*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

ÉCOLE CENTRALE LYON

Flow stability and Introduction to turbulence

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Centrale Lyon – 2A FLE-1a – 20-01-2025

http://acoustique.ec-lyon.fr

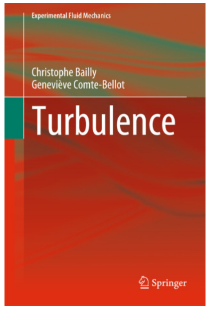
Introduction to turbulence

Textbooks

Batchelor, G.K., 1967, An introduction to fluid dynamics, *Cambridge University Press*, Cambridge.

Bailly C. & **Comte Bellot** G., 2003 Turbulence, *CNRS éditions*, Paris (out of print).
———, 2015, *Turbulence* (in english), Springer, Heidelberg.
(360 pages, 147 illustrations, Foreword by Charles Meneveau, 53 € for ECL students)

Bailly C. & Comte Bellot G., 2003, *Turbulence* (in french), CNRS éditions, Paris.
———, 2015, *Turbulence* (in english), Springer, Heidelberg.



Springer, ISBN 978-3-319-16159-4,
360 pages, 147 illustrations.
(discount for students, 53 €)

Candel S., 1995, Mécanique des fluides, *Dunod Université*, 2nd édition, Paris.

Cousteix, J., 1989, Turbulence et couche limite, Cépaduès, Toulouse.

Davidson P.A., 2004, *Turbulence. An introduction for scientists and engineers*, Oxford University Press, Oxford.

Davidson, P.A., **Kaneda**, Y., **Moffatt**, H.K. & **Sreenivasan**, K.R., Edts, 2011, *A voyage through Turbulence*, Cambridge University Press, Cambridge.

Guyon E., **Hulin** J.P. & **Petit** L., 2001, Physical hydrodynamics, *EDP Sciences / Editions du CNRS*, première édition 1991, Paris - Meudon.

Textbooks (cont.)

Hinze J.O., 1975, Turbulence, *McGraw-Hill International Book Company*, New York, 1ère édition en 1959.

Landau L. & **Lifchitz** E., 1971, Mécanique des fluides, *Editions MIR, Moscou*.
Also *Pergamon Press*, 2nd edition, 1987.

Lesieur M., 2008, Turbulence in fluids : stochastic and numerical modelling, *Kluwer Academic Publishers, 4th revised and enlarged ed.*, Springer.

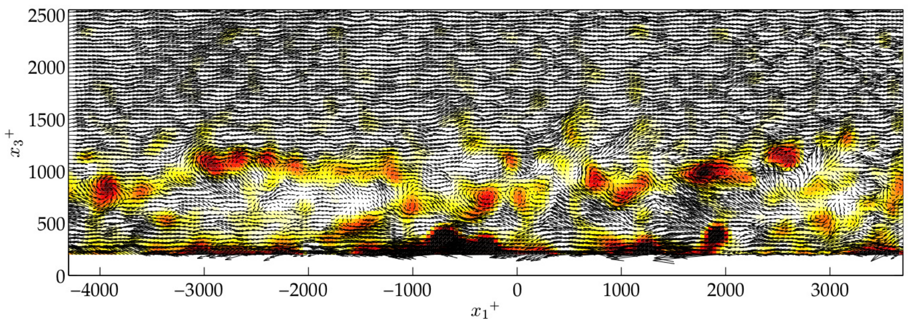
Pope S.B., 2000, Turbulent flows, *Cambridge University Press*.

Tennekes H. & **Lumley** J.L., 1972, A first course in turbulence, *MIT Press*, Cambridge, Massachussetts.

Van Dyke M., 1982, An album of fluid motion, *The Parabolic Press*, Stanford, California.

White F., 1991, Viscous flow, *McGraw-Hill, Inc.*, New-York, first edition 1974.

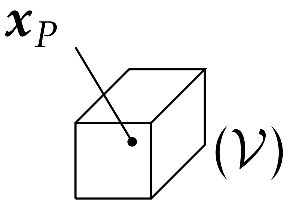
Identification of vortical structures
(When is a region of vorticity a vortex?)



2-D 2-C PIV snapshot, $\boldsymbol{u} - 0.85\boldsymbol{U}_\infty$, colored by vorticity magnitude ω_2, from Salze *et al.* (2015)

Introduction : deformation of a fluid particle

Taylor series for the velocity in the vicinity of a fluid particle at $\boldsymbol{x}_P$ (at a given time t)



$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_P) + \nabla\boldsymbol{u}(\boldsymbol{x}_P)\cdot(\boldsymbol{x}-\boldsymbol{x}_P) + \cdots$$

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{x}_P) + \left.\frac{\partial u_i}{\partial x_j}\right|_{x_P}(x_j - x_{Pj}) + \cdots$$

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{D_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}}$$

