*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

ÉCOLE CENTRALE LYON

Flow stability and Introduction to turbulence

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http://acoustique.ec-lyon.fr

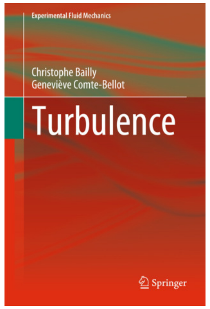
Introduction to turbulence

Textbooks

Batchelor, G.K., 1967, An introduction to fluid dynamics, *Cambridge University Press*, Cambridge.

Bailly C. & **Comte Bellot** G., 2003 Turbulence, *CNRS éditions*, Paris (out of print).
———, 2015, *Turbulence* (in english), Springer, Heidelberg.
(360 pages, 147 illustrations, Foreword by Charles Meneveau, 53 € for ECL students)

Bailly C. & Comte Bellot G., 2003, *Turbulence* (in french), CNRS éditions, Paris.
———, 2015, *Turbulence* (in english), Springer, Heidelberg.



Springer, ISBN 978-3-319-16159-4,
360 pages, 147 illustrations.
(discount for students, 53 €)

Candel S., 1995, Mécanique des fluides, *Dunod Université*, 2nd édition, Paris.

Cousteix, J., 1989, Turbulence et couche limite, Cépaduès, Toulouse.

Davidson P.A., 2004, *Turbulence. An introduction for scientists and engineers*, Oxford University Press, Oxford.

Davidson, P.A., **Kaneda**, Y., **Moffatt**, H.K. & **Sreenivasan**, K.R., Edts, 2011, *A voyage through Turbulence*, Cambridge University Press, Cambridge.

Guyon E., **Hulin** J.P. & **Petit** L., 2001, Physical hydrodynamics, *EDP Sciences / Editions du CNRS*, première édition 1991, Paris - Meudon.

Textbooks (cont.)

Hinze J.O., 1975, Turbulence, *McGraw-Hill International Book Company*, New York, 1ère édition en 1959.

Landau L. & **Lifchitz** E., 1971, Mécanique des fluides, *Editions MIR, Moscou*.
Also *Pergamon Press*, 2nd edition, 1987.

Lesieur M., 2008, Turbulence in fluids : stochastic and numerical modelling, *Kluwer Academic Publishers, 4th revised and enlarged ed.*, Springer.

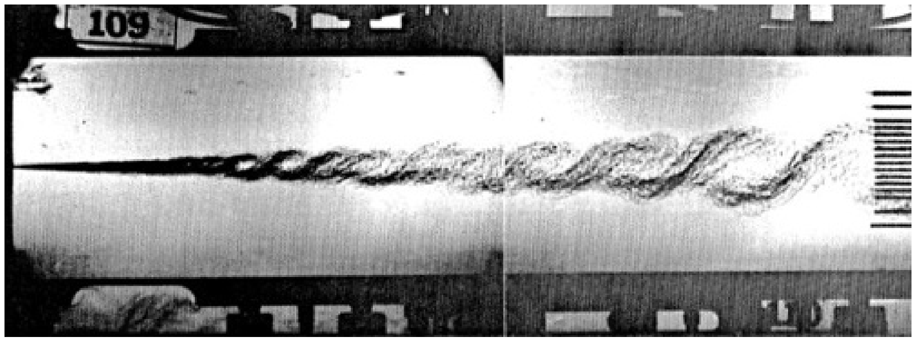
Pope S.B., 2000, Turbulent flows, *Cambridge University Press*.

Tennekes H. & **Lumley** J.L., 1972, A first course in turbulence, *MIT Press*, Cambridge, Massachussetts.

Van Dyke M., 1982, An album of fluid motion, *The Parabolic Press*, Stanford, California.

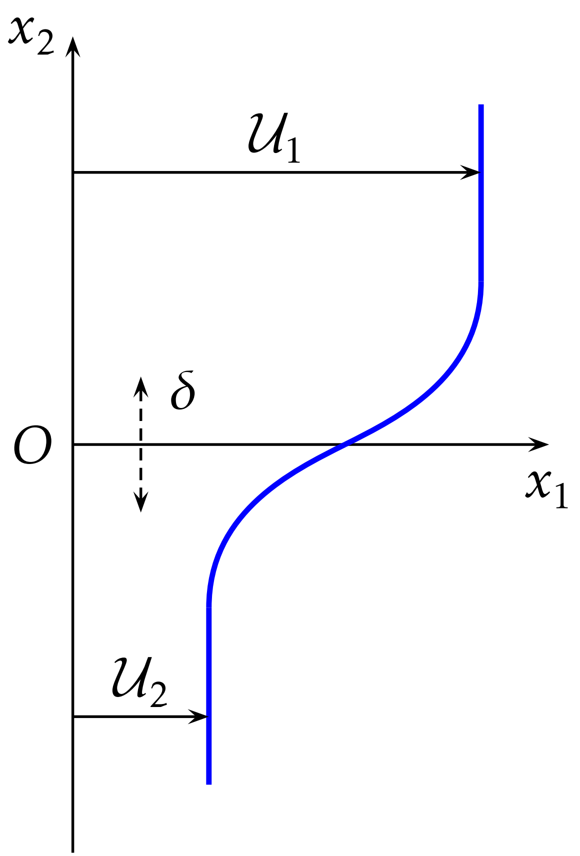
White F., 1991, Viscous flow, *McGraw-Hill, Inc.*, New-York, first edition 1974.

Free shear flows
(flows in the absence of walls!)



Composite schlieren of the free shear layer, with the two streams injected parallel to each other. The upper stream is 100% N2 while the lower is a mixture of 1/3 He and 2/3 Ar, with $M_1 = 0.59$ and $M_2 = 0.29$ Hall, Dimotakis & Rosemann, 1993, *AIAA J.*, **31**(12), 2247-2254.

