General outline of the course

- **Statistical description of turbulent flows**
  - Introduction
  - Turbulent signals & signal processing, statistical approach
  - Reynolds-averaged Navier-Stokes eqs, kinetic energy and dissipation
  - Introduction to turbulence models

- **Some turbulent flows**
  - Free shear flows (mixing layer, jet, wake)
  - Self-similar solutions
  - Wall-bounded flows

- **Turbulent fields**
  - Identification of (coherent) turbulent structures
  - Experimental techniques
## Schedule

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<td>TD4 stability</td>
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<td>Friday 06/12/2019</td>
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*Laptop required for small classes!*
# Textbooks


(360 pages, 147 illustrations, Foreword by Charles Meneveau, 53 € for ECL students)


Springer, ISBN 978-3-319-16159-4,
360 pages, 147 illustrations.
(discount for students, 53 €)


Textbooks (cont.)


Introduction
Turbulent flows are part of everyday life!

- Geophysical flows
  astrophysics, climate, weather, environment, hydraulics
- Transportation industry: space, aeronautics, marine & submarine
  and also sport applications
- Transport of fluids (energy industry, chemistry), production of energy
- Biology (physiology, biomechanics, medicine)
- Complex flows (two-phase flows, including solid particles, ...)

External aerodynamics • Noise of turbulent flows (aeroacoustics) • Sound propagation (atmosphere, ocean) • Fluid-solid coupling and vibroacoustics • Combustion (reactive flows)

Fluid mechanics is involved in many societal challenges
Introduction

- Turbulent flows
  - unsteady aperiodic motion
  - unpredictable behaviour
  - presence of a wide range of scales (eddies)

*Turbulence* appears when the source of the kinetic energy which drives the fluid motion is able to overcome viscosity effects.
- Simulation of a cube-shaped piece of the universe
  (500 million light-years long on each side)

‘Horizon project’, Cosmological hydrodynamics; Institut d’Astrophysique de Paris (Pichon) & CEA (Teyssier)

Numerical simulation of structure formation in an expanding universe: \( N \)-body simulation (red temperature) with 70 billions particles
- **Weather satellite images**
  - www.meteofrance.com  •  www.meteo-lyon.net
  - https://www.meteoblue.com/fr/meteo/semaine/lyon_france_2996944

Intertropical convergence zone
Annual mean temperature in Lyon - Bron Airport - from 1921 to 2018
(from Météo France, Le Monde 08.01.2019)

Evolution de la température par année depuis 1921
Lyon (69) : de 10 °C à 14,5 °C
Cyclone Katrina - Sept. 2005 - Category 5

Wind gusts of 280 km/h (average during 1 minute in USA), 80% of New Orleans was flooded, Dixon et al., 2006, *Nature*, 441, 586-587

(1464 people died in the hurricane and subsequent floods according to the Louisiana Department of Health)
Earth’s (land and marine) surface temperature from 1850 to 2017 expressed as ‘anomaly’ from 1961-90 in dashed line data from www.cru.uea.ac.uk
Near-surface wind speeds 10 meters above the Atlantic Ocean
Data collected by the SeaWinds scatterometer on-board NASA’s QuikSCAT satellite (NASA’s Jet Propulsion Laboratory)
Eruption of the subglacial Grimsvötn volcano, Iceland, on 21 May 2011
An initial large plume of smoke and ash rose up to about 17 km height.
Courtesy of Thördís Högnadóttir, Institute of Earth Sciences, University of Iceland
Propeller hydrodynamics

(propeller cavitation)
**Hydrodynamics : azimuth thruster**

Azimuth thruster: configuration of marine propellers placed in pods that can be rotated to any horizontal angle (azimuth), making a rudder unnecessary. It is equipped with a new-generation exhaust gas cleaning system (multi-stream scrubbers) and also features a hull lubrication system allowing the ship to float on air bubbles (created around the hull) thus reducing drag and increasing fuel efficiency.

