the early 70s for sound propagation in the ocean (Fred Tappert).

To remove rapid evolutions of p at the wavelength scale, a further change of variable is often used, by introducing the "envelope" of the pressure wave ψ:

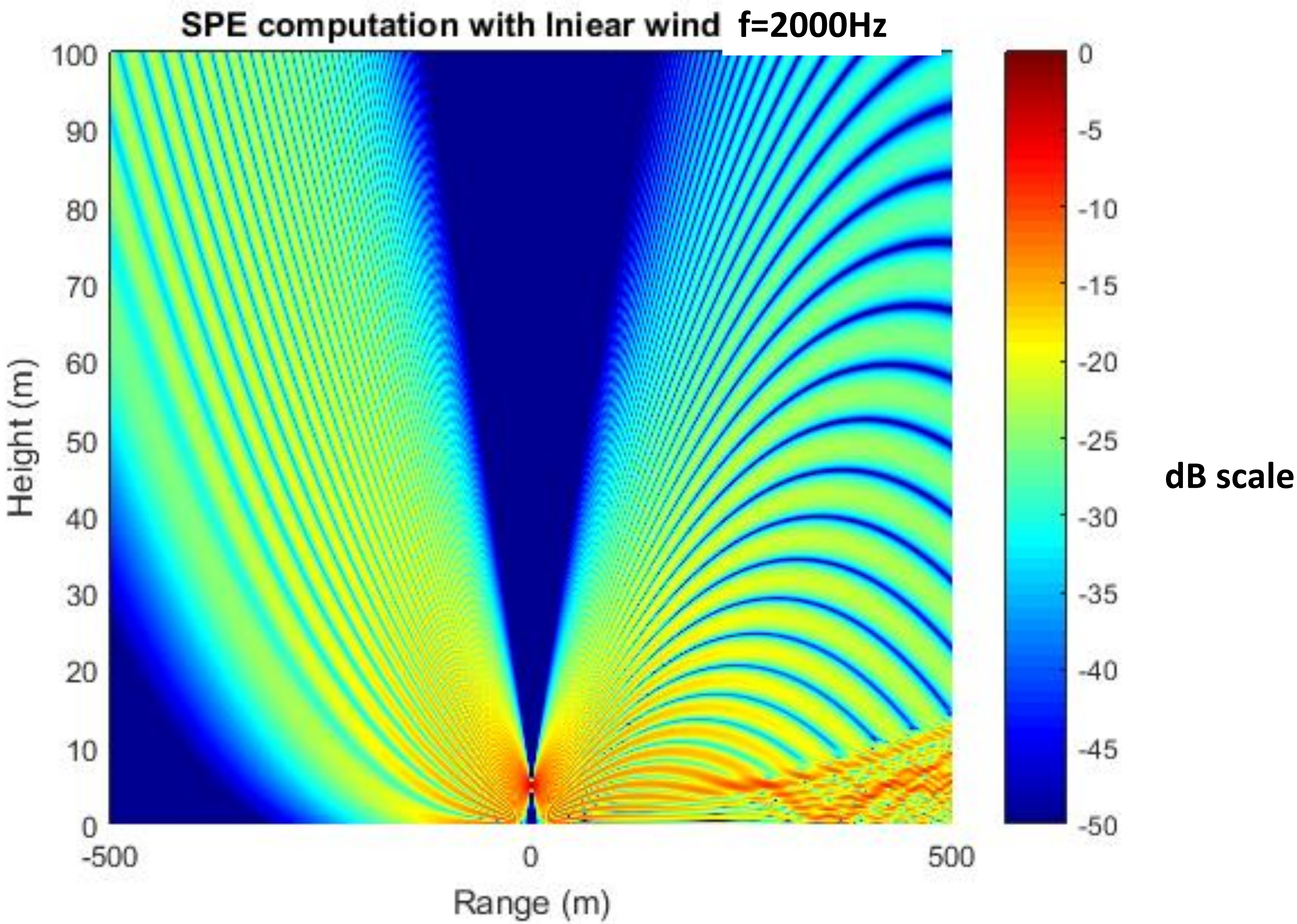
$$p = \psi e^{-ik_0 x_1}$$

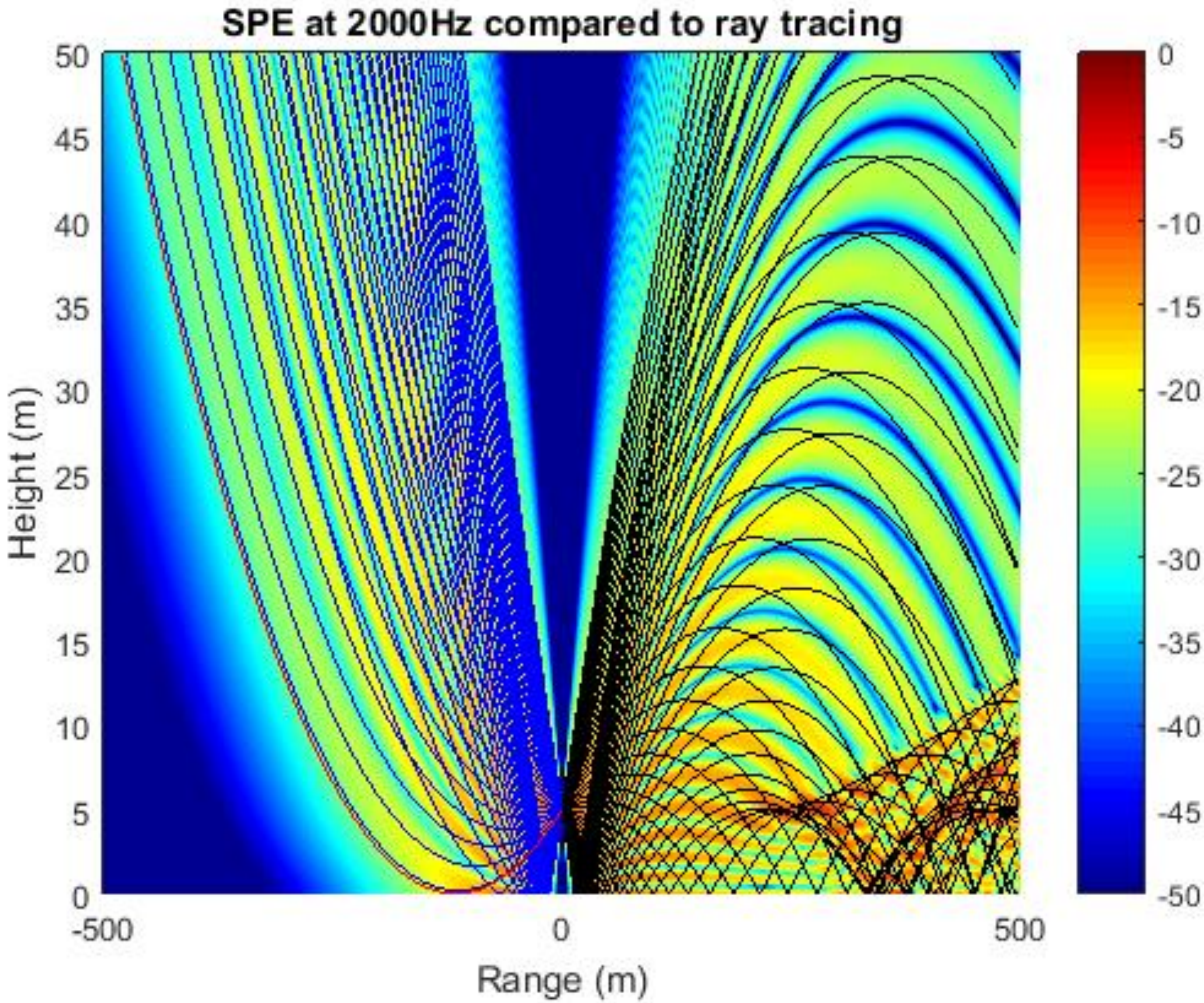
$$-2ik_0\frac{\partial \psi}{\partial x_1} + \frac{\partial^2 \psi}{\partial x_2^2} + k_0^2\left(n^2 - 1\right)\psi = 0$$

This equation can be solved via spatial Fourier transforms (split step Fourier algorithm, Green's function parabolic eq., GFPE) or finite differences (Crank-Nicholson PE, CNPE) or FE.


SPE computation in linear wind (f=250Hz)
Height (m)
Range (m)
100
90
80
70
60
50
40
30
20
10
0
-500
0
500
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50

dB scale


SPE computation with lniear wind f=2000Hz
Height (m)
Range (m)
100
90
80
70
60
50
40
30
20
10
0
-500
0
500
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50
dB scale


SPE at 2000Hz compared to ray tracing
Height (m)
Range (m)
50
45
40
35
30
25
20
15
10
5
0
-500
0
500
0
-5
-10
-15
-20
-25
-30
-35
-40
-45
-50

The SPE is considered to give excellent results for propagation angles lower than 15-20° . In some cases a higher range of validity is important.

One simple idea would be to consider a higher order Taylor expansion of the square root operator (the resulting PE would involve 4th order derivatives in the plane normal to x_1).

In practice a different approach is used. The square root is approximated by a rational function expansion, or Padé approximant.