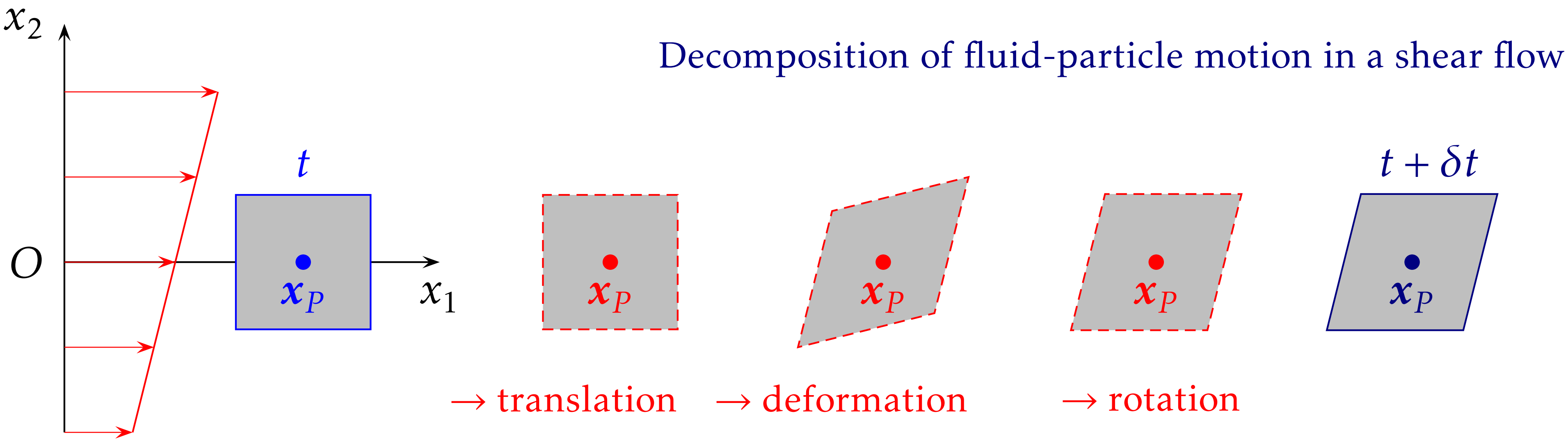
- $\overline{\overline{\boldsymbol{D}}}$ symmetric part of $\nabla\boldsymbol{u}$
- $\overline{\overline{\boldsymbol{\Omega}}}$ antisymmetric part of $\nabla\boldsymbol{u}$

Ω_{ij} is associated with the rotation of the fluid particle

$$\overline{\overline{\boldsymbol{\Omega}}} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix} \qquad \boldsymbol{\Omega} \equiv \frac{1}{2}\nabla\times\boldsymbol{u} \quad (\neq \overline{\overline{\boldsymbol{\Omega}}})$$

Deformation of vorticity

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}_P) + \underbrace{\overline{\overline{\boldsymbol{D}}}(\boldsymbol{x}_P)\cdot(\boldsymbol{x}-\boldsymbol{x}_P)}_{\text{deformation}} + \underbrace{\boldsymbol{\Omega}(\boldsymbol{x}_P)\times(\boldsymbol{x}-\boldsymbol{x}_P)}_{\text{rotation}} + \ldots$$



The vorticity vector is defined as $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$

When $\boldsymbol{\omega} = 0$, absence of vorticity, the flow is irrotational

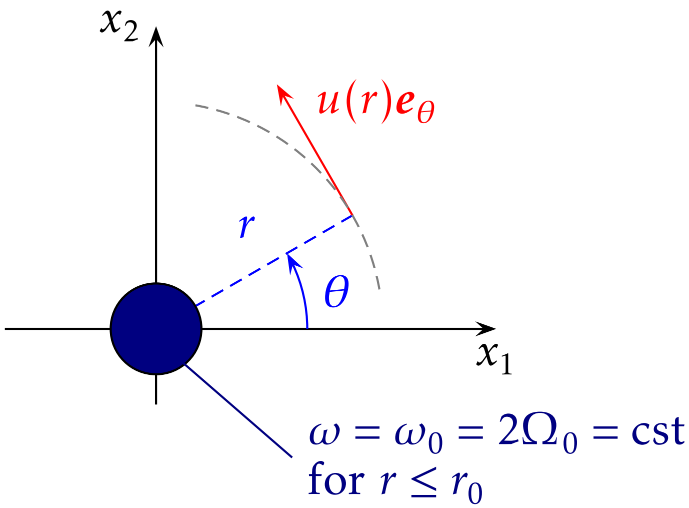
The vorticity vector $\boldsymbol{\omega}$ is twice the angular velocity of the solid-body rotation motion of the fluid particle, $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$.