Cruise vessel *Harmony of the Seas* (2016)
Aeronautics

Tip vortex behind an airplane

Fleet Air Arm Corsair III in 1944, (unintended) visualization of the propeller wake

Boeing 767-370/ER
Emirates A380-800 over Arabian Sea on Jan 7th 2017, wake turbulence sends business jet in uncontrolled descent

The CL-604 passed 1000 feet below an Airbus A380-800 while enroute over the Arabian Sea, when a short time later (1-2 minutes) the aircraft encountered wake turbulence sending the aircraft in uncontrolled roll turning the aircraft around at least 3 times, both engines flamed out, the Ram Air Turbine could not deploy possibly as result of G-forces and structural stress, the aircraft lost about 10,000 feet until the crew was able to recover the aircraft exercising raw muscle force, restart the engines and divert to Muscat.
Building’s energy efficiency: Pearl River Tower
(Guangzhou, China, 2011, height 309.6 m, 71-story skyscraper)
Reynolds' experience (1883): laminar versus turbulent régime

The general results were as follows:

1. When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube, Fig. 3.

2. If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity.

3. As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake, the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water, as in Fig. 4.

Any increase in the velocity caused the point of break down to approach the trumpet, but with no velocities that were tried did it reach this.

On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies, as in Fig. 5.
Control parameter: the Reynolds number

\[ \text{Re}_D \sim \frac{\text{diffusion time}}{\text{convection time}} \sim \frac{D^2/\nu}{D/U_d} \]

The transition from a laminar to a turbulent state occurs for

\[ \text{Re}_D = \frac{\rho U_d D}{\mu} = \frac{U_d D}{\nu} \sim 2300 \]

\( D \) characteristic length of the mean shear
\( U_d \) bulk velocity

This concept of a turbulent régime (wrt a laminar state) was introduced by Boussinesq (1872, 1877) and Reynolds (1883, 1894)
Reynolds number and mock-up

\[
\text{full scale } \text{Re}'_c = \frac{\rho'_\infty U'_\infty c'}{\mu'_\infty} \quad \text{mock-up } \text{Re}_c = \frac{\rho_\infty U_\infty c}{\mu_\infty}
\]

In aeronautics, \( U_\infty \simeq U'_\infty \) (Mach number similitude), but \( c \equiv c_{\text{mock-up}} \) is smaller than \( c' \) \( \implies \) need to increase \( \text{Re}_c \) to reach the nominal value \( \text{Re}'_c \), and thus to correctly measure \( C_D = C_D(\text{Re}'_c) \)

How to increase the Reynolds number (for a perfect gas)?

\[
\text{Re}_c = \frac{\rho_\infty U_\infty c}{\mu_\infty} \quad \rho_\infty = \frac{p_\infty}{r T_\infty} \quad q_\infty \equiv \frac{1}{2} \rho_\infty U_\infty^2 = \frac{\gamma}{2} p_\infty M_\infty^2
\]

\( p_\infty \uparrow \) pressurized wind tunnel (inducing mechanical constraints with the increase of the dynamic pressure \( q_\infty \)), see Superpipe in slide 57!

\( T_\infty \downarrow \) cryogenic wind tunnel (\( \mu_\infty \downarrow \), see next slide)

A320 (180 passengers, 50 M\( \text{\euro} \)) in cruise condition

Reynolds number \( \text{Re}'_c = 3 \times 10^7 \)

At an altitude of 10000 m, \( \mu'_\infty / \rho'_\infty = 3.53 \times 10^{-5} \text{ m}^2\text{s}^{-1} \)

\( M_\infty = 0.8, \ U'_\infty = 240 \text{ m.s}^{-1}, \ c' = 4.4 \text{ m} \)
Dynamic viscosity $\mu(T)$

Sutherland’s law (1893)

$$\mu \simeq \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{T_0 + T_s}{T + T_s}$$

Air

$T_0 = 273$ K  \hspace{1em} $T_s = 111$ K

$\mu_0 = 1.716 \times 10^{-5}$ kg/(m.s)

White (1988)

For common gases, $\mu = \mu(T)$ but the kinematic viscosity $\nu = \mu/\rho = \nu(p, T)$

For air at $T = 20^\circ$ C and $p = 1$ bar, one has $\nu = 1.5 \times 10^{-5}$ m$^2$.s$^{-1}$

$\nu = 10^{-6}$ m$^2$.s$^{-1}$ for water
European Transonic Windtunnel (ETW, Cologne, Germany)