The classical (1,1) approximation is given by

$$\sqrt{1+x} \sim 1+\frac{x}{2}-\frac{x^2}{8}+\ldots$$

$$\sqrt{1+x} \sim \frac{1+px}{1+qx} = 1+(p-q)x-q(p-q)^2x^2+\ldots$$

By matching expansions up to order 2, we get

$$\left.\begin{array}{l} p-q=\dfrac{1}{2} \\ q(p-q)=\dfrac{1}{8} \end{array}\right\} \Rightarrow q=\frac{1}{4}; p=\frac{3}{4}$$

$$\sqrt{1+x} \sim \frac{1+\frac{3}{4}x}{1+\frac{1}{4}x} \quad \text{ou} \quad \left(1+\frac{1}{4}x\right)\sqrt{1+x} \sim 1+\frac{3}{4}x$$

And the resulting parabolic equation (for p) is given by

$$\left(1+\frac{\varepsilon}{4}+\frac{\mu}{4}\right)Q=1+\frac{3\varepsilon}{4}+\frac{3\mu}{4}$$

$$\frac{n^2+3}{4}\frac{\partial p}{\partial x_1}+\frac{1}{4k_0^2}\frac{\partial^3 p}{\partial x_1 \partial x_2^2}=-ik_0\frac{3n^2+1}{4}p-\frac{3i}{4k_0}\frac{\partial^2 p}{\partial x_2^2}$$

This 3rd order Wide Angle PE (but 1st order in x_1!) is due to Claerbout (with application to seismic migration) and is considered to be accurate for propagation angles up to 45°.

It is usually solved by a Crank-Nicholson finite difference approach. (Wide Angle CNPE).

This equation can be generalized to include the effects of density gradients and of flow (with some approximations).

For details, we refer to the book by V. Ostashev & K. Wilson, Acoustics in Moving Inhomogeneous Media (2nd Edition), CRC Press, 2015.

See also Computational Ocean Acoustics (2nd Edition), F. Jensen, W. Kuperman, M. Porter & H. Schmidt, Springer, 2011.

Inclusion of (volume) random fluctuations in numerical simulations of sound propagation in the atmosphere

Up to now we have considered that the propagation medium was inhomogeneous and possibly moving, but deterministic; we have not considered the intrinsic randomness of the atmosphere ("turbulence"). We have taken into account only "mean values" of the local speed of sound and of the wind. The inclusion of randomness can however have some very significant effects on the mean acoustic level (notably in shadow zones) and on the 2-point coherence of the pressure field, which has a profound impact on the detection and localization of acoustic sources using microphone arrays.

We will consider here only "line of sight" propagation, not scattering off the main axis of propagation. In such a case methods based on parabolic equations are the most appropriate numerical tools for long range propagation, but FDTD techniques (based on LEEs) can also be considered (see lecture by Didier Dragna).

The main difficulty when dealing with random fluctuations of the propagation medium is that we have to estimate statistically averaged values of the pressure, intensity (pressure squared), and product of pressure fluctuations computed at 2 different points, or coherence functions (to limit ourselves to 2-point statistics).

Two approaches are possible:

- **Obtaining and solving equations for these averaged values**
- **Solving "deterministic" equations for a large number of individual realizations (or snapshots) of the random medium and performing averaging in a post-processing step.**

Equations

Relatively small random fluctuations will be superimposed on the deterministic (averaged) model of the atmosphere. In the (standard) parabolic equation framework, the equation to be considered for the wave envelope (or complex amplitude) writes as

$$-2ik\frac{\partial \Psi}{\partial x_1} + \nabla_T^2 \Psi + \left(n_{eff}^2 - 1 + 2k^2 \varepsilon\right)\Psi = 0$$

$\nabla_T^2 \Psi$ **is the "transverse Laplacian" (Laplacian in the plane (x_2, x_3); random fluctuations are intrinsically 3D).**

The random fluctuations enter through $\varepsilon = -\frac{T'}{T_0} - 2\frac{v_{x_1}'}{c_0}$

T' and v' are the fluctuations of temperature and of the component of the velocity along the principal direction of propagation, x_1. They are random processes characterized by a spatial correlation function or a spatial Fourier spectrum. Often Gaussian (simple form with only one characteristic length scale) or, better, von Karman (with outer and inner scales and a typical Kolmogorov inertial range in-between) spectra are used.