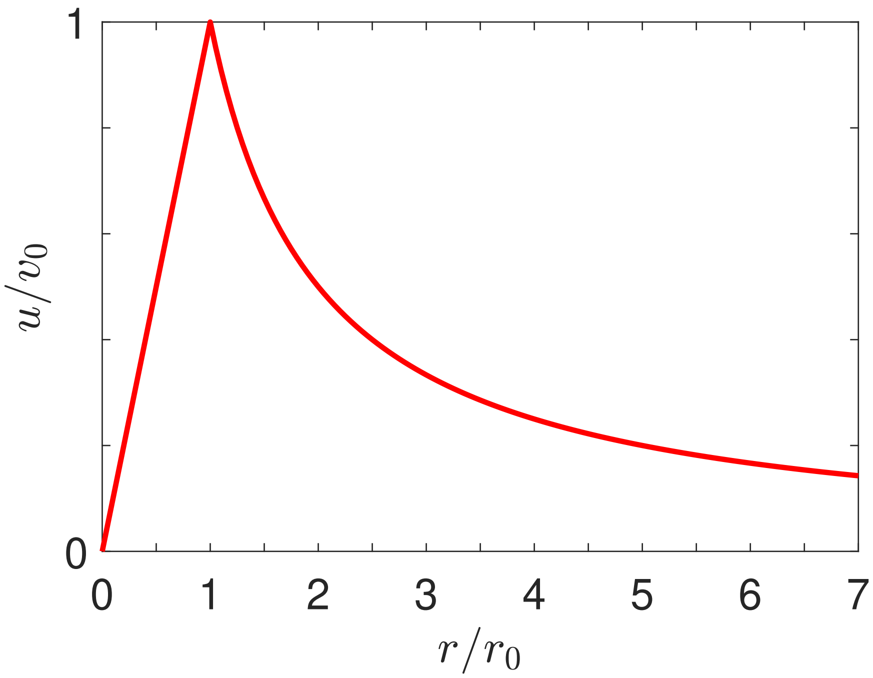
Example of the Rankine vortex (1858)

Rankine (1820-1872)

$$\begin{cases} u(r) = v_0 \dfrac{r}{r_0} = \Omega_0 r & r \leq r_0 \\ u(r) = v_0 \dfrac{r_0}{r} = \Omega_0 r_0 \dfrac{r_0}{r} & r > r_0 \end{cases}$$

$(v_0 = \Omega_0 r_0 = \omega_0 r_0 / 2)$

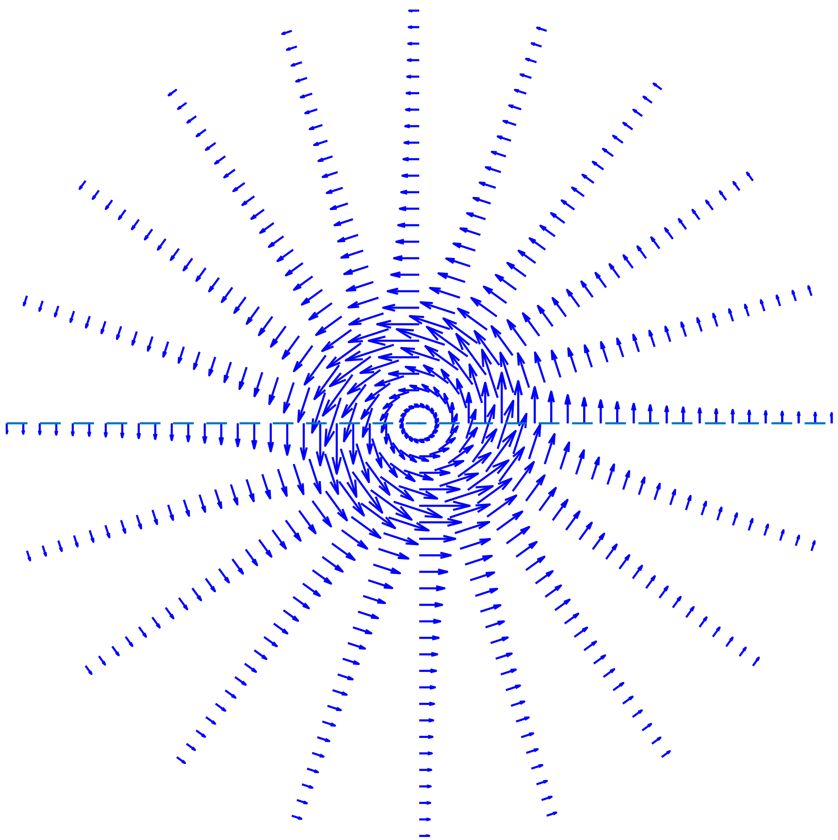




Solid body motion inside the vortex itself, *i.e.* for $r \leq r_0$ in the vortical region

Irrotational flow outside, for $r > r_0$: the localized circular patch of vorticity produces a velocity field away from the vortical region

Example of the Rankine vortex (cont.)



Pressure field ?

Inviscid steady flow, Euler's equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r}$$

$$\begin{cases} p = p_\infty - \dfrac{\rho v_0^2}{2}\left(2 - \dfrac{r^2}{r_0^2}\right) & r \leq r_0 \\ p = p_\infty - \dfrac{\rho v_0^2}{2}\dfrac{r_0^2}{r^2} & r > r_0 \end{cases}$$

More generally, can vortex structures be identified with local pressure minimum ?