Test Section: 2.4 m × 2.0 m; length 9 m
Mach Number Range: 0.15 - 1.3
Pressure range: 1.25 - 4.5 bar
Temperature range: 110 - 313 K

http://www.etw.de/
Aerodynamics of cars and trucks
Optifuel Lab 3 - Renault Trucks laboratory vehicle - aims to reduce fuel consumption by 13% (FALCON, collaborative project including LMFA)
Turbulent subsonic (round) jet

- Prasad & Sreenivasan (1989)
  \[ Re_D \approx 4000 \]

- Dimotakis et al. (1983)
  \[ Re_D \approx 10^4 \]

- Kurima, Kasagi & Hirata (1983)
  \[ Re_D \approx 5.6 \times 10^3 \]

- Ayrault, Balint & Schon (1981)
  \[ Re_D \approx 1.1 \times 10^4 \]

- Mollo-Christensen (1963)
  \[ Re_D = 4.6 \times 10^5 \]
Importance of entrainment for free shear flows

Visualization with smoke wires
$Re_D \approx 5.4 \times 10^4$
Courtesy of H. Fiedler (1987)

Entrainment by a turbulent round jet from a wall
$Re_D = 10^6$
Florent, J. Méc. (1965)
**Drag coefficient $C_D$ for a smooth cylinder**

\[
\begin{align*}
R_e D &= 138 & R_e D &= 168 & R_e D &\approx 2 \times 10^5 \\
R_e D &= 0.16 & R_e D &= 1.54, Van Dyke
\end{align*}
\]
Wake behind a circular cylinder / Karman vortex street

These Karman vortices formed over the islands of Broutona, Chirpoy, and Brat Chirpoyev ("Chirpoy’s Brother"), all part of the Kuril Island chain found between Russia’s Kamchatka Peninsula and Japan.

Alexander Selkirk Island in the southern Pacific Ocean.
Interaction between wakes

Transport efficiency

Turbulence $\sim$ increase of diffusion, reduction of flow separation regions, decrease of pressure drag (versus increase in wall friction), increase in thermal exchanges

The laminar boundary layer in the upper photograph separates from the sharp corner whereas the turbulent boundary layer in the second photograph remains attached (from Van Dyke, fig. 156)
**Elite cyclist: reduction of drag...**

... when a cyclist rides **in front** of a car

For a 50 km individual time trial: $3 \leq d \leq 10 \text{ m} \implies 1 \text{ mm} \rightarrow 4 \text{ s time reduction!}

Recommendation for UCI, $d \geq 30 \text{ m}$

(Ref: Blocken & Toparlar, *J. Wing. Eng. Ind. Aerodyn.*, 2015)
Both indicial and boldface notations are used to indicate vectors

vector $\mathbf{U} \equiv \vec{U}$, $i$-th component $U_i$, norm $U$, $U^2 = \mathbf{U} \cdot \mathbf{U}$

gravity $\mathbf{g}$, $g_i = -g \delta_{3i}$, $\mathbf{g} = (g_1, g_2, g_3) = (0, 0, -g)$, $g = 9.81 \text{ m.s}^{-2}$

density $\rho$ (kg.m$^{-3}$)

$\delta_{ij}$ Kronecker delta

Einstein summation convention

When an index variable appears twice in a single term (dummy index), it implies summation of that term over all the values of the index.

Scalar product between two vectors $\mathbf{a}$ and $\mathbf{b}$ (index $i$ repeated)

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i = \sum_{i=1}^{3} a_i b_i$$

Short quiz $\delta_{ij} a_j =? \quad \delta_{ij} \delta_{ij} =?$
Differential operators (expressed in Cartesian coordinates)

Gradient

\[ b = \nabla f \equiv \text{grad} f \quad b_i = \frac{\partial f}{\partial x_i} \]

Divergence

\[ \nabla \cdot \mathbf{U} = \text{div}(\mathbf{U}) = \sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial U_i}{\partial x_i} \]

Laplacian

\[ \nabla^2 f = \Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_i} = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_i} \]

Curl

\[ \nabla \times \mathbf{U} = \text{rot} \mathbf{U} \]

Remark – The dot symbol (\cdot) is never decorative in all previous notations!
Reynolds-Averaged Navier-Stokes equations
Fluctuating velocity signal in the shear layer of a subsonic round jet
(measured by crossed-wire probes at $x_1 = 2D$, $x_2 = D/2$, $x_3 = 0$)
Nozzle diameter $D = 50$ mm, exit velocity $U_j = 30$ m.s$^{-1}$
$\sim$ Reynolds number $Re_D = 10^5$

$u'_1(t)$
Fluctuating velocity signal in the shear layer of a jet (cont.)

