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ÉCOLE CENTRALE LYON

Flow stability and Introduction to turbulence

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http://acoustique.ec-lyon.fr

Introduction to turbulence

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(360 pages, 147 illustrations, Foreword by Charles Meneveau, 53 € for ECL students)

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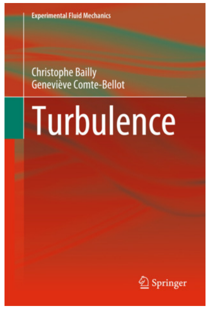

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360 pages, 147 illustrations.
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Reynolds decomposition

(see slides of UE FLE in first year, revision of Chapter 5)

Navier-Stokes equations

As a reminder, Navier-Stokes Eqs. for an incompressible flow,

$$\begin{cases} \nabla \cdot \boldsymbol{u} = 0 \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\dfrac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \end{cases}$$

The acceleration of the fluid particle $D\boldsymbol{u}/Dt$ is balanced by two terms on the right hand-side :

- acceleration of the fluid particle towards the low pressure regions of the flow
- viscous diffusion of the momentum

Mean and fluctuating quantities

The statistical mean $\bar{F}(\boldsymbol{x}, t)$ of a variable $f(\boldsymbol{x}, t)$ is defined as

$$\bar{F}(\boldsymbol{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f^{(i)}(\boldsymbol{x}, t)$$

where $f^{(i)}$ is the i-th realization : convenient when manipulating equations but difficult to implement in practice, or even impossible for geophysical flows !

We approximate the ensemble mean $\bar{F}$ of $f = \bar{F} + f'$ by a sufficiently long time average of one realization only :

$$\bar{F}(\boldsymbol{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\boldsymbol{x}, t')dt'$$

Time average makes sense only if turbulence is stationary, that is statistics are independent of time (refer to signal processing, associated with the hypothesis of ergodicity).

Reynolds decomposition

For a given variable f, Reynolds decomposition $f = \bar{F} + f'$ into mean and fluctuating (deviation) components is introduced.

- Centered fluctuating field

  $f \equiv \bar{F} + f' \quad \text{with} \quad \overline{f'} = 0 \qquad (f' = f - \bar{F} \quad \text{and} \quad \overline{f'} = \bar{F} - \bar{F} = 0)$

- Rule for the product of two any variables (f and g here),

  $fg \equiv (\bar{F} + f')(\bar{G} + g') = \bar{F}\bar{G} + \bar{F}g' + f'\bar{G} + f'g'$

  and thus, $\overline{fg} = \bar{F}\,\bar{G} + \bar{F}\,\overline{g'} + \overline{f'}\,\bar{G} + \overline{f'g'} = \bar{F}\,\bar{G} + \overline{f'g'}$

$$\boxed{\overline{fg} = \bar{F}\,\bar{G} + \overline{f'g'}} \tag{1}$$

Reynolds decomposition for velocity : $U_i \equiv \bar{U}_i + u_i'$ with $\overline{u_i'} = 0$

$\bar{U}_i$ part which can be reasonably calculated

u_i' part which must be modelled (turbulent fluctuations)

- **Reynolds Averaged Navier-Stokes (RANS) equations**

Assumptions (to simplify) : incompressible flow $\nabla \cdot \boldsymbol{U} = 0$
and homogeneous fluid, constant density ρ

How to determine the transport equation of the mean quantities ?

First, substitute the Reynolds decomposition and then,
average the equation,

$$\frac{\partial(\bar{U}_i + u_i')}{\partial x_i} = 0 \qquad \overline{\frac{\partial(\bar{U}_i + u_i')}{\partial x_i}} = 0 \qquad \Longrightarrow \qquad \frac{\partial \bar{U}_i}{\partial x_i} = 0 \tag{2}$$

Second, substract the averaged equation from the instantaneous one,
which provides

$$\frac{\partial U_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial \bar{U}_i}{\partial x_i} = 0 \qquad \Longrightarrow \qquad \frac{\partial u_i'}{\partial x_i} = 0$$

The mean flow field $\bar{\boldsymbol{U}}$ is incompressible, and so is the fluctuating field $\boldsymbol{u}'$

Reynolds Averaged Navier-Stokes (RANS) equations

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho U_i U_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \tau_{ij} = 2\mu D_{ij}$$