Identification of vortices in turbulent flow

Equation for the pressure (incompressible flow, $\rho = \text{cst}$), by taking the divergence of Navier-Stokes equation

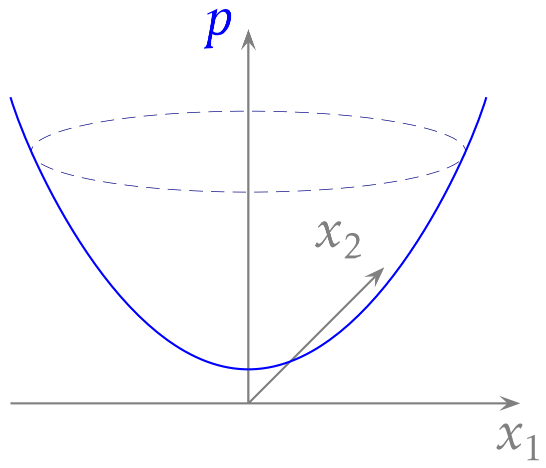
$$-\frac{1}{\rho}\nabla^2 p = \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

Using the previous decomposition of the velocity gradient tensor (the product of a symmetric and an antisymmetric tensor is zero)

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = (D_{ij} + \Omega_{ij})(D_{ij} - \Omega_{ij}) = D_{ij}D_{ij} - \Omega_{ij}\Omega_{ij} = \overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}} - \overline{\overline{\boldsymbol{\Omega}}} : \overline{\overline{\boldsymbol{\Omega}}}$$

$$\frac{1}{\rho}\nabla^2 p = \overline{\overline{\boldsymbol{\Omega}}} : \overline{\overline{\boldsymbol{\Omega}}} - \overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}}$$

A vortex may then be defined by a concentrated flow region dominated by $\overline{\overline{\boldsymbol{\Omega}}} : \overline{\overline{\boldsymbol{\Omega}}}$, and consequently we expect $\nabla^2 p > 0$ (positive curvature, local minimum)



Identification of vortices (cont.)

The source term of Poisson's equation for the pressure is one of the three invariants (invariant, that is independent of the orientation of the coordinate system) of the velocity gradient tensor $A_{ij} \equiv \partial u_i / \partial x_j = (\nabla \boldsymbol{u})_{ij}$

The three invariants of a second-order tensor ($\overline{\overline{\boldsymbol{A}}}$ here) are given by

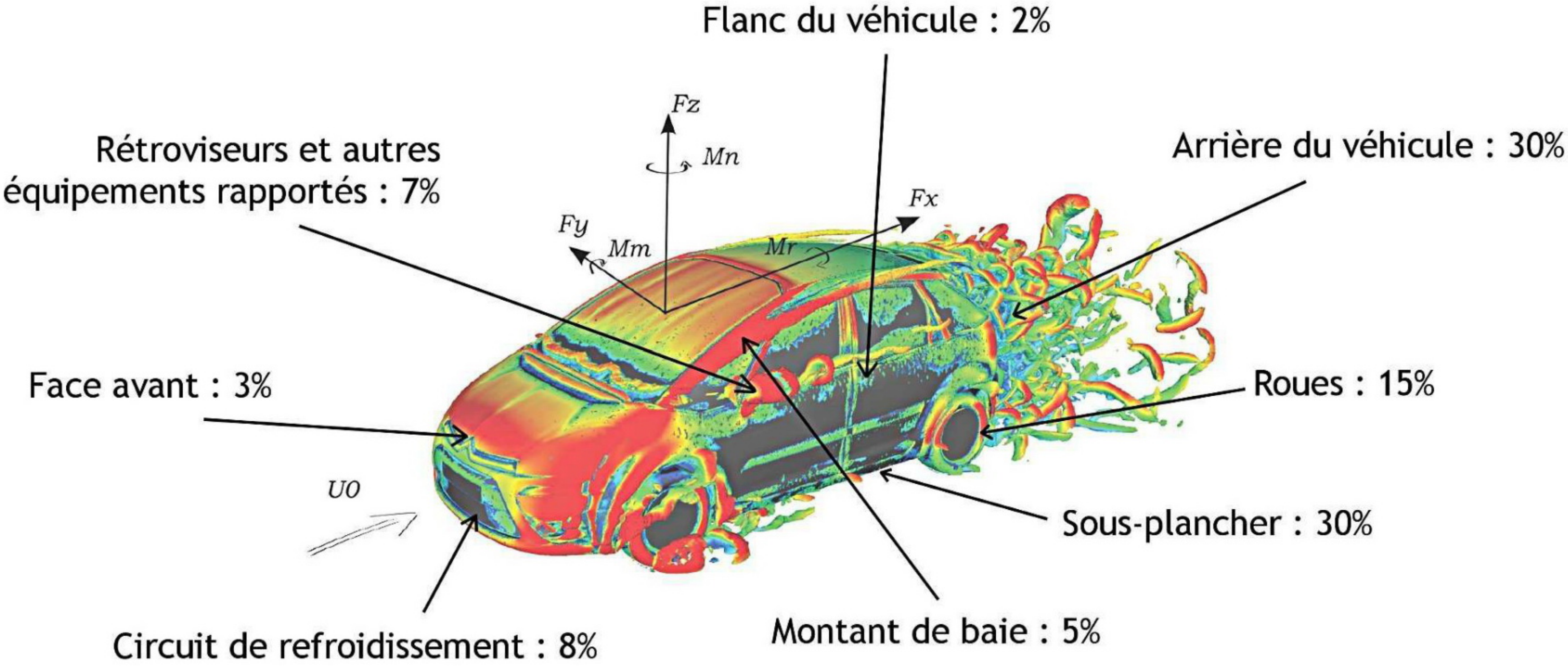
$$\begin{cases} P = \mathrm{tr}(\overline{\overline{\boldsymbol{A}}}) = A_{ii} \\ Q = \dfrac{1}{2}\left[\mathrm{tr}^2(\overline{\overline{\boldsymbol{A}}}) - \mathrm{tr}(\overline{\overline{\boldsymbol{A}}}^2)\right] = \dfrac{1}{2}\left[(A_{ii})^2 - A_{ij}A_{ji}\right] \\ R = \det(\overline{\overline{\boldsymbol{A}}}) = \dfrac{1}{6}\mathrm{tr}^3(\overline{\overline{\boldsymbol{A}}}) - \dfrac{1}{2}\mathrm{tr}(\overline{\overline{\boldsymbol{A}}})\mathrm{tr}^2(\overline{\overline{\boldsymbol{A}}}) + \dfrac{1}{3}\mathrm{tr}(\overline{\overline{\boldsymbol{A}}}^3) \end{cases}$$

The incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ leads to $P = 0$ and to $Q = -A_{ij}A_{ji}/2$. Hence, the previous pressure equation reads

$$\frac{1}{\rho}\nabla^2 p = \overline{\overline{\boldsymbol{\Omega}}} : \overline{\overline{\boldsymbol{\Omega}}} - \overline{\overline{\boldsymbol{D}}} : \overline{\overline{\boldsymbol{D}}} = 2Q$$

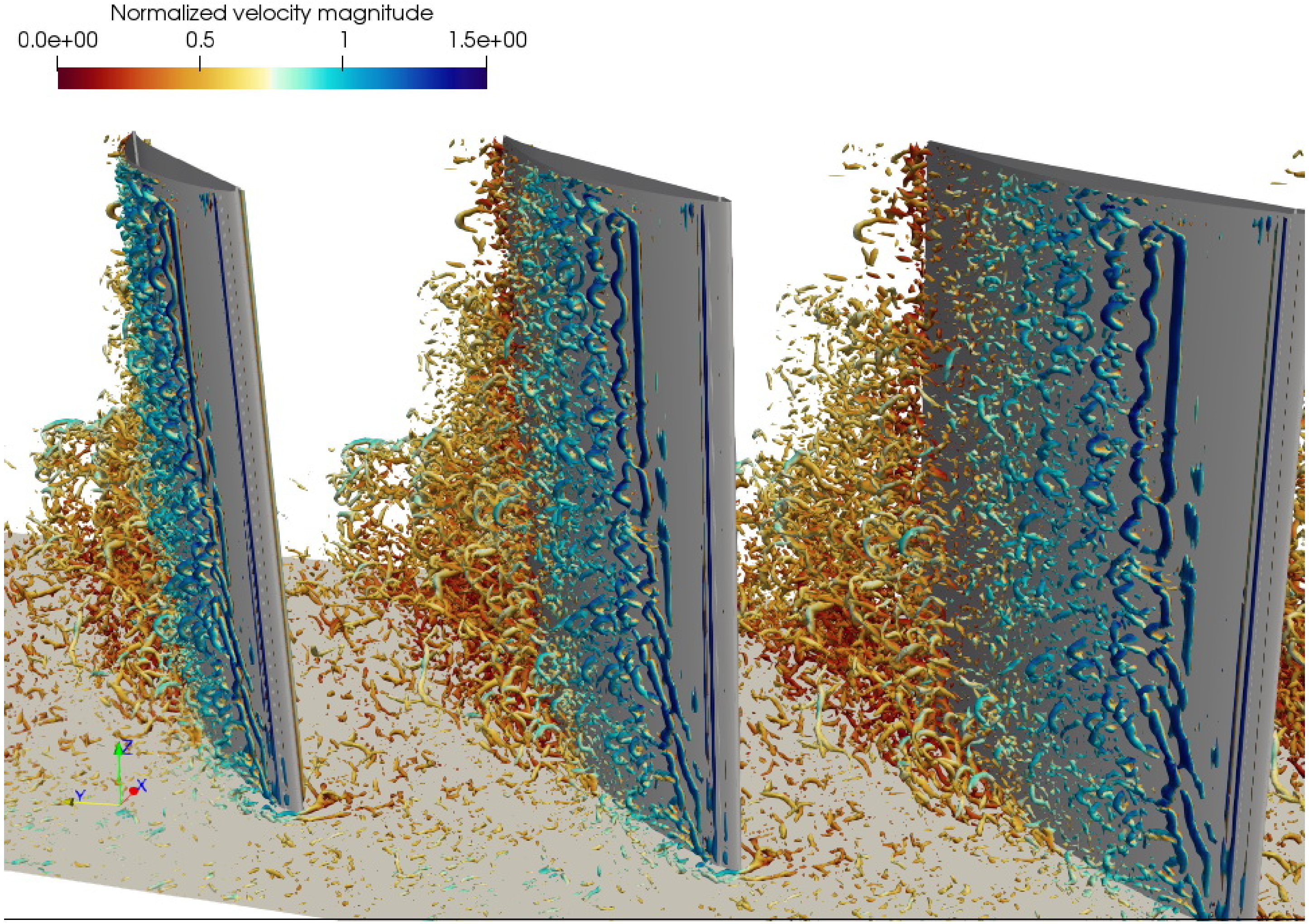
Vortical structures are thus expected to be identified for high positive values of the invariant Q, leading to the so-called Q-criterion