$u'_1(t)$ and $u'_2(t)$ with $\xi = u'_\alpha(t)/u_{\alpha\text{rms}}$

$$S_{u'_1} \approx 0.21 \quad T_{u'_1} \approx 2.74$$

$$S_{u'_2} \approx 0.24 \quad T_{u'_2} \approx 2.76$$
Fluctuating velocity signal in the shear layer of a jet (cont.)

Skewness $S$ and flatness or kurtosis $T$ factors

Centered variable $x'_i \equiv x_i - \bar{X}_i$  $x'_{i,rms} \equiv \sqrt{x_i'^2} = \sigma_x$ standard deviation

$$S_{x_i} \equiv \frac{x_i'^3}{x_{i,rms}'^3} \quad T_{x_i} \equiv \frac{x_i'^4}{x_{i,rms}'^4}$$

For a Gaussian (normal) distribution

$$p(\xi) = \frac{1}{\sqrt{2\pi}\sigma_\xi} \exp\left(-\frac{\xi^2}{2\sigma_\xi^2}\right)$$

$$\sigma_\xi^2 \equiv (\xi - \bar{\xi})^2 \quad S_\xi = 0 \quad T_\xi = 3$$
The statistical mean $\bar{F}(x, t)$ of a variable $f(x, t)$ is defined as

$$\bar{F}(x, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f^{(i)}(x, t)$$

where $f^{(i)}$ is the $i$-th realization: convenient when manipulating equations but difficult to implement in practice!

Temporal average

$$\bar{F}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(x, t') dt'$$

for a stationary turbulent field, i.e. when time $t$ does not enter into $\bar{F}$ (hypothesis of ergodicity)

Spatial average

$$\bar{F}(t) = \lim_{V \to \infty} \frac{1}{V} \int_{V} f(x', t) dx'$$

for a homogeneous turbulent field, i.e. when position $x$ does not enter into $\bar{F}$

The statistical mean is a linear operator, which commutes with time and space derivative operators: rules of Reynolds
Some important properties

\[ f \equiv \bar{F} + f' \quad \text{with} \quad \bar{f}' = 0 \quad (f' = f - \bar{F} \quad \text{and} \quad \bar{f}' = \bar{F} - \bar{F} = 0) \]

Product of two variables \( f \) and \( g \),

\[ fg \equiv (\bar{F} + f')(\bar{G} + g') = \bar{F}\bar{G} + \bar{F}g' + f'\bar{G} + f'g' \]

and thus,

\[ \bar{f}g = \bar{F}\bar{G} + \bar{F}g' + f'\bar{G} + f'g' = \bar{F}\bar{G} + f'g' \]

Rule for the product of two variables,

\[ \bar{f}g = \bar{F}\bar{G} + f'g' \quad (1) \]

The Reynolds decomposition

\[ U_i \equiv \bar{U}_i + u'_i \quad \text{with} \quad \bar{u}'_i = 0 \]

\( \bar{U}_i \) part which can be reasonably calculated

\( u'_i \) part which must be modelled (turbulent fluctuations)
The Reynolds Averaged Navier-Stokes (RANS) equations

Assumptions (to simplify): *incompressible flow* $\nabla \cdot \mathbf{U} = 0$ and *homogeneous fluid*, constant density $\rho$

How to determine the transport equation of the mean quantities?