By introducing the Reynolds decomposition, and taking the average

$U_i \equiv \bar{U}_i + u_i'$ $\qquad p \equiv \bar{P} + p'$ $\qquad \tau_{ij} \equiv \bar{\tau}_{ij} + \tau_{ij}'$

$$\frac{\partial(\rho \bar{U}_i)}{\partial t} + \frac{\partial(\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho \overline{u_i' u_j'}\right) \qquad (3)$$

The term $-\rho\overline{u_i'u_j'}$ is Reynolds stress tensor, unknown, thus leading to a closure problem for Eqs. (2) and (3). Generally this term is larger than the mean viscous stress tensor $\bar{\boldsymbol{\tau}}$ except for wall bounded flows, where the viscosity effects become preponderant close to the wall (no-slip condition, $\overline{u_i'u_j'} = 0$ at the wall)

Turbulent kinetic energy and dissipation

The turbulent kinetic energy k_t and the turbulent dissipation ϵ are two key quantities to examine turbulence dynamics. By introducing the Reynolds decomposition, using the rule (1), we obtain

for the kinetic energy

$$\frac{\overline{U_i U_i}}{2} = \frac{\bar{U}_i \bar{U}_i}{2} + \frac{\overline{u'_i u'_i}}{2} \qquad k_t \equiv \frac{\overline{u'_i u'_i}}{2}$$

k_t is the turbulent kinetic energy

for the dissipation

$$2\nu \overline{D_{ij} D_{ij}} = 2\nu \bar{D}_{ij} \bar{D}_{ij} + 2\nu \overline{d'_{ij} d'_{ij}} \qquad \epsilon \equiv 2\nu \overline{d'_{ij} d'_{ij}}$$

ϵ ($m^2.s^{-3}$) is the dissipation rate of k_t ($m^2.s^{-2}$) induced by the molecular viscosity

- **Concept of turbulent viscosity for turbulence models**
  introduced by Boussinesq (1877)

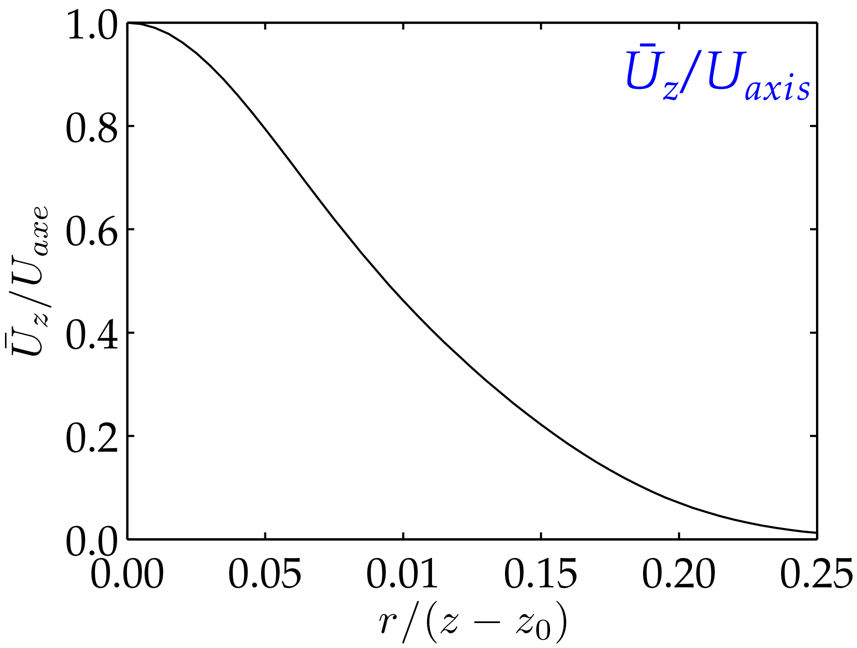
  To model Reynolds stress tensor $-\rho\overline{u_i'u_j'}$, one defines by analogy with the viscous stress tensor $\overline{\overline{\boldsymbol{\tau}}}$,