- **As an illustration : total drag breakdown of a car, and iso-surfaces of Q-criterion colored with velocity magnitude**



(Fiabane, PhD thesis, 2011, ENSTA-PSA)

Illustration : corner separation in a compressor cascade



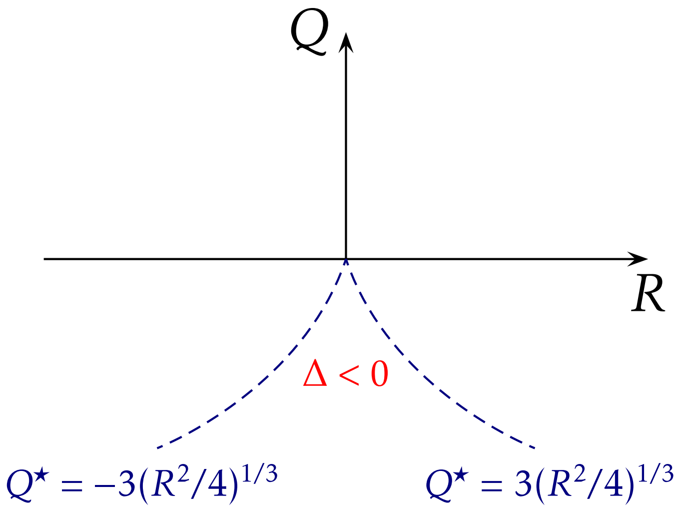
Lattice Boltzmann simulations of corner separation flow in a compressor cascade. Instantaneous isosurface of Q-criterion, colored by velocity magnitude. Turbulent structures develop around the blades and accumulate in the separation zone.

Boudet, Lévêque & Touil, 2022, *J. Turbomachinary*, **144**

Identification of vortices (cont.)

The eigenvalues λ_i of $\overline{\overline{A}}$ are the roots of the characteristic equation
$\lambda^3 - P\lambda^2 + Q\lambda - R = 0$,
with $P = \lambda_1 + \lambda_2 + \lambda_3$ ($P = \nabla \cdot \boldsymbol{u} = 0$ for incompressible flow),
$Q = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$ and $R = \lambda_1\lambda_2\lambda_3$

Hence, the characteristic equation reads $\lambda^3 + Q\lambda - R = 0$. In introducing the discriminant $\Delta = Q^3/27 + R^2/4$, one finds three real values λ_i for $\Delta < 0$, or two complex conjugate values $\lambda_{1,2} = \sigma \pm i\omega$ and one real value λ_3 for $\Delta > 0$.