First, substitute the Reynolds decomposition and then, average the equation!

$$\frac{\partial (\bar{U}_i + u'_i)}{\partial x_i} = 0 \quad \frac{\partial (\bar{U}_i + u'_i)}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial \bar{U}_i}{\partial x_i} = 0$$

By subtracting both equations,

$$\frac{\partial U_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial \bar{U}_i}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial u'_i}{\partial x_i} = 0$$

The mean flow field is incompressible, and so is the fluctuating field.
The Reynolds Averaged Navier-Stokes (RANS) equations

\[
\frac{\partial (\rho U_i)}{\partial t} + \frac{\partial}{\partial x_j} \left( \rho U_i U_j \right) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad \tau_{ij} = 2\mu D_{ij}
\]

By introducing the Reynolds decomposition, and taking the average
\[ u_i \equiv \bar{U}_i + u'_i \quad p \equiv \bar{P} + p' \quad \tau_{ij} \equiv \bar{\tau}_{ij} + \tau'_{ij} \]

\[
\frac{\partial (\rho \bar{U}_i)}{\partial t} + \frac{\partial (\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \bar{\tau}_{ij} - \rho u'_i u'_j \right)
\]

The term \(-\rho u'_i u'_j\) is the Reynolds stress tensor, unknown, thus leading to a closure problem. Generally this term is larger than the mean viscous stress tensor except for wall bounded flows, where the viscosity effects become preponderant close to the wall (no-slip condition)
Turbulent kinetic energy and dissipation

The turbulent kinetic energy and the turbulent dissipation are two key quantities. By introducing the Reynolds decomposition, using the rule (1),

**Kinetic energy**

\[
\overline{U_i U_i} = \overline{U_i U_i} + \frac{u'_i u'_i}{2}
\]

\[
k_t \equiv \frac{u'_i u'_i}{2} \text{ (m}^2 \text{s}^{-2})
\]

\(k_t\) is the turbulent kinetic energy

**Dissipation**

\[
2\nu \overline{D_{ij} D_{ij}} = 2\nu \overline{\bar{D}_{ij} \bar{D}_{ij}} + 2\nu \overline{d'_{ij} d'_{ij}}
\]

\[
\epsilon \equiv 2\nu \overline{d'_{ij} d'_{ij}} \text{ (m}^2 \text{s}^{-3})
\]

\(\epsilon\) is the dissipation rate of \(k_t\) induced by the molecular viscosity
Concept of turbulent viscosity for turbulence models introduced by Boussinesq (1877)

Modeling of the Reynolds stress tensor \(-\rho u'_i u'_j\). By analogy with the viscous stress tensor \(\overline{\tau}\), one defines

\[
-\rho u'_i u'_j = 2 \mu_t \overline{D}_{ij} - \frac{2}{3} \rho k_t \delta_{ij} = \mu_t \left( \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right) - \frac{2}{3} \rho k_t \delta_{ij}
\]

where \(\mu_t = \mu_t(x,t)\) is the turbulent viscosity, a property of the flow field (and not of the fluid as for the molecular viscosity).

The introduction of a turbulent viscosity for closing the Reynolds stress tensor is an assumption, not always verified by turbulent flow. Here, it is also assumed that the turbulent viscosity remains positive (thus inducing specific behaviours in terms of energy transfer).
Illustration for a free subsonic round jet
M = 0.16 and Re_D = 9.5 \times 10^4 (from Hussein, Capp & George, 1994)
Concept of turbulent viscosity for turbulence models (cont.)

Reynolds Averaged Navier-Stokes (RANS) equations

\[
\frac{\partial \bar{U}_i}{\partial x_i} = 0
\]

\[
\frac{\partial (\rho \bar{U}_i)}{\partial t} + \frac{\partial (\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial (\bar{P} + \frac{2}{3}\rho k_t)}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2(\mu + \mu_t)\bar{D}_{ij}\right)
\]

How to compute the turbulent viscosity \( \nu_t(x, t) \)?

From dimensional arguments, the product of a velocity scale by a length scale, for example \( \nu_t \sim k_t^{1/2} \times k_t^{3/2}/\epsilon \sim k_t^2/\epsilon \), and then write a transport equation for \( k_t \) and \( \epsilon \) to obtain the famous \( k_t - \epsilon \) model. There are about 200 turbulent viscosity models published in the literature!
Turbulent kinetic energy budget