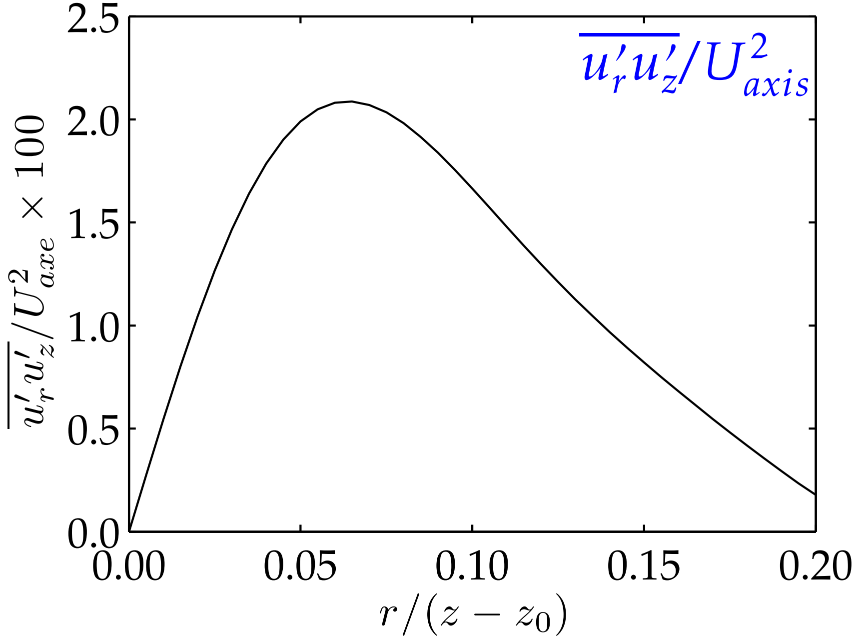
$$\boxed{-\rho\overline{u_i'u_j'} = 2\mu_t\bar{D}_{ij} - \frac{2}{3}\rho k_t\delta_{ij} = \mu_t\left(\frac{\partial\bar{U}_i}{\partial x_j} + \frac{\partial\bar{U}_j}{\partial x_i}\right) - \frac{2}{3}\rho k_t\delta_{ij}} \qquad (4)$$

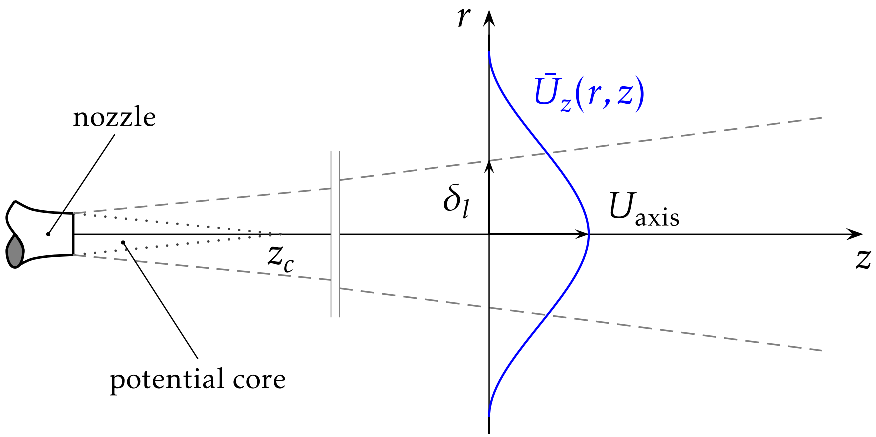
  where $\mu_t = \mu_t(\boldsymbol{x}, t)$ is the turbulent viscosity, a property of the flow field, and not of the fluid as for the molecular viscosity. It is thus expected that $\mu_t = \mu_t(\mathrm{Re})$.

  The introduction of a turbulent viscosity for closing Reynolds stress tensor is an assumption, so not always verified by turbulent flows. In addition, it is also assumed in Eq. (4) that the turbulent viscosity remains positive (thus inducing specific behaviours in terms of energy transfer)

- **Illustration for a free subsonic round jet**
  M = 0.16 and $\mathrm{Re}_D = 9.5 \times 10^4$ (from Hussein, Capp & George, 1994)







$$\rho\overline{u_r'u_z'} \simeq -\mu_t \frac{\partial \bar{U}_z}{\partial r}$$

self-similar solution $\dfrac{r}{\delta_l} \sim \dfrac{r}{z-z_0}$

Concept of turbulent viscosity for turbulence models (cont.)

Reynolds Averaged Navier-Stokes (RANS) equations

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0$$

$$\frac{\partial(\rho \bar{U}_i)}{\partial t} + \frac{\partial(\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial(\bar{P} + \frac{2}{3}\rho k_t)}{\partial x_i} + \frac{\partial}{\partial x_j}\Big(2(\mu + \mu_t)\bar{D}_{ij}\Big)$$

How to compute the turbulent viscosity $\nu_t(\boldsymbol{x}, t)$?

From dimensional arguments, the product of a velocity scale by a length scale, for example $\nu_t \sim k_t^{1/2} \times k_t^{3/2}/\epsilon \sim k_t^2/\epsilon$, and then write a transport equation for k_t and ϵ to obtain the famous $k_t - \epsilon$ model.