Statistical moments

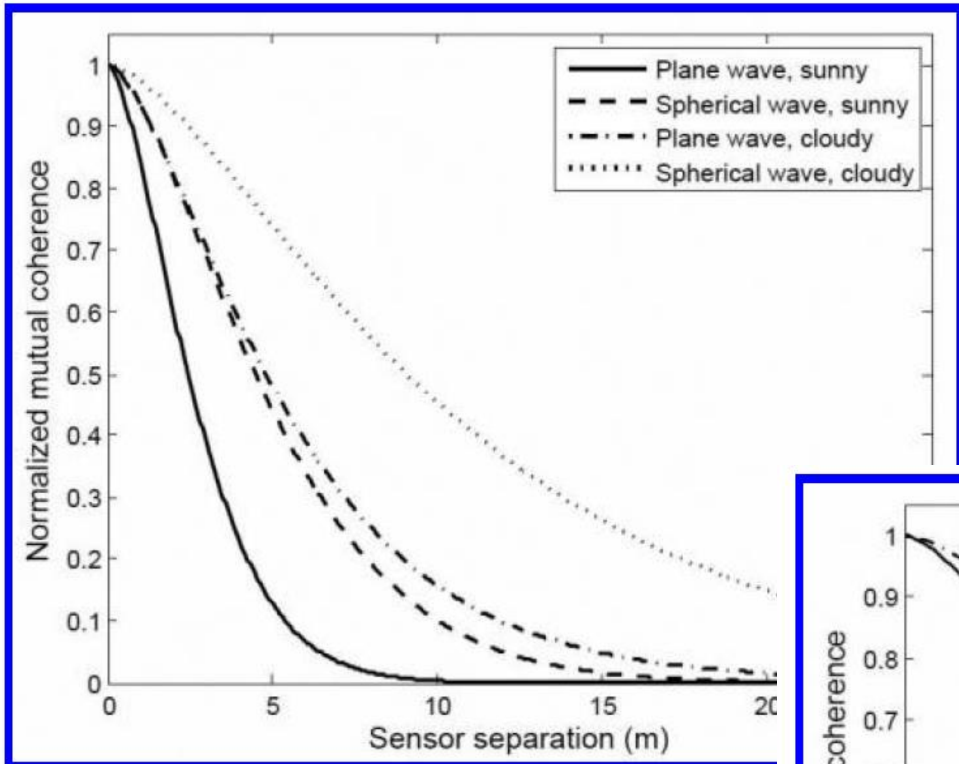
The acoustic pressure field is written as the sum of the statistical mean value of the complex amplitude (the mean field) and a fluctuating part

$$p(x_1, x_T) = \langle p(x_1, x_T) \rangle + p'(x_1, x_T)$$

The main quantity of interest is the second order moment for a given horizontal range, at the same transverse position (mean-squared pressure, acoustic intensity) or at 2 different positions (mutual coherence function).

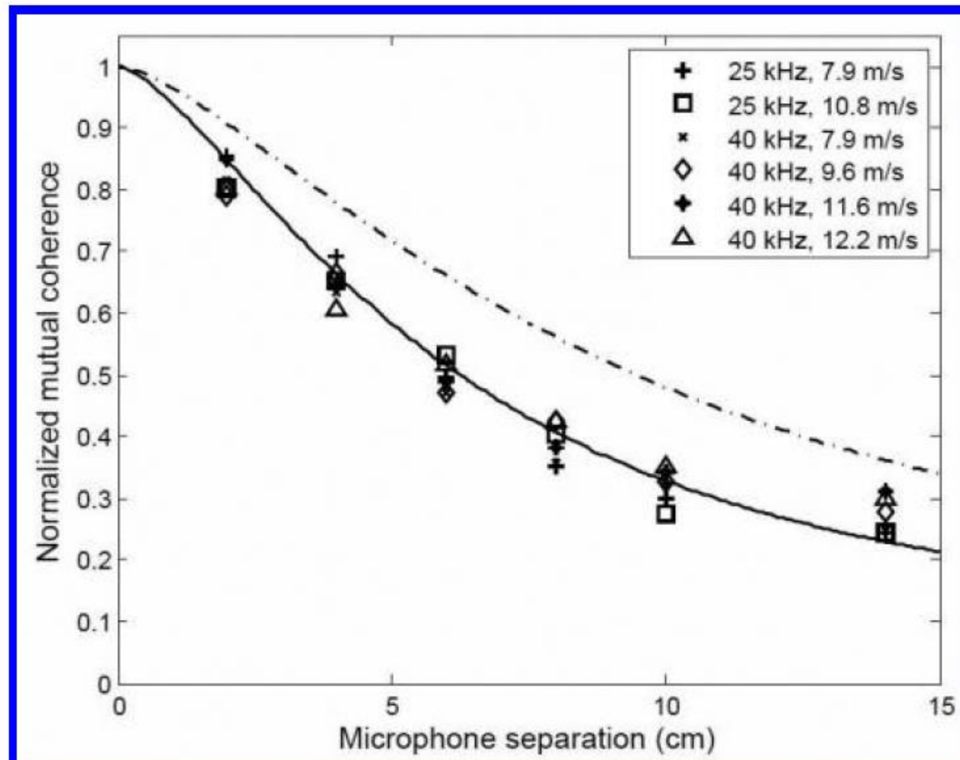
$$\Gamma(x, x_{T_1}, x_{T_2}) = \langle p'(x, x_{T_1}) p'^{*}(x, x_{T_2}) \rangle$$

Closed form solutions can be found for free field propagation in a statistically homogeneous medium *(a Markov hypothesis for the random field has to be used)* and for specific initial conditions. For a plane wave for example the coherence function depends only on the separation distance between the 2 observation points and can be expressed as a function of the spatial spectrum of the random fluctuations. An example of coherence functions is given in the next slide.



Computed coherence functions for moderate wind, f=500Hz. From Ostashev & Wilson, 2015.