- **Almost parallel and two-dimensional flow :** slow evolution in the x_1 direction, and statistics independent of the spanwise coordinate x_3



$\mathcal{U}_1$ and $\mathcal{U}_2$ high- and low-speed

$u_m \equiv \dfrac{\mathcal{U}_1 + \mathcal{U}_2}{2}$ convection velocity scale

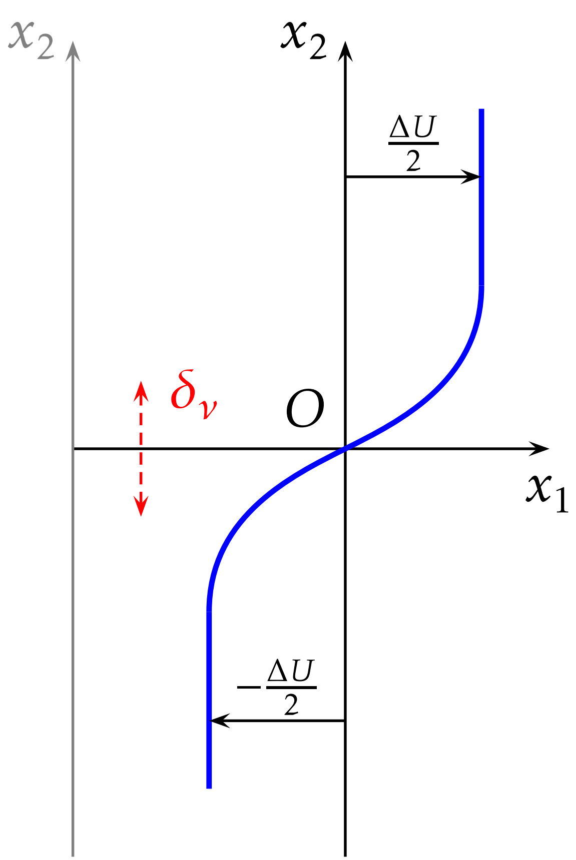
$\Delta U \equiv \mathcal{U}_1 - \mathcal{U}_2$ velocity difference, which characterizes the turbulent diffusion

$\delta(x_1)$ mixing layer width

L scale of change in the x_1 direction
l scale of change in the x_2 direction

$l/L \ll 1$ is a small parameter (parallel flow), and $\mathrm{Re} = \Delta U \delta/\nu \gg 1$ (inviscid flow)

- **Digression : solution for the laminar viscous diffusion**
  in the frame moving at u_m



In the local convected frame,

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial x_2^2}$$

Self-similar solution

$$\eta = \frac{x_2}{2\sqrt{\nu t}} \qquad U = \frac{\Delta U}{2} f(\eta) \qquad \begin{cases} f = 0 & \eta = 0 \\ f = 1 & \eta \to \infty \end{cases}$$

$$f'' + 2\eta f' = 0$$

$$U(x_2, t) = \frac{\Delta U}{2\sqrt{\pi \nu t}} \int_0^{x_2} e^{-\tilde{x}_2^2/(4\nu t)} d\tilde{x}_2 \qquad \delta_\nu \sim \sqrt{\nu t}$$

Solution never observed alone in practice : unstable flow, leading to the development of Kelvin-Helmholtz instability waves (a fundamentally inviscid process)

Mean velocity field

Conservation of mass, which provides the transverse velocity scale V

(where $\sim$ stands for order of magnitude)

$$\frac{\partial \bar{U}_1}{\partial x_1} + \frac{\partial \bar{U}_2}{\partial x_2} = 0 \qquad \frac{\partial \bar{U}_1}{\partial x_1} \sim \frac{\Delta U}{L} \quad \Longrightarrow \quad V \sim \frac{l}{L}\Delta U$$

Reynolds-averaged Navier-Stokes equation in the x_1 direction

$$\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_1} - \frac{\partial \overline{u_1' u_1'}}{\partial x_1} - \frac{\partial \overline{u_1' u_2'}}{\partial x_2} + \nu \nabla^2 \bar{U}_1$$

– for the convection terms,

$$\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2} = u_m \frac{\partial \bar{U}_1}{\partial x_1} + \underbrace{(\bar{U}_1 - u_m)\frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2}}_{\sim \Delta U \frac{\Delta U}{L}}$$

Mean velocity field (cont.)

– for the Reynolds stress,
with $u \sim \Delta U$ for the scale of velocity fluctuations,

$$-\frac{\partial \overline{u_1' u_2'}}{\partial x_2} \sim \frac{u^2}{l} \sim \frac{(\Delta U)^2}{l}$$

Then, by considering the balance between the two dominant (red) terms

$$u_m \frac{\Delta U}{L} \sim \frac{(\Delta U)^2}{l} \quad \Longrightarrow \quad \Delta U \sim \frac{l}{L} u_m$$

– for the pressure term, using the RANS Eq. in the x_2 direction

$$\underbrace{\bar{U}_1 \frac{\partial \bar{U}_2}{\partial x_1}}_{\sim \frac{l}{L}\frac{u_m \Delta U}{L}} + \bar{U}_2 \frac{\partial \bar{U}_2}{\partial x_2} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_2} - \frac{\partial \overline{u_1' u_2'}}{\partial x_1} \underbrace{- \frac{\partial \overline{u_2' u_2'}}{\partial x_2}}_{\sim \frac{u_m \Delta U}{L}} + \nu \nabla^2 \bar{U}_2$$

- **Mean velocity field (cont.)**

  – for the pressure term (cont.)

  and by integration in the transverse direction,

  $\bar{P} + \rho\overline{u_2'^2} \simeq \text{cst} = p_\infty \quad \Longrightarrow \quad \frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_1} \simeq 0$ in the RANS Eq. in the x_1 direction

Finally, the equation governing the mean velocity is

$$u_m \frac{\partial \bar{U}_1}{\partial x_1} \simeq -\frac{\partial \overline{u_1' u_2'}}{\partial x_2} \qquad (5)$$

Self-similar solution

We now look for self-similar solutions of Eq. (5)

Definition of the mixing layer width $\delta(x_1) = x_2^{0.9} - x_2^{0.1}$
where x_2^{α} is the transverse location such that $\bar{U}_1 = \mathcal{U}_2 + \alpha \Delta U$