There are about 200 turbulent viscosity models published in the literature !
(see Wilcox and Durbin books among others)

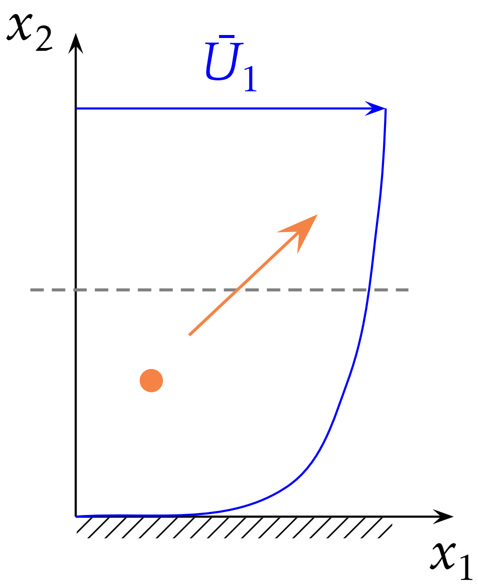
- **Turbulent kinetic energy budget**
  (the demonstration can be found in textbooks)

$$\underbrace{\frac{\partial(\rho k_t)}{\partial t} + \frac{\partial(\rho k_t \bar{U}_j)}{\partial x_j}}_{\text{advection by the mean flow}} = \underbrace{-\rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j}}_{\text{production } \mathcal{P}} \underbrace{- \rho \epsilon}_{\text{dissipation}}$$

$$+ \underbrace{\frac{\partial}{\partial x_k}\left(-\frac{1}{2}\rho \overline{u_i' u_i' u_k'} - \overline{p' u_k'} + \overline{u_i' \tau_{ik}'}\right)}_{\text{transport terms}}$$

Case of homogeneous turbulence ?
(when turbulence statistics are independent of space)

Heuristic interpretation of the production term $\mathcal{P}$



$$\begin{cases} u_2' > 0 \\ u_1' < 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

$$\begin{cases} u_2' < 0 \\ u_1' > 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

$\mathcal{P} \simeq -\rho \overline{u_1' u_2'} \dfrac{d\bar{U}_1}{dx_2} > 0$ is thus expected !

The production term $\mathcal{P}$ is – in general – a transfer from the shear mean flow $\bar{\boldsymbol{U}}$ to the turbulent field $\boldsymbol{u}'$; but becomes always a positive transfer term using a turbulent viscosity model (4), a drawback of turbulence models