(the demonstration can be found in textbooks)

\[
\frac{\partial (\rho k_t)}{\partial t} + \frac{\partial (\rho k_i \overline{U}_j)}{\partial x_j} = -\rho u'_i u'_j \frac{\partial \overline{U}_i}{\partial x_j} - \rho \epsilon + \frac{\partial}{\partial x_k} \left( -\frac{1}{2} \rho u'_i u'_j u'_k - p' u'_k + u'_i \tau'_{ik} \right)
\]

advection by the mean flow

production \( P \)

dissipation

transport terms

Case of homogeneous turbulence?
Heuristic interpretation of the production term $P$

\[
\mathcal{P} \approx -\rho u'_1 u'_2 \frac{d\bar{U}_1}{dx_2} > 0 \text{ is thus expected!}
\]

\[
\begin{aligned}
\begin{cases}
  u'_2 > 0 \\
  u'_1 < 0
\end{cases}
\quad & \quad u'_1 u'_2 < 0 \\
\begin{cases}
  u'_2 < 0 \\
  u'_1 > 0
\end{cases}
\quad & \quad u'_1 u'_2 < 0
\end{aligned}
\]

The production term $\mathcal{P}$ is – in general – a transfer from the shear mean flow $\bar{U}$ to the turbulent field $u'$ (but becomes always a positive transfer term using a turbulent viscosity model, a possible drawback of turbulence models).
Wall-bounded flows

El Khoury et al. (2013), $Re^+ = 1000$
Zero-pressure-gradient boundary layer on a flat plate

\[ U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} \sim \nu \frac{\partial^2 U_1}{\partial x_2^2} \]

- convection
- molecular diffusion

\[ \frac{U_1^2}{L} \sim \nu \frac{U_1}{\delta^2} \]

\[ \frac{\delta^2}{L^2} \sim \frac{\nu}{U_1 L} \]

Parabolic laminar boundary layer \((\delta^2 \sim \nu t)\)
The Blasius solution (PhD 1908, supervised by Prandtl)

Self-similar solution for laminar flow over a flat plate

\[
\frac{\delta}{x_1} \approx 4.92 \frac{1}{\sqrt{\text{Re}_{x_1}}} \quad (\delta \sim x_1^{1/2})
\]

Friction coefficient

\[
C_f = \frac{\tau_w}{\frac{1}{2}\rho U_{e1}^2} \sim \frac{\mu U_1}{\rho U_1^2} \sim \frac{\nu}{U_1 \delta} \sim \frac{1}{\text{Re}_\delta^{1/2}}
\]

\[
C_f \approx \frac{0.664}{\sqrt{\text{Re}_{x_1}}}
\]
**Boundary layer transition**

Transition is experimentally observed when the Reynolds number (based on the distance from the leading edge and the free stream velocity) is of the order of

$$Re_{x_1} = \frac{x_1 U_{e1}}{\nu} \simeq 3.2 \times 10^5$$

or equivalently for a Reynolds number based on the boundary layer thickness $\delta$,

$$Re_\delta = \delta U_{e1}/\nu \simeq 2800$$

Turbulent spot growing by destabilizing the surrounding laminar flow

Visualization by laser-induced fluorescence (LIF) Gad-el-Hak *et al.* (1981)
Turbulent boundary layer

F. Laadhari (LMFA)

\[ \text{Re}_{\delta} \approx 1000 \quad U_{e1} = 2.1 \text{ m.s}^{-1} \quad \delta \approx 7 \text{ cm} \quad u_{\tau} \approx 0.1 \text{ m.s}^{-1} \quad x_1 \approx 3 \text{ m} \]

Convection time \( \frac{L}{U} \approx \frac{\delta}{u'} \) turbulent diffusion time

Turbulence intensity \( u'/U_{e1} \approx 10^{-1} \), no reasoning to easily estimate \( C_f \)
Zero-pressure-gradient boundary layer on a flat plate
Transition for $\text{Re}_{x_1} \approx 3.2 \times 10^5$ or equivalently for $\text{Re}_\delta = U_{e1} \delta / \nu \approx 2800$

In laminar regime, molecular diffusion $\tau \sim \delta^2 / \nu$ in the transverse direction, compared with the turbulent regime, turbulent diffusion $\delta / u'$ with $u' \approx 0.1 U_{e1}$

Lee, Kwon, Hutchins & Monty (Melbourne University)
Two main classes of wall flows: confined flows & external flows