Hence, from this definition, we have $\bar{U}_1(\pm\delta/2) = u_m \pm 0.4\Delta U$

Self-similar variable $\eta = \dfrac{x_2 - \bar{x}_2}{\delta}$
where $\bar{x}_2(x_1)$ is usually the line along which $\bar{U}_2 = 0$ (the flow is indeed not symmetric about $x_2 = 0$ for various reasons)

$$\begin{cases} \bar{U}_1 = u_m + \Delta U\, f(\eta) \\ \overline{u_1' u_2'} = (\Delta U)^2 g(\eta) \end{cases}$$

We expect that the nondimensionalized quantities f and g are indeed functions of η only (see tutorials)

Champagne, Pao & Wygnanski (*J. Fluid Mech.*, 1976)

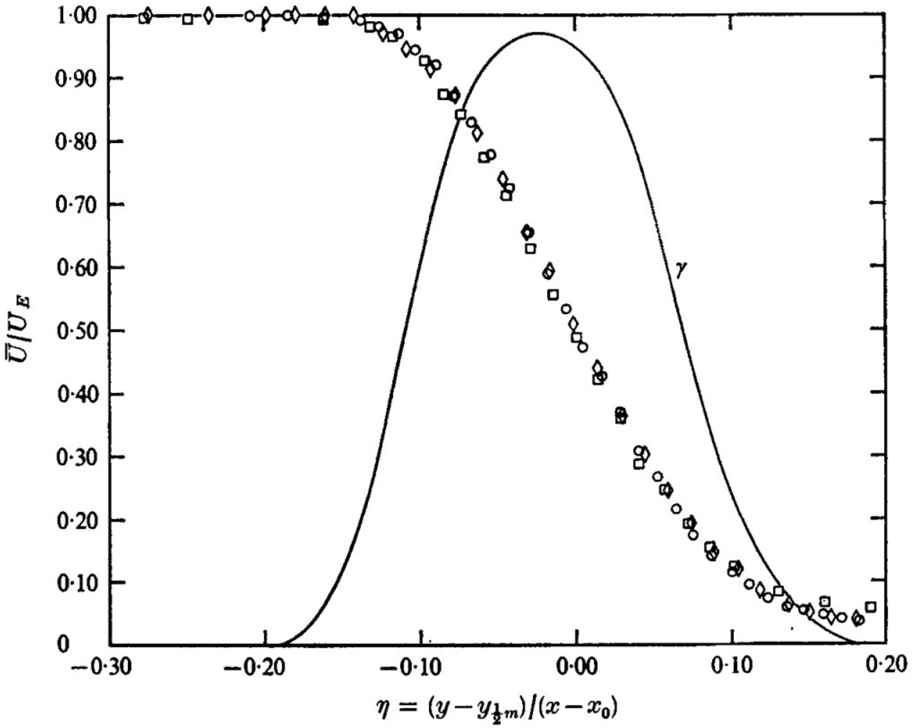


FIGURE 2. Development of mean velocity field. x: ◇, 39·5 cm; □, 49·5 cm; ○, 59·5 cm. γ = intermittency.

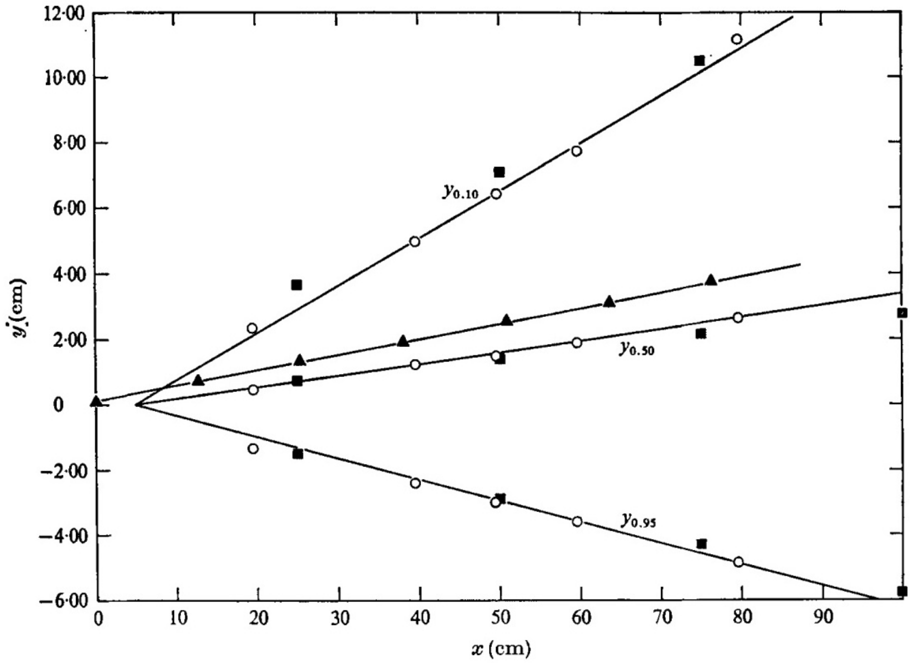
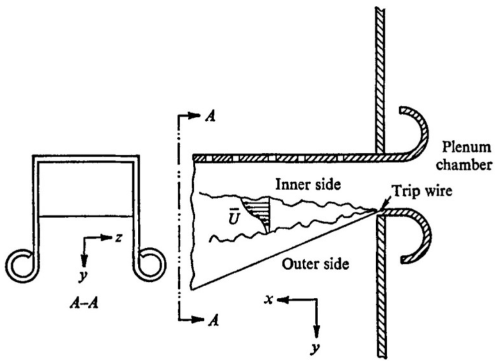