Scales and energy cascade

Scales

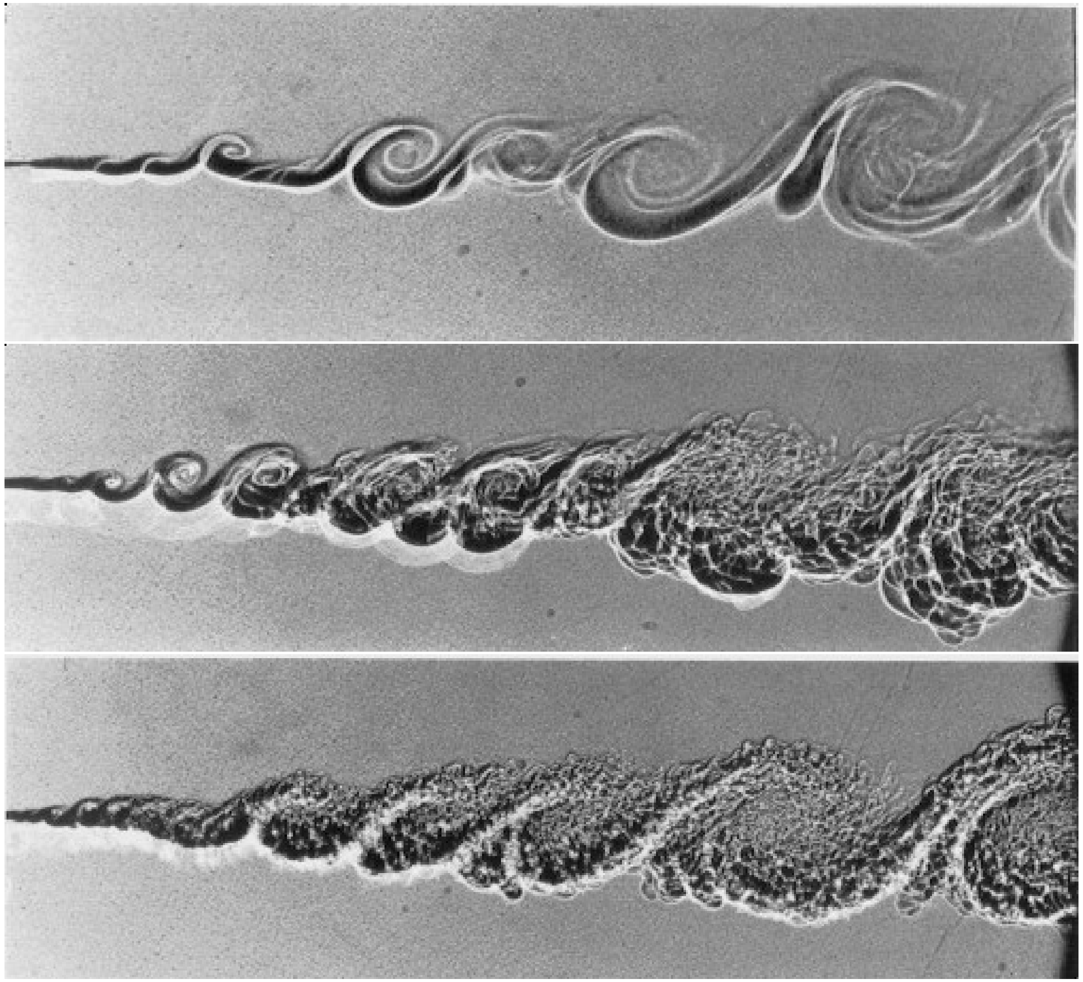
Large scale in $\mathcal{O}(L, u')$ – associated with the geometry (cavity, cylinder, jet, wake, car, airfoil, pipe, ...) and thus with the size of the flow itself

$\leadsto$ energy transfer (basically) from large scale structures to small scale structures, but this transfer is stopped by the molecular viscosity when

$$\frac{\partial \boldsymbol{u}'}{\partial t} \sim \nu \nabla^2 \boldsymbol{u}'$$

Small scale in $\mathcal{O}(l_\eta, u_\eta)$ – known as Kolmogorov (viscous) scales with $\mathrm{Re} = u_\eta l_\eta / \nu = 1$. Kolmogorov's length scale l_η plays a fundamental role in experiments (sampling frequency) as well as in numerical simulations (grid size)

Turbulent mixing layer (Brown & Roshko, 1974)

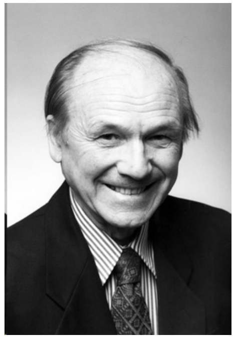


(Shadowgraphs with a spark source)