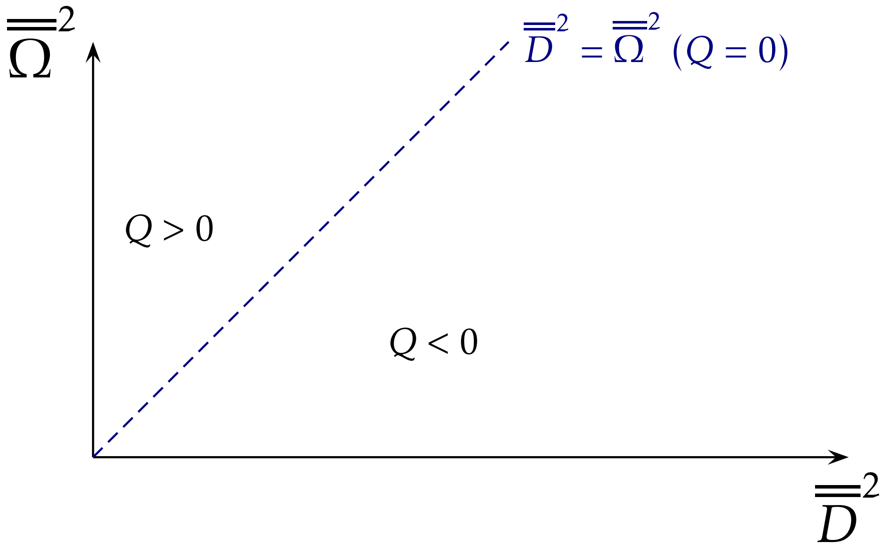


Illustration with an incompressible 2-D flow

Stream function $\psi = ax_1^2 + bx_2^2$ and velocity field

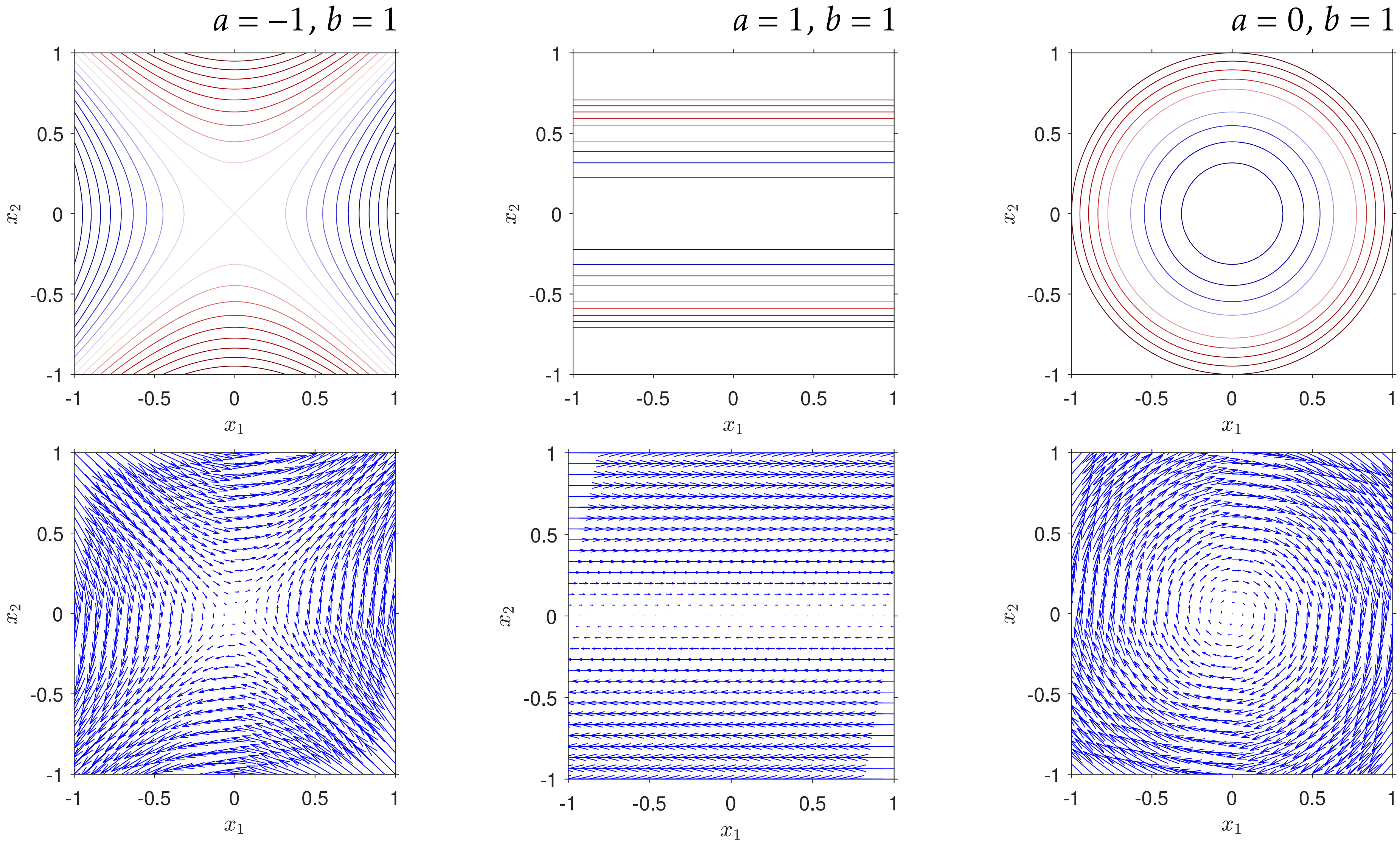


Illustration with an incompressible 2-D flow

$$\psi = \frac{\omega - s}{4} x_1^2 + \frac{\omega + s}{4} x_2^2 \qquad \begin{cases} u_1 = \dfrac{\partial \psi}{\partial x_2} = \dfrac{\omega + s}{2} x_2 \\ u_2 = -\dfrac{\partial \psi}{\partial x_1} = -\dfrac{\omega - s}{2} x_1 \\ \omega_3 = \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} = -\nabla^2 \psi = -\dfrac{\omega - s}{2} - \dfrac{\omega + s}{2} = -\omega \end{cases}$$

$$\overline{\overline{\boldsymbol{D}}} = \begin{pmatrix} 0 & s/2 \\ s/2 & 0 \end{pmatrix} \qquad \overline{\overline{\boldsymbol{\Omega}}} = \begin{pmatrix} 0 & \omega/2 \\ -\omega/2 & 0 \end{pmatrix}$$

$$\frac{1}{\rho} \nabla^2 p = \Omega_{ij}\Omega_{ij} - D_{ij}D_{ij} = \frac{1}{2}(\omega^2 - s^2) = 2Q$$