Laboratory measurements of coherence functions for velocity turbulence (Ostashev, Blanc-Benon & Juvé, 1998)



Mean-squared pressure *(and transverse coherence function)* in a refractive atmosphere

For propagation in a refractive atmosphere above an impedance ground, a closed form equation for the second moment was derived by Wilson & Ostashev, 2001, and numerically solved for 2D propagation in a vertical plane.
While conceptually appealing, this approach is highly computationally demanding, requiring much more computation time than the "snapshot" method to be presented below. We show here a result taken from the cited paper and comparing PE results with and without turbulence.

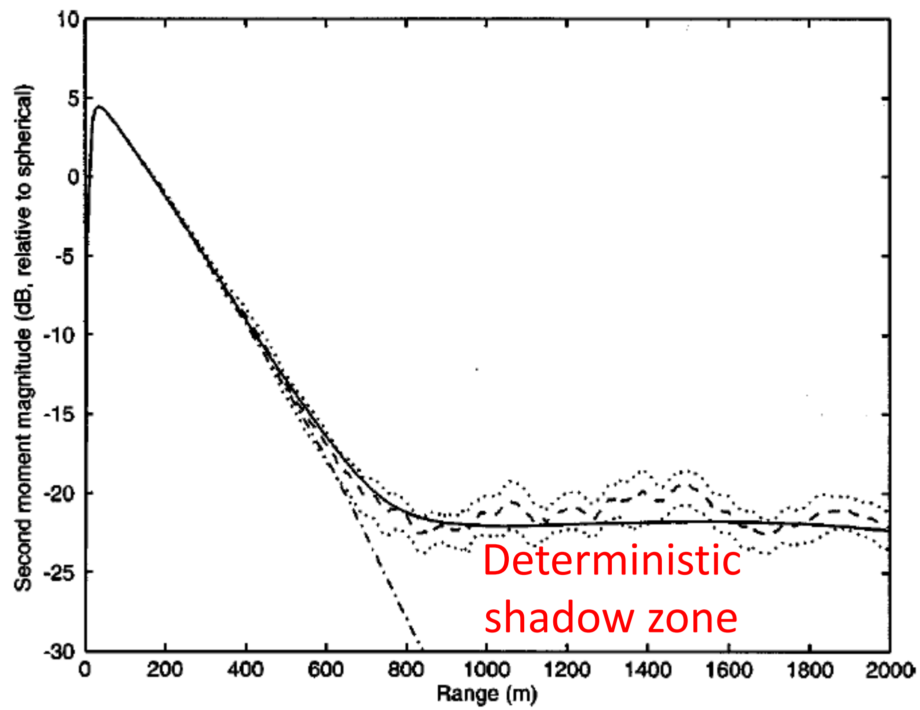


FIG. 4. Comparison of second moments (mean-square pressures) for upward refraction calculated using various methods. The receiver height is 2.6 m. Dash–dotted line is the calculation without turbulence. Solid line is the second-moment equation with turbulence. Dashed line is the average of 40 runs of the standard PE using random snapshots of the turbulence. Dotted lines enclose the 90% confidence interval for the 40 runs.

The method of snapshots

The “snapshot” method is numerically very simple to implement; it consists of solving the classical PE (SPE or WAPE) for a series of realizations of the random fluctuations “added” to the deterministic part of the index of refraction (Monte Carlo approach).

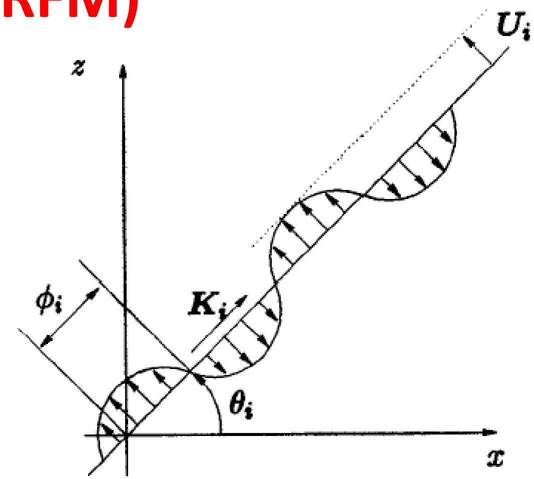
The only difficulty is to generate realistic random fluctuations (without using unsteady CFD); several *kinematic* methods (i.e. not based on the dynamical eq. of motion, but synthesized from prescribed second-order spatial statistics) exist, most of them using a sum of (spatial) Fourier modes. This method allows an easy and exact description of (incompressible) velocity fluctuations. One disadvantage is the difficulty to take into account non homogeneous random fields.

The PE code is run on a sufficient number of realizations to obtain statistical convergence; typically N=100 realizations are sufficient for estimating the mean acoustic level, but a larger number is required for estimating the coherence function or pdfs.