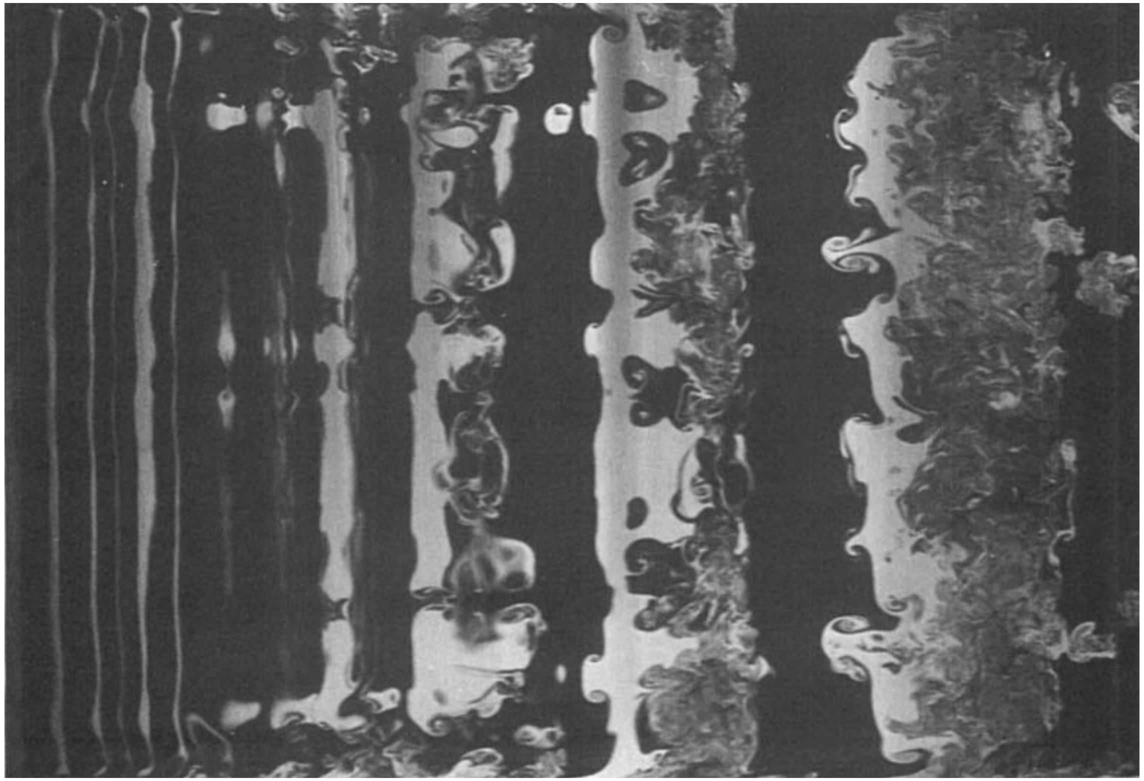
FIGURE 3. The growth of the mixing region with downstream distance. ▲, W & F; ■, Patel; ○, present results.

$$\mathcal{U}_1 = U_E = 8\ \mathrm{m.s}^{-1}, \mathcal{U}_2 = 0$$

(the flow, running from the right to the left, spreads preferentially into the low-speed stream)



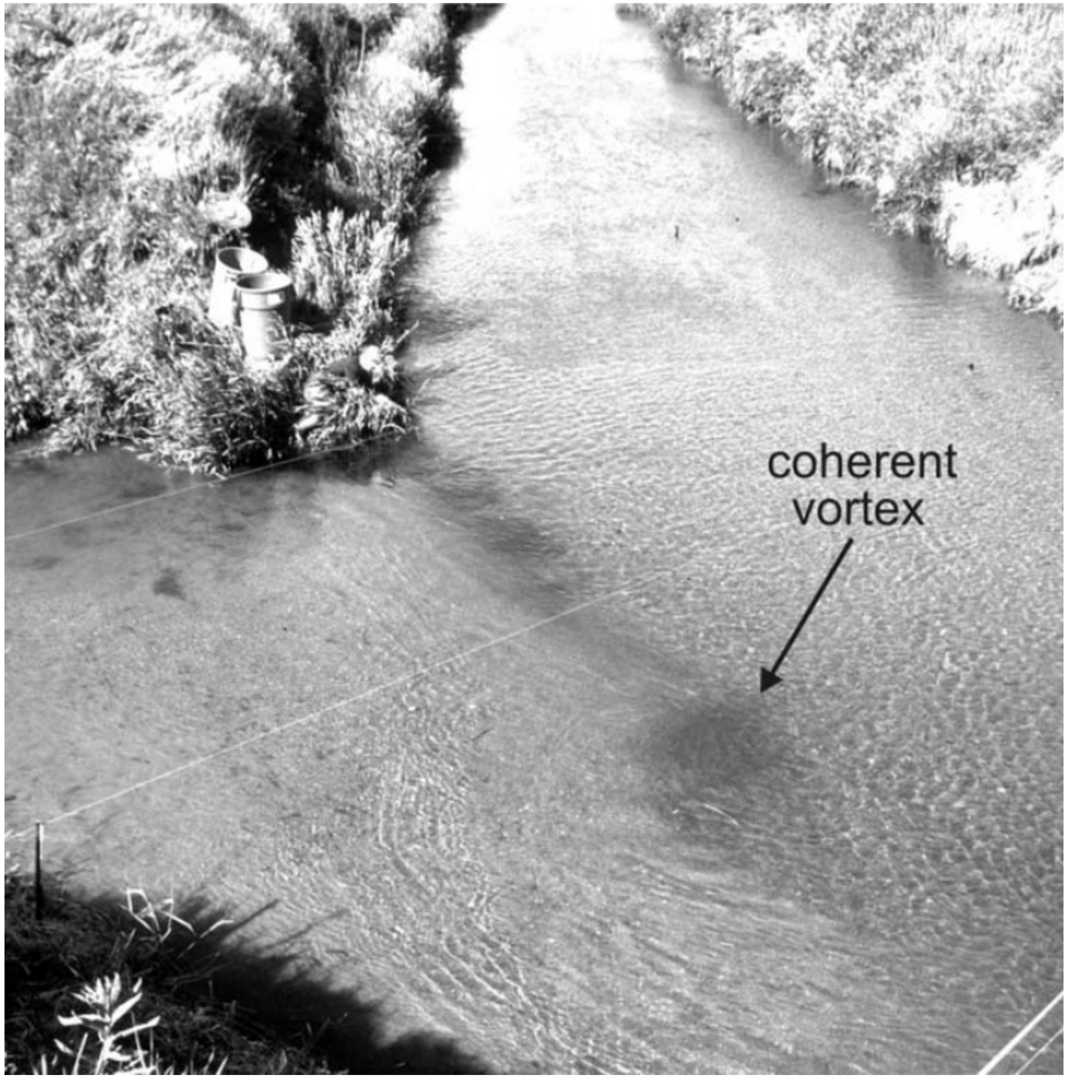
Flow visualization by laser-induced fluorescence in a cross-section plane