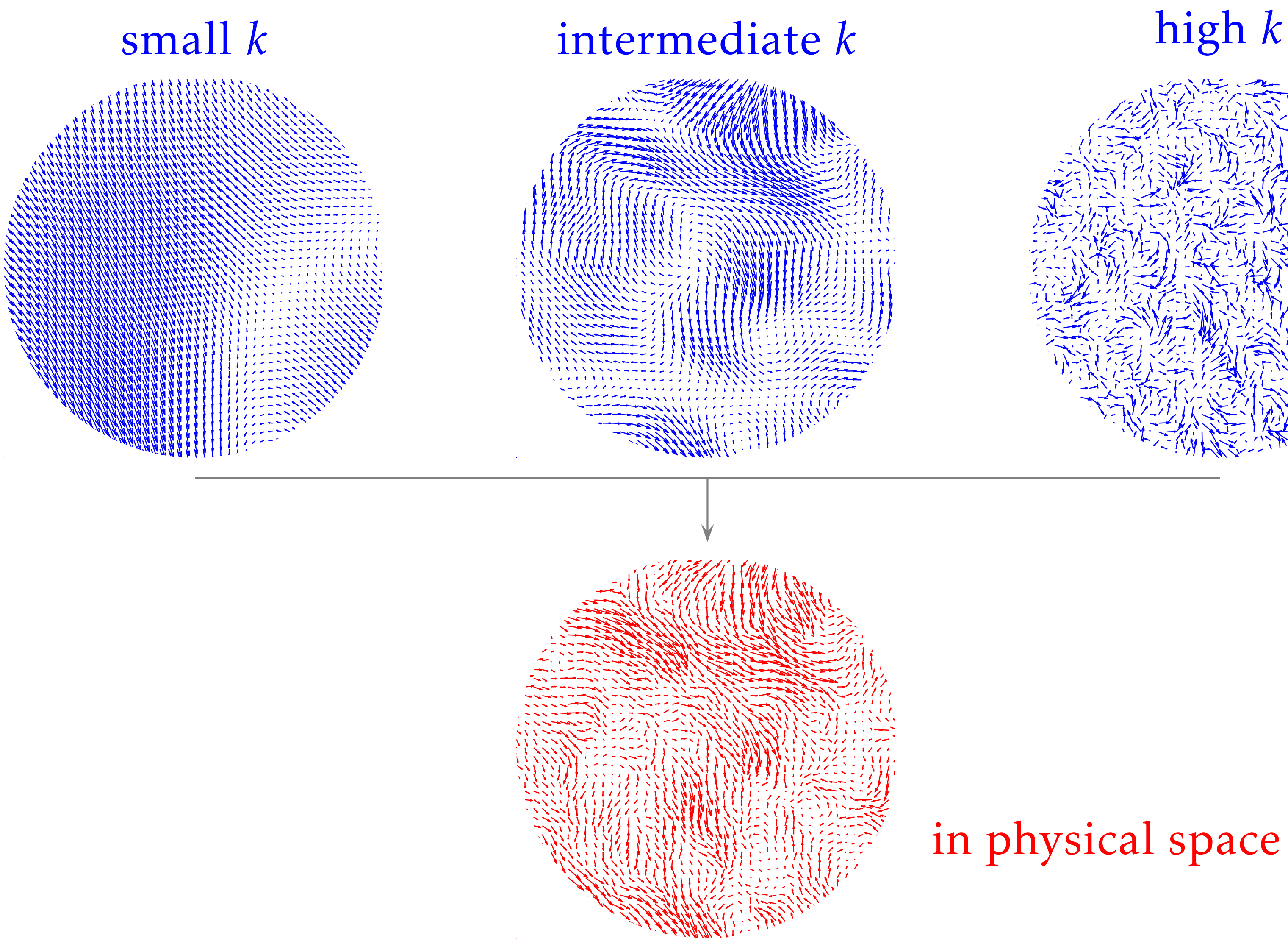
Energy cascade in a mixing layer by increasing the Reynolds number (through pressure and velocity, ×2 for each view)

More small-scale structures are produced without basically altering the large-scale ones (linked to the transition process, as shown by Winant & Browand, 1974)

Anatol Roshko (1923-2017)