The numerical cost is important as the not only we have to consider a large number of runs, but also as the step of the numerical grid has to be reduced in order to well describe the influence of the smaller random scales (however it is not useful in practice to resolve scales lower than say $\lambda/10$).

Modeling of random fluctuations: Random Fourier Modes (RFM)*

$$\vec{v}(\vec{x}) = \sum_{i=1}^{N} \vec{U}(\vec{K}^i)\cos(\vec{K}^i \cdot \vec{x} + \phi^i)$$

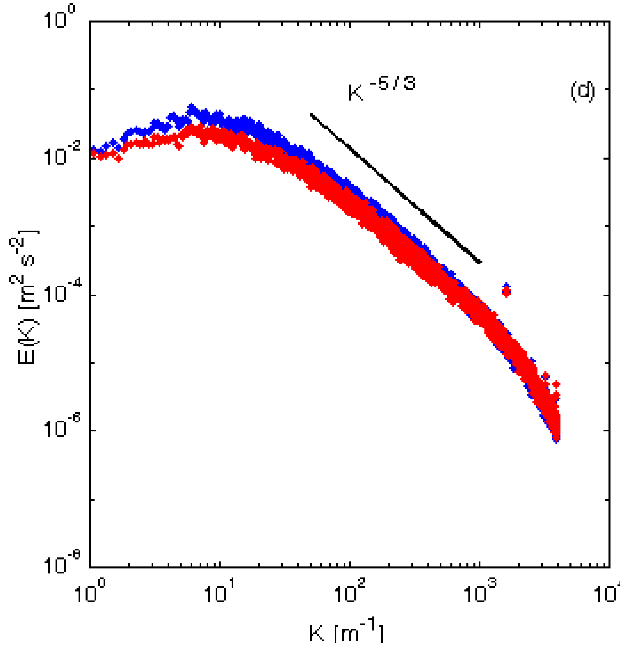


- **Wave-vector direction and phase ϕ of the modes are uniformly distributed over [0, π] to ensure isotropy and statistical homogeneity**

- **Amplitudes of modes are fixed according to a given form of the energy spectrum.**

The modified von Karman spectrum is a good approximation of experiments

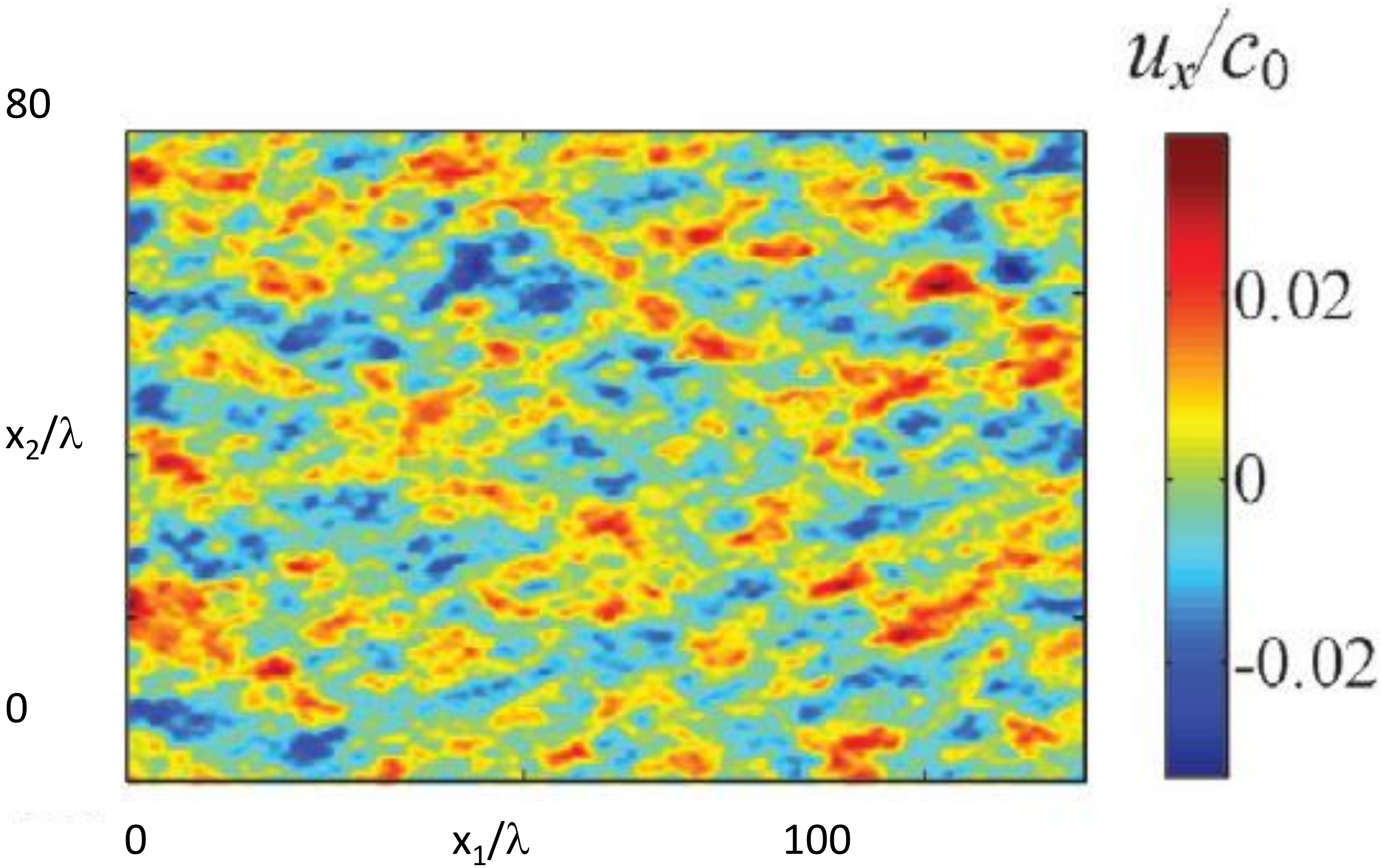
$$E = \frac{8v^2}{9} \frac{K^3}{L_0^{2/3}(K^2 + L_0^{-2})^{14/6}} \exp\left(-\left(\frac{Kl_0}{2}\right)^2\right)$$