Experiments performed in water, $U_1 = 57$ cm.s^{-1}, $U_2 = 23$ cm.s^{-1}, the lower low-speed fluid is marked with fluorescein dye

Bernal & Roshko, 1986, *J. Fluid Mech.*, **170**

Shear layer at a stream confluence



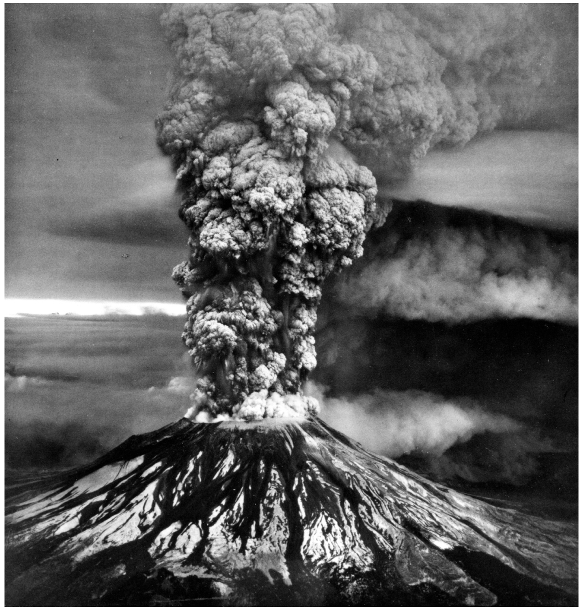
Kaskaskia River - Copper Slough confluence in East Central Illinois

(Rhoads & Sukhodolov, 2004, *Water Resources Research*)

- **Rhône and Saône rivers meeting point, Lyon**

Lyon 2021-05-08

Incursion into the field of forced plumes !



Mount St. Helens Eruption on 16 May 1980 (Washington, USA). *Courtesy of Longview Daily News, Washington* (Rodi, 1982)

Eruption of the subglacial Grimsvötn volcano, Iceland, on 21 May 2011
An initial large plume of smoke and ash rose up to about 17 km height.
Courtesy of Thördïs Högnadóttir, Institute of Earth Sciences, University of Iceland
(Bailly & Comte-Bellot, 2015)

Incursion into the field of forced plumes (cont.)

Smoke plume in the presence of a near surface temperature inversion, which hinders vertical turbulent motion and drives the plume horizontally. *The Quays Shopping Centre*, near to Tiur and Cloghoge, taken by Eric Jones (Bailly & Comte-Bellot, 2015)

Two examples of application

1. Forced plume in atmosphere. At what altitude does the difference between the temperature of the plume and the temperature of the quiescent surroundings become less than 1 deg. ?

Data to solve the problem later

$D = 1$ m

$U_P = 3$ m.s^{-1}

$\Theta_P = 273$ K

$T_0 = 273$ K

2. Thermal pollution. At a river mouth, fresh water is pumped out to sea in a large round pipe and released at the bottom. At what depth must the fresh water be released to avoid raising the temperature in the first 30 m below the surface by more than 1 deg. ?

Data to solve the problem later

$Q_v = 10$ m^3.s^{-1}

$\rho_P = 1.0 \times 10^3$ kg.m^{-3}

$T_P = 35°$C

$\rho_0 = 1.03 \times 10^3$ kg.m^{-3}

$T_0 = 5°$C

Boussinesq approximation (1903)

- Density fluctuations which appear in gouverning equations result principally from thermal (as opposed to pressure $\sim \mathrm{M}^2$) effects, $\rho \simeq \rho(T)$. Its Taylor series leads to

$$\rho \simeq \rho_0(1 - \beta(T - T_0)) \qquad \beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\bigg|_p = \frac{1}{T_0}$$

(thermal expansion coefficient for ideal gas)

- In the equations for the momentum and mass conservation, density variations may be neglected except in the buoyancy force $\rho \boldsymbol{g}$

These approximations are valid if the velocity perturbations are small ($\mathrm{M} \ll 1$), and when the variations in $\rho_0(z)$ and $T_0(z)$ of the quiescent surroundings are small over the vertical size of the flow. As an exercise, you can check from hydrostatic balance that the temperature varies by 1 degree every 100 meters for an adiabatic atmosphere (6.5 degree /km with a more realistic model)

Gouverning equations

Viscosity and conductivity are neglected (free jet flow, no wall, Re and Pe $\gg 1$). The hydrostatic balance $-\nabla p_0 + \rho_0 \boldsymbol{g} = 0$ is also substracted from the momentum conservation, which leads to

$$\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho_0} \nabla (p - p_0) - \frac{1}{T_0}(T - T_0)\boldsymbol{g}$$

$$\nabla \cdot \boldsymbol{U} = 0$$

$$\frac{\partial (T - T_0)}{\partial t} + \boldsymbol{U} \cdot \nabla (T - T_0) = 0$$

Reynolds decomposition

$$\begin{cases} T - T_0 = \bar{T} + \theta' \\ p - p_0 = \bar{P} + p' \\ \boldsymbol{u} = \bar{\boldsymbol{U}} + \boldsymbol{u}' \quad (\text{no wind, } \boldsymbol{U}_0 = 0) \end{cases}$$

Boundary-layer approximation applied to free turbulent flow : the divergence of the thin shear layer is slow.