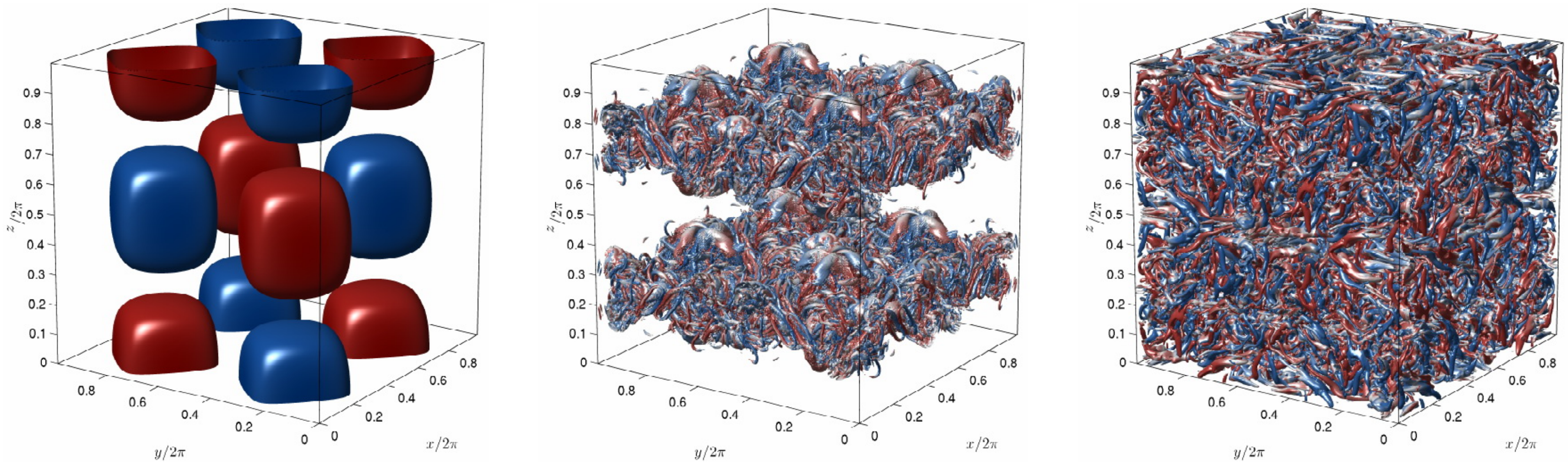


- **Local nature of the energy cascade in space**



- **Taylor-Green vortex flow**
  Re = 3000 on a 384^3 grid at times $t = 0$, $t = 9$ and $t = 18$ (dimensionless variables)
  From Fauconnier *et al.* (2013)

Vortex structures colored by z-vorticity



Introduction to the energy cascade

From Navier-Stokes Eqs, we can calculate the power developed by the viscous friction,

$$P = \boldsymbol{u}' \cdot \nu \nabla^2 \boldsymbol{u}' = u_i' \nu \frac{\partial^2 u_i'}{\partial x_j \partial x_j} = \nu \frac{\partial}{\partial x_j}\left(u_i' \frac{\partial u_i'}{\partial x_j} \right) - \nu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}$$

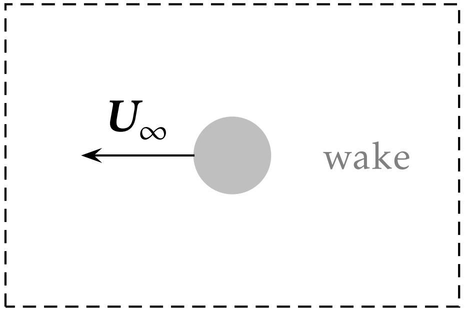
$$= \underbrace{\nu \nabla^2 \left(\frac{\boldsymbol{u}'^2}{2} \right)}_{\text{(a)}} - \underbrace{\nu \nabla \boldsymbol{u}' : \nabla \boldsymbol{u}'}_{\text{(b)}}$$

(a) = diffusion of the kinetic energy

(b) = rate ϵ of dissipation (per unit mass) of the kinetic energy on average, see slide 45

High velocity gradients (and thus small scale activity) are required to ensure dissipation

Introduction to the energy cascade



e.g. energy dissipated by the motion at U_∞ of a sphere of diameter D

$$F_D \cdot U_\infty \propto C_D(\mathrm{Re}_D)\rho U_\infty^3 D^2$$

The power developed by the drag force F_D, that is $F_D U_\infty$, is balanced by the energy dissipated with the flow $\rho\mathcal{V} \times \epsilon$ where $\mathcal{V} \propto D^3$