K

** Other techniques can be used, RFG (Frehlich), spatial filtering of random fields …*

Typical realization of a random velocity field
Fourier modes synthesis-von Karman energy spectrum

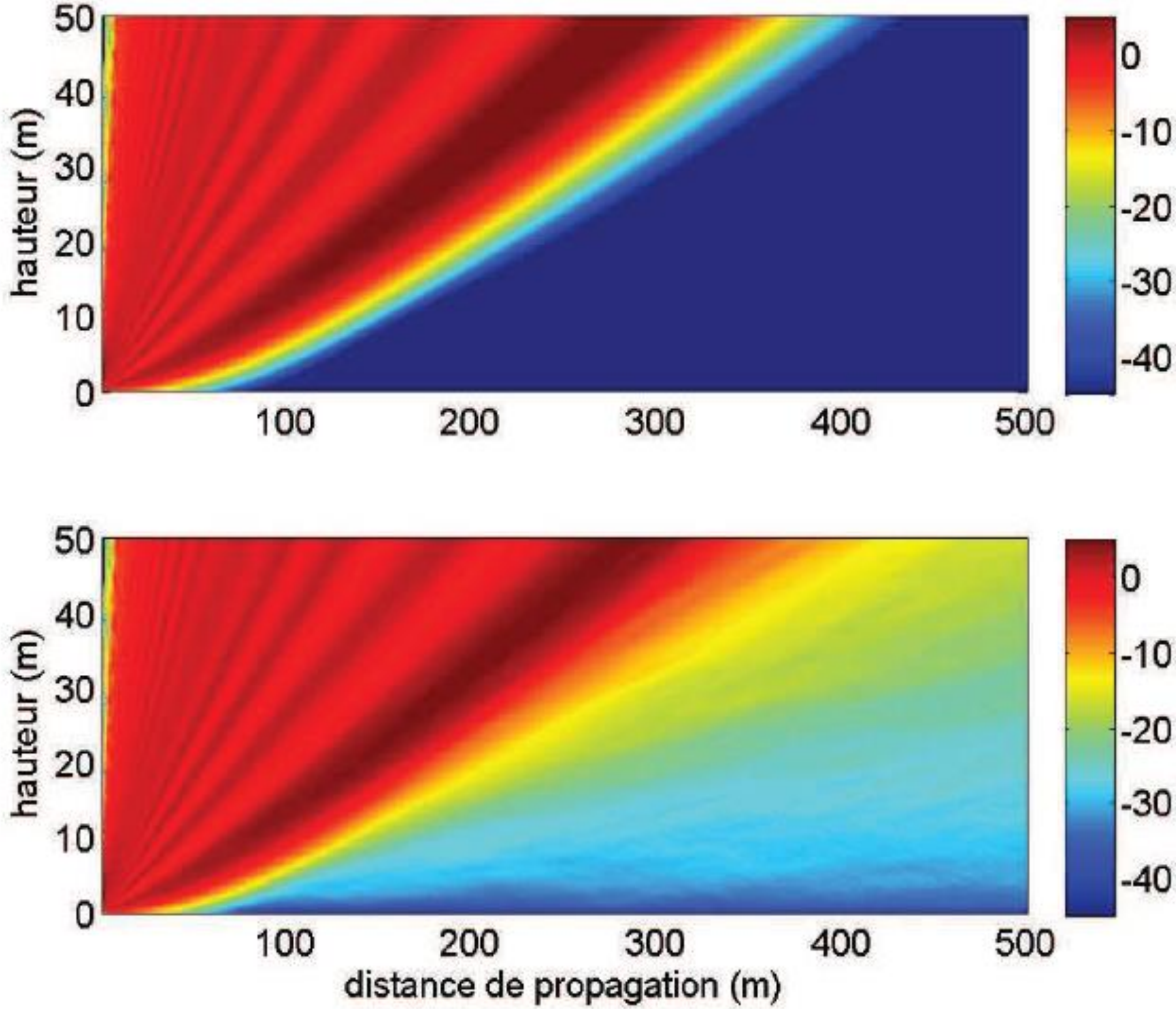


Mach number of longitudinal fluctuations

Numerical simulation of sound propagation in an upward refracting atmosphere (f=800Hz, WAPE)