- **Gouverning equations (cont.)**
  Reynolds-averaged Euler Eqs. including the buoyancy force

$$\frac{1}{r}\frac{\partial(r\bar{U}_r)}{\partial r} + \frac{\partial \bar{U}_z}{\partial z} = 0 \qquad (6)$$

$$\frac{1}{r}\frac{\partial(r\bar{U}_r\bar{U}_z)}{\partial r} + \frac{\partial(\bar{U}_z\bar{U}_z)}{\partial z} = -\frac{1}{r}\frac{\partial(r\overline{u_r'u_z'})}{\partial r} + \frac{\bar{T}}{T_0}g \qquad (7)$$

$$\frac{1}{r}\frac{\partial(r\bar{U}_r\bar{T})}{\partial r} + \frac{\partial(\bar{U}_z\bar{T})}{\partial z} = -\frac{1}{r}\frac{\partial(r\overline{u_r'\theta'})}{\partial r} \qquad (8)$$

Densimetric Froude number Fr

ambient fluid : ρ_0, T_0
forced plume : D, U_P, ρ_P, $T_P = T_0 + \Theta_P \quad (\rho_P T_P = \rho_0 T_0)$

$$\mathrm{Fr} = \left(\frac{\text{inertial forces}}{\text{buoyancy forces}}\right)^{1/2} = \left(\frac{U_P^2/D}{g\Theta_P/T_0}\right)^{1/2} = \frac{U_P}{\sqrt{D\,g(\rho_0/\rho_P - 1)}} \qquad (9)$$

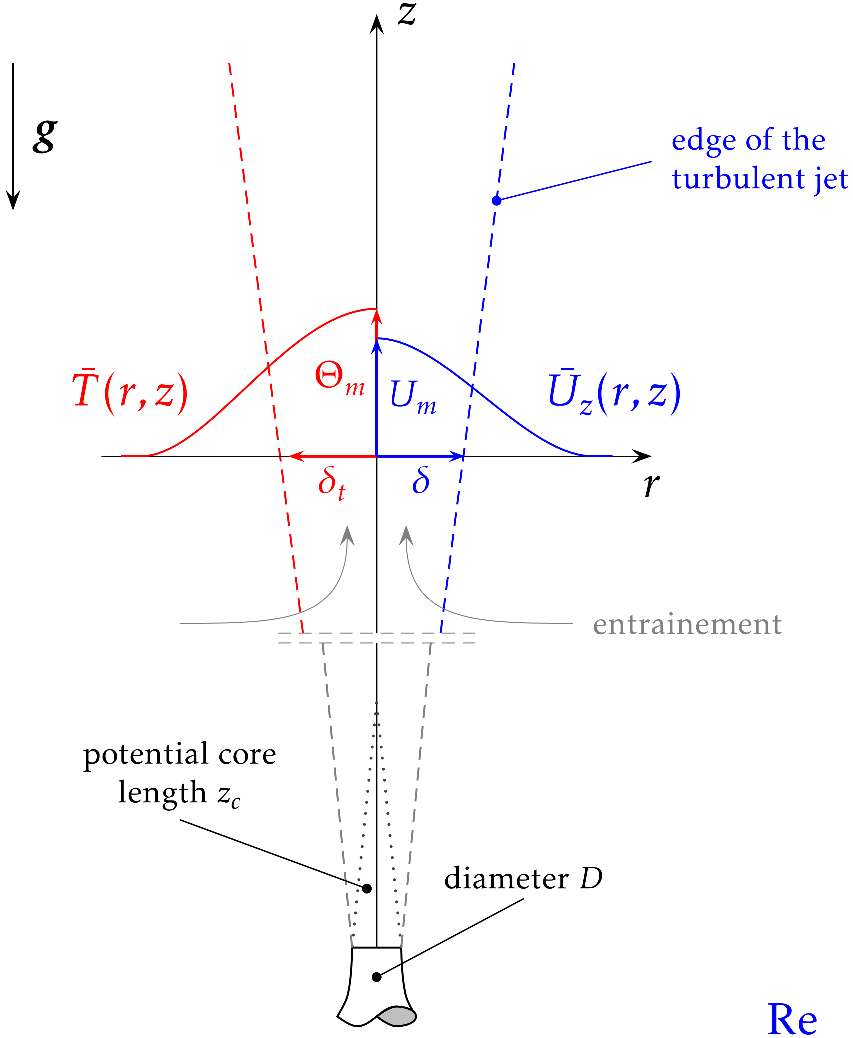
Froude number for forced plumes

Sketch of the mean velocity and temperature fields in the region where the flow has reached a self-similar state ($z/D \geq 10$)

Pure plume : $\mathrm{Fr} = 0$

Buoyant jet / forced plume :
$0 < \mathrm{Fr} < \infty$

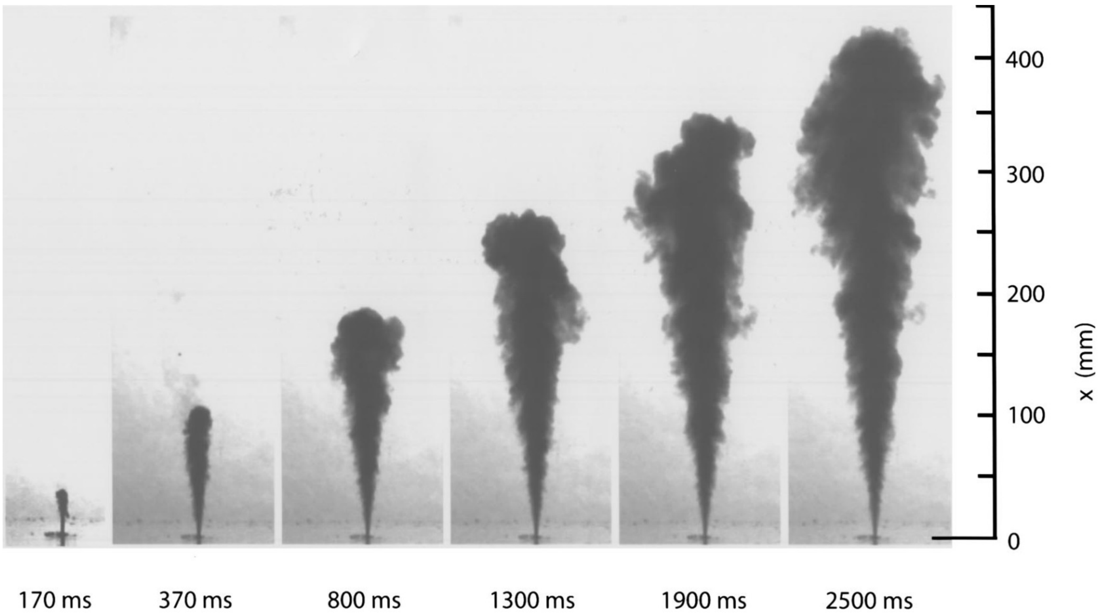
Pure jet : $\mathrm{Fr} = \infty$



Fr Re

Froude number for forced plumes (cont.)