- Flat-plate boundary layer
  \[ \text{Re}_\delta = \frac{U_{e1} \delta}{\nu} \]
  Fully turbulent for \( \text{Re}_\delta \geq 2800 \)

- Channel flow
  \[ \text{Re}_{2h} = \frac{U_d 2h}{\nu} \]
  (\( U_d \) bulk velocity)
  Fully turbulent for \( \text{Re}_{2h} \geq 1800 \)
  Homogeneous flow along \( x_1 \)
Wall-bounded flows

Relation between the skin-friction coefficient $C_f$ and the friction factor $\psi$

\[
\tau_w = \mu \frac{dU}{dr} \bigg|_{r=R} \quad C_f \equiv \frac{\tau_w}{\frac{1}{2} \rho U_d^2} \quad \Delta p \equiv \psi \frac{L}{D} \frac{1}{2} \rho U_d^2
\]

\[
\int_S \rho u \cdot (u \cdot n) \, ds = -\int_S pn \, ds + \int_S \overline{\tau} \cdot n \, ds
\]

\[
0 = (-p_1 + p_2) \frac{\pi D^2}{4} + \pi DL \mu \frac{dU}{dr} \bigg|_{r=R}
\]

\[
\Delta p \equiv p_1 - p_2 = 4c_f \frac{L}{D^2} \rho U_d^2 = \psi \frac{L}{2} \rho U_d^2 
\]

\[
\psi \equiv 4c_f
\]
Wall-bounded flows

Skin-friction coefficient $C_f = \tau_w / (\rho U_d^2/2)$ for a circular pipe

- Oregon facility
- Princeton Superpipe

McKeon et al. (2004)
(The Reynolds number is increased through the pressure)

- Laminar regime $C_f = 16/\text{Re}_D$
- Blasius’ relationship, $C_f \approx 0.0791 \text{Re}_D^{-1/4}$
- $1/C_f^{1/2} \approx 3.860 \log_{10}(\text{Re}_D C_f^{1/2}) - 0.088$
Short overview

Two different scales must be introduced to correctly describe a turbulent boundary layer

**Outer scales** – $u'$ with $u'/U_\ell \approx 0.1$ and $\delta$

The convection along the mean flow is balanced by the turbulent diffusion in the transverse direction

**Inner scales** – $u_\tau$ and $l_v = \nu/u_\tau$

$u_\tau$ is the friction velocity defined from the mean wall shear stress $\bar{\tau}_w = \rho u_\tau^2$

($\bar{\tau}_w = \mu d\bar{U}_1/dx_2$ at the wall)
Wall-bounded flows

Short overview (cont.)

Ratio between the two length scales,

\[ Re^+ = \frac{\delta}{l_v} = \frac{u_\tau \delta}{\nu} \]

Kármán number

Notations in wall unit

\[ x_2^+ = \frac{x_2}{l_v} = \frac{x_2 u_\tau}{\nu} \quad \overline{U}_1^+ = \frac{\overline{U}_1}{u_\tau} \]
Mean velocity profile the logarithmic law (inner scales)

\[ \bar{U}_1 = \frac{1}{\kappa} \ln x_2^+ + B \]

log-law
\[ x_2^+ \geq 30 \& \frac{x_2}{\delta} \leq 0.20 \]

For a zero-pressure-gradient boundary layer,
\[ \kappa \simeq 0.384 \quad B \simeq 4.17 \]

(Bailly & Comte-Bellot, Turbulence, Springer, 2015, data from Osterlünd, 1999)
Scales and energy cascade
Scales

Large scale in $\mathcal{O}(L, u')$ – associated with the geometry (cavity, cylinder, jet, wake, car, airfoil, pipe, ...) and thus with the size of the flow itself.

$\sim$ energy transfer between (basically from) large scale structures to small scale structures, but this transfer is stopped by the molecular viscosity

$$\frac{\partial u'}{\partial t} \sim \nu \nabla^2 u'$$

Small scale in $\mathcal{O}(l_\eta, u_\eta)$ – known as the Kolmogorov scales ($\text{Re} = u_\eta l_\eta / \nu = 1$). The Kolmogorov length scale $l_\eta$ plays a fundamental role in experiments (sampling frequency) as well as in numerical simulations (grid size).