Without turbulence

With turbulence



Level relative to free field (dB)

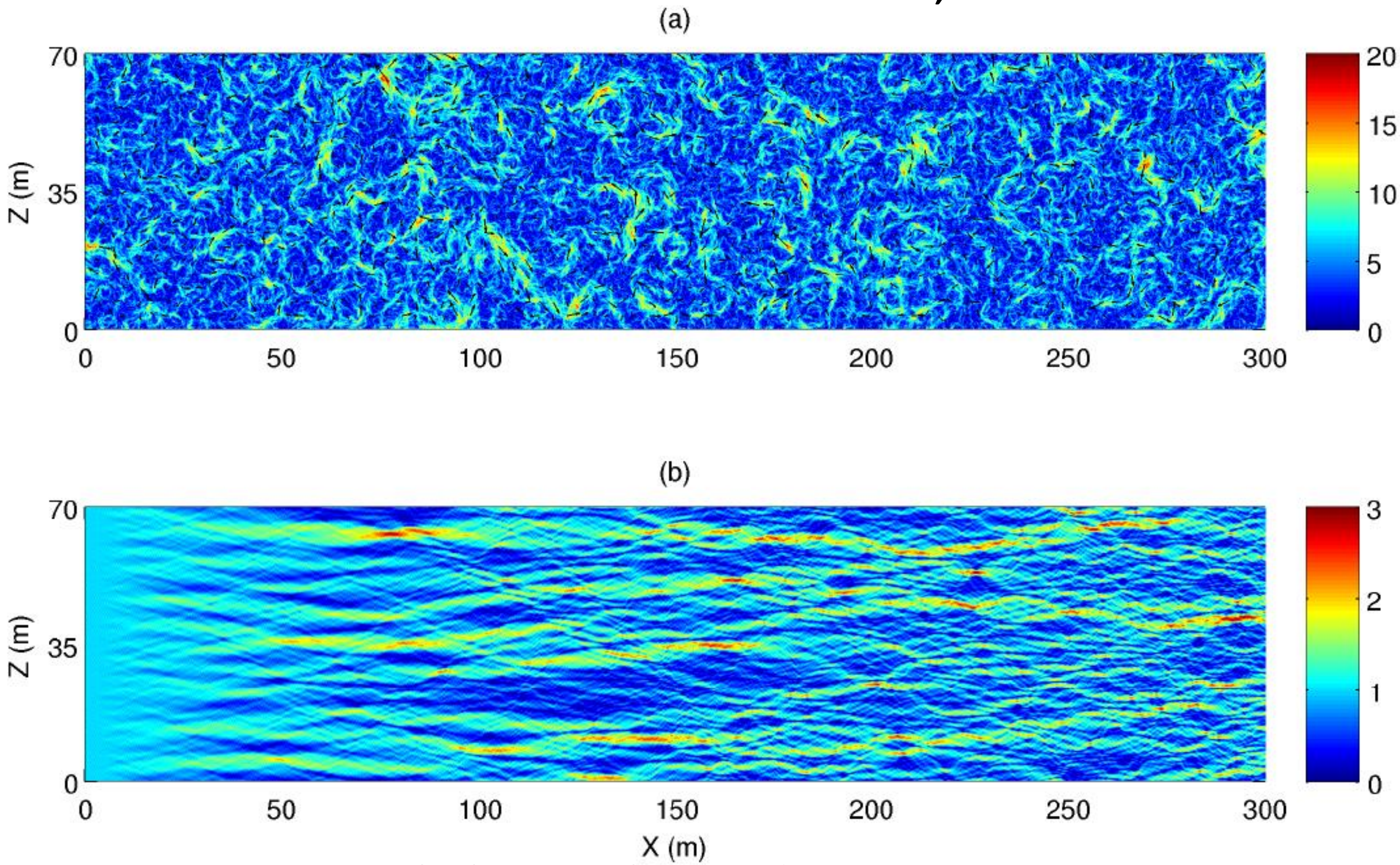
And a time domain computation to make a link with tomorrow's lecture

Propagation of an initially plane wave in a realization of wind turbulence

(a) Wind amplitude generated by RFG model

(b) FDTD, f=300Hz

Ehrhardt & al., JASA 2013



Thank you for attention

Any questions?

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