Visualization of starting plumes

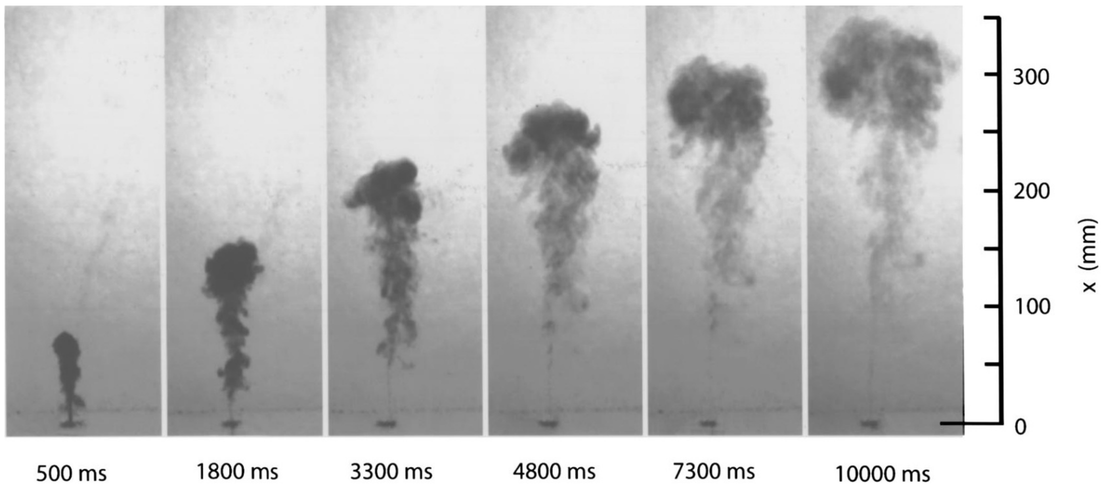


Re = 9000, Fr = 58.5

$D = 3.2$ mm

$U_P = 3700$ mm.s^{-1}

$\rho_P/\rho_0 = 1.150$



Re = 4000, Fr = 29.3

$D = 3.2$ mm

$U_P = 1850$ mm.s^{-1}

$\rho_P/\rho_0 = 1.150$

Diez, Sangras & Faeth, 2003, *J. Heat Transfer*, **125**

Derivation of the self-similar solution

Only one length scale ($\delta_t \simeq \delta$) is considered to find the self-similar solution

$$\eta = \frac{r}{\delta_l(z)} \qquad \begin{cases} \bar{U}_z = U_m(z)\, f(\eta) \\ \bar{U}_r = U_m(z)\, g(\eta) \\ -\eta\overline{u_r' u_z'} = U_m^2(z)\, h(\eta) \\ \bar{T} = \Theta_m(z)\, F(\eta) \\ -\overline{u_r'\theta'} = U_m(z)\Theta_m(z)\, H(\eta) \end{cases} \tag{10}$$

At first, we are interested in the asymptotic behavior of the functions $U_m(z)$ (velocity on the plume axis), $\Theta_m(z)$ (temperature on the plume axis) and $\delta_l(z)$ (the half-width of the plume), and not in the expression of the profiles f, g, h, F and H.

Self-similar solution (cont.)

Eq. (7) $\Longrightarrow$

$$-r\overline{u'_r u'_z} = \int_0^r \frac{\partial \bar{U}_z \bar{U}_z}{\partial z} r'dr' - \bar{U}_z \int_0^r \frac{\partial \bar{U}_z}{\partial z} r'dr' - \int_0^r \frac{\bar{T}}{T_0} g r'dr'$$

$$= \frac{\partial}{\partial z}\left\{ U_m^2 \delta^2 \int_0^\eta f^2 \eta' d\eta' \right\} - U_m f \frac{\partial}{\partial z}\left\{ U_m \delta^2 \int_0^\eta f \eta' d\eta' \right\} - \Theta_m \delta^2 \frac{g}{T_0} \int_0^\eta F \eta' d\eta'$$

with the change of variable $\eta' = r'/\delta(z)$

$$= \frac{d}{dz}(U_m^2 \delta^2) \int_0^\eta f^2 \eta' d\eta' + U_m^2 \delta^2 f^2(\eta) \eta \frac{d\eta}{dz}$$

$$- U_m f \frac{d}{dz}(U_m \delta^2) \int_0^\eta f \eta' d\eta' - U_m^2 \delta^2 f^2(\eta) \eta \frac{d\eta}{dz} - \Theta_m \delta^2 \frac{g}{T_0} \int_0^\eta F \eta' d\eta'$$

by using $\displaystyle \frac{d}{dx}\int_{a(x)}^{b(x)} I(x,y)dy = \int_{a(x)}^{b(x)} \frac{\partial I}{\partial x} dy + I(x,b(x))\frac{db}{dx} - I(x,a(x))\frac{da}{dx}$

Self-similar solution (cont.)