$$\epsilon \propto C_D(\mathrm{Re}_D)\frac{U_\infty^3}{D}$$

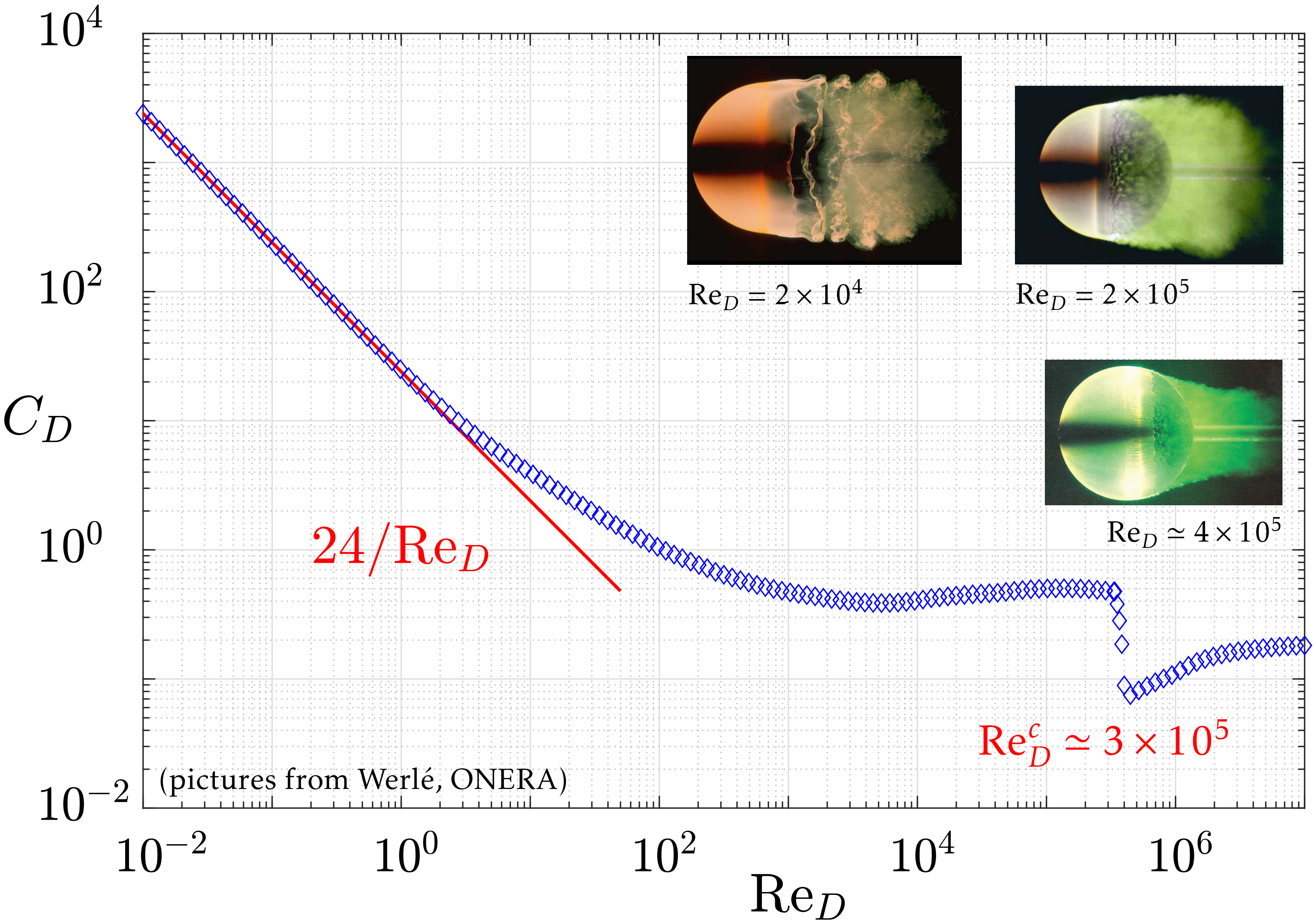
- As $\mathrm{Re}_D \to 0$, $\nu \nearrow$ as well as energy which must be dissipated to move inside the fluid

- As $\mathrm{Re}_D \to \infty$, $C_D = C_D(\mathrm{Re}_D) \simeq \mathrm{cst}$

  $\epsilon \propto \dfrac{U_\infty^3}{D}$ is independent of the viscosity

Drag coefficient for a smooth sphere

(adapted from Clift, Grace, & Weber, 1978)



and paradox of the energy cascade ...

For high Reynolds number flows when ν goes to zero, the rate of dissipation per unit mass ϵ becomes independent of the viscosity ν

$\epsilon = \nu\overline{|\nabla \boldsymbol{u}'|^2} \simeq \text{cst}$, leading to possible singularities for the velocity gradients $\nabla \boldsymbol{u}' \nearrow$ (fragmentation of fine scales, still a research topic in turbulence)

Energy cascade - Kolmogorov (1941)

Kinetic energy is not conserved from scales to scales, but the rate of transfer of this energy ϵ is conserved

$$\begin{cases} \dfrac{l_\eta u_\eta}{\nu} = 1 \qquad \epsilon = \dfrac{u_\eta^3}{l_\eta} \qquad l_\eta = \nu^{3/4}\epsilon^{-1/4} \\ \epsilon = \dfrac{u'^3}{L} \qquad \mathrm{Re}_L = \dfrac{u'L}{\nu} \qquad \dfrac{L}{l_\eta} \sim \mathrm{Re}_L^{3/4} \end{cases}$$

As an illustration

Soccer ball

$D = 22$ cm

$U_\infty = 100$ km.h$^{-1} \simeq 27.8$ m.s^{-1}

$\mathrm{Re}_D \simeq 4.1 \times 10^5$

$u' \simeq 0.15 U_\infty$

$L \simeq D/2$

$\mathrm{Re}_L \simeq 3.1 \times 10^4$

$l_\eta \simeq L/\mathrm{Re}_L^{3/4} \simeq 4.7 \times 10^{-5}$ m !

$$k = \frac{\omega}{U_\infty} \quad f = \frac{U_\infty}{l_\eta} \simeq 5.8 \times 10^5 \text{ Hz}$$