Topologie des lignes de courant $\psi = \text{cst}$
elliptiques $(\omega - s)(\omega + s) > 0 \Longrightarrow Q > 0 \Longrightarrow \nabla^2 p > 0$
hyperboliques $(\omega - s)(\omega + s) < 0 \Longrightarrow Q < 0 \Longrightarrow \nabla^2 p < 0$

Illustration with an incompressible 2-D flow (cont.)

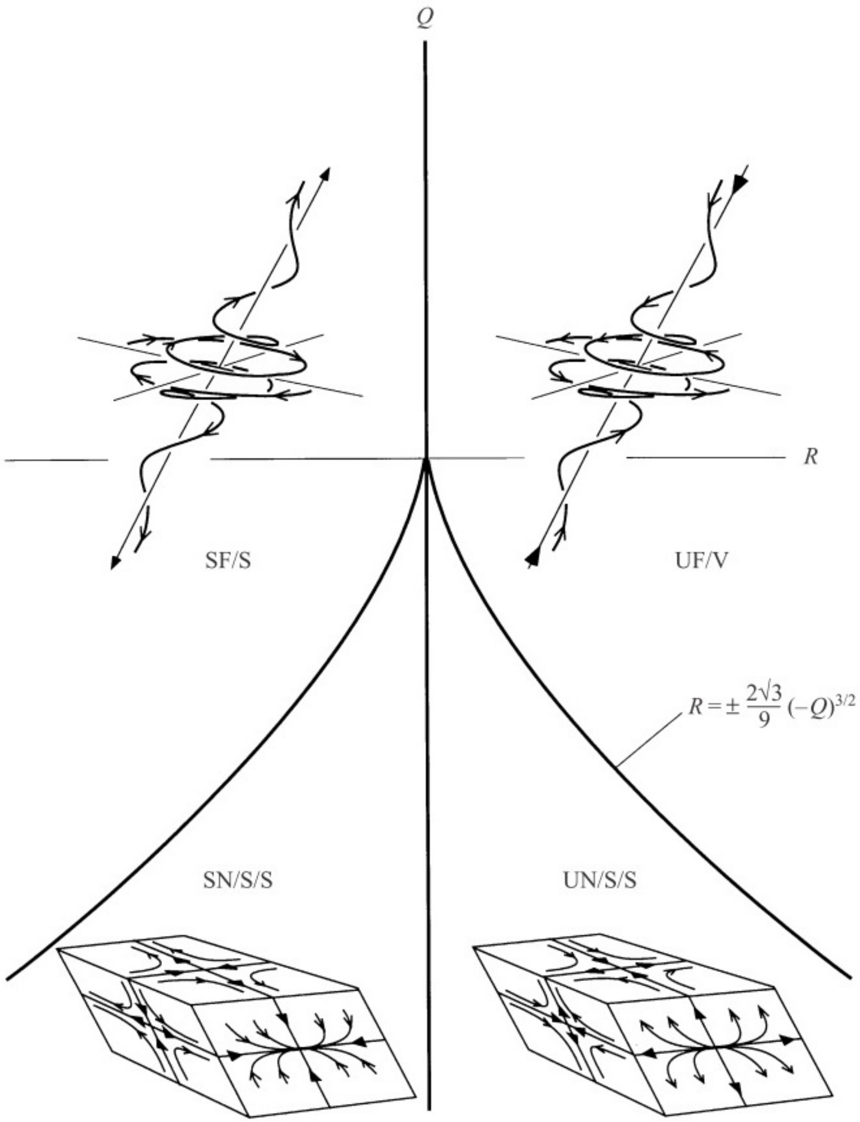
rotation pure, $s = 0$ et par conséquent $Q < 0$
cisaillement pur, $\omega = 0$ et par conséquent $Q > 0$
cisaillement simple $\omega = s$, et $Q = 0$

Champ de pression

$$\begin{cases} u_2 \dfrac{\partial u_1}{\partial x_2} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_1} \\ u_1 \dfrac{\partial u_2}{\partial x_1} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_2} \end{cases} \qquad \begin{cases} -\dfrac{\omega^2 - s^2}{4} x_1 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_1} \\ -\dfrac{\omega^2 - s^2}{4} x_2 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x_2} \end{cases}$$

$$p = p_0 + \frac{\omega^2 - s^2}{8}(x_1^2 + x_2^2)$$

Local topologies for incompressible flow



From Ooi *et al.*, 1999, *J. Fluid Mech.*