The Reynolds shear stress is finally recast as,

$$\underbrace{-\eta\frac{\overline{u_r'u_z'}}{U_m^2} = h(\eta)}_{\text{given by Eq. (10)}} = \frac{(U_m^2\delta^2)'}{U_m^2\delta}\int_0^\eta f^2\eta' d\eta' - \frac{(U_m\delta^2)'}{U_m\delta} f\int_0^\eta f\eta' d\eta' - \frac{\theta_m\delta}{U_m^2}\frac{g}{T_0}\int_0^\eta F\eta' d\eta'$$

A self-similar solution can be obtained only if the red terms are constant terms, and not terms functions of z, which leads to the following behaviours for a plume,

$$\delta \sim z \qquad U_m \sim z^m \qquad \Theta_m \sim z^{2m-1}$$

Self-similar solution (cont.)

Conservation of the heat flux in a cross-section to determine m

By integration of Eq. (8) in the radial direction,

$$\underbrace{\left[r\bar{U}_r\bar{T}\right]_0^\infty}_{=0} + \int_0^\infty \frac{(\partial \bar{U}_z\bar{T})}{\partial z} r' dr' = \underbrace{\left[-r\overline{u'_r\theta'}\right]_0^\infty}_{=0}$$

Therefore,

$$\int_0^\infty \bar{U}_z\bar{T} r' dr' = \text{cst} \qquad \Longrightarrow \qquad U_m\Theta_m\delta^2 \int_0^\infty f F \eta' d\eta' = \text{cst}$$

A self-similar solution is then only possible if

$$\delta \sim z \qquad U_m \sim z^{-1/3} \qquad \Theta_m \sim z^{-5/3}$$

- Faster decrease in temperature than in velocity (for a single free round jet, we have $\delta \sim z$ and $U_m \sim z^{-1}$)
- The Froude number value is a constant, $\text{Fr}^2 = (U_m^2/\delta)/(g\Theta_m/T_0) = \text{cst}$

Self-similar solution (cont.)

Experimental data collected by Rodi (1986)

In the plume region,
for $z/z^\star > 5.0$ with $z^\star/D = \mathrm{Fr}^{1/2}\,(\rho_P/\rho_0)^{1/4}$

$$\frac{U_m}{U_P} \simeq 3.5\,\mathrm{Fr}^{-1/3}\left(\frac{\rho_P}{\rho_0}\right)^{1/3}\left(\frac{z}{D}\right)^{-1/3} \qquad \frac{\Theta_m}{\Theta_P} \simeq 9.35\,\mathrm{Fr}^{1/3}\left(\frac{\rho_P}{\rho_0}\right)^{1/3}\left(\frac{z}{D}\right)^{-5/3}$$

Two examples of application

1. Forced plume in atmosphere. At what altitude does the difference between the temperature of the plume and the temperature of the quiescent surroundings become less than 1 deg. ?

Data to solve the problem

$D = 1$ m

$U_P = 3$ m.s^{-1}

$\Theta_P = 273$ K

$T_0 = 273$ K

Solution.

$T_P = T_0 + \Theta_P = 2T_0 \quad \Longrightarrow \quad \rho_P = \frac{1}{2}\rho_0$ (constant pressure in the plume, then using $p = \rho r T$)

From its definition (9),

$$\mathrm{Fr}^2 = \frac{3^2/1}{9.81 \times 1} \simeq 0.92 \text{ (plume)}$$

Numerically,

$$\frac{z}{D} \simeq \left(\frac{1}{9.35\,\mathrm{Fr}^{1/3}}\frac{\Theta_m}{\Theta_P}\right)^{-3/5}\left(\frac{\rho_0}{\rho_P}\right)^{-1/5}$$

$$\simeq \left(\frac{1}{9.35 \times 0.92^{1/3}}\frac{1}{273}\right)^{-3/5} 2^{-1/5}$$

$$\simeq 95 \text{ m}$$

(we verify that the ambient temperature T_0 is almost constant along this distance)

Two examples of application

2. Thermal pollution. At a river mouth, fresh water is pumped out to sea in a large round pipe and released at the bottom. At what depth must the fresh water be released to avoid raising the temperature in the first 30 m below the surface by more than 1 deg. ?

Data to solve the problem

$Q_v = 10\ \mathrm{m}^3.\mathrm{s}^{-1}$

$\rho_P = 1.0 \times 10^3\ \mathrm{kg.m}^{-3}$

$T_P = 35°\mathrm{C}$

$\rho_0 = 1.03 \times 10^3\ \mathrm{kg.m}^{-3}$

$T_0 = 5°\mathrm{C}$

Solution.

Two contribution for the buoyancy force : temperature T_P versus T_0, and also density of fresh water (fw) versus salt water (sw)

Mixing of salt and fresh water by entrainment inside the plume to reach $T = 6°\mathrm{C}$; from the conservation of energy,

$$M_{\mathrm{fw}} C_p T_P + M_{\mathrm{sw}} C_p T_0 = (M_{\mathrm{fw}} + M_{\mathrm{sw}}) C_p T$$

$$\frac{M_{\mathrm{sw}}}{M_{\mathrm{fw}}} = \frac{T_P - T}{T - T_0} = \frac{35 - 6}{6 - 5} = 29$$

In other words, 1 kg of fresh water has entrained 29 kg of salt water in order to reach $T = 6°\mathrm{C}$. For this temperature (and this altitude), the plume density is then estimated at

$$\rho = \frac{\rho_{\mathrm{fw}} + 29\rho_{\mathrm{sw}}}{30} \simeq 1.029 \times 10^3\ \mathrm{kg.m}^{-3}$$

Two examples of application

Solution (cont.)

For an ideal gas, $p = \rho r T$, and for $p = \text{cst}$, it yields by logarithmic differentiation

$$-\frac{\Delta\rho}{\rho} = \frac{\Delta T}{T}$$

and for a liquid, the boyancy force is $-g\Delta\rho/\rho_0$, where $\Delta\rho$ takes into account temperature and density effects here. Hence, by analogy with the gas plume, we still use for the liquid plume

$$\frac{\rho - \rho_0}{\rho_P - \rho_0} \simeq 9.35\,\mathrm{Fr}^{1/3}\left(\frac{\rho_P}{\rho_0}\right)^{1/3}\left(\frac{z}{D}\right)^{-5/3}$$

Numerically, $z \simeq 70$ m, and the required distance from the nozzle to the surface is thus 100 m.