The ratio $L/l_\eta$ is a function of the Reynolds number $\text{Re}_L = u'L/\nu$,

where $u'^2 \approx 2k_t/3$

$$\frac{L}{l_\eta} \sim \text{Re}_L^{3/4}$$
Scales (cont.)

Time scale – Memory time of turbulence $L/u'$

This time should not be confused with the passing time which can be easily observed in a fixed frame,

$$\frac{D}{Dt} = 0 \implies -i\omega + i\bar{U}_1 k_1 = 0 \quad \text{leading to} \quad \frac{\omega}{k_1} = \bar{U}_1$$
Turbulent mixing layer (Brown & Roshko, 1974)

Energy cascade in a mixing layer by increasing the Reynolds number (through pressure and velocity, \( \times 2 \) for each view)

More small-scale structures are produced without basically altering the large-scale ones (linked to the transition process, as shown by Winant & Browand, 1974)

Anatol Roshko (1923-2017)

(Shadowgraphs with a spark source)
Local nature of the energy cascade in space

- small $k$
- intermediate $k$
- high $k$

in physical space
Taylor-Green vortex flow
Re = 3000 on a $384^3$ grid at times $t = 0$, $t = 9$ and $t = 18$ (dimensionless variables)
From Fauconnier et al. (2013)

Vortex structures colored by $z$-vorticity
High fidelity flow/noise simulation in a **physically and numerically controlled** environment

Isothermal turbulent jet at \( M = 0.9 \) and 
\[ Re_D = 10^5, \]
\[ 1.1 \times 10^9 \text{ points}, \]
\[ 0 \leq r \leq 7.5D \text{ and } \]
\[ -D \leq z \leq 20D \]

Pressure fluctuations \((\pm 55 \text{ Pa})\) and 
normalized vorticity \((|\omega| \times \delta \theta / U_c)\)

(Bogey, 2017)
Paradox of the energy cascade ...

e.g. Energy dissipated by the motion at $U_\infty$ of a sphere of diameter $D$

Consider the power developed by the drag force $F_D U_\infty$ is balanced by the energy dissipated with the flow, that is $\rho V \epsilon$

$$\epsilon \approx \frac{1}{\rho V} \frac{\rho U_\infty^2}{2} SC_D (Re_D) U_\infty \propto C_D (Re_D) \frac{U_\infty^3}{D}$$

with $V \sim D^3$, $S \sim D^2$ and where $Re_D \approx \text{cst}$ as $Re_D$ goes to infinity.

For high Reynolds number flows $Re_D \gg 1$, the rate of dissipation per unit mass $\epsilon$ becomes independent of the viscosity $\nu$, whereas $\epsilon = \nu |\nabla u|^2$

$Re_D \nearrow$, $\nu \searrow$ and $|\nabla u| \nearrow$ leading to singularities inside the flow
Drag coefficient for a smooth sphere

(adapted from Clift, Grace, & Weber, 1978)

\[ C_D = \frac{24}{Re_D} \]

\[ Re_D = 2 \times 10^4 \]

\[ Re_D = 2 \times 10^5 \]

\[ Re_D \approx 4 \times 10^5 \]

\[ Re_D^c \approx 3 \times 10^5 \]

(pictures from Werlé, ONERA)
Free shear flows: self-similar solution of the plane mixing layer

Composite schlieren of the free shear layer, with the two streams injected parallel to each other. The upper stream is 100% N2 while the lower is a mixture of 1/3 He and 2/3 Ar, with $M_1 = 0.59$ and $M_2 = 0.29$. Hall, Dimotakis & Rosemann, 1993, *AIAA J.*, 31(12), 2247-2254.
Almost parallel and two-dimensional flow: slow evolution in the $x_1$ direction (and statistics independent of the spanwise coordinate $x_3$)

- $u_m \equiv \frac{U_1 + U_2}{2}$ convection velocity scale
- $\Delta U \equiv U_1 - U_2$ velocity difference, which characterizes the diffusion
- $\delta(x_1)$ mixing layer width
- $L$ scale of change in the $x_1$ direction
- $l$ scale of change in the $x_2$ direction
- $l/L \ll 1$ is a small parameter (parallel flow), and $Re = \Delta U \delta/\nu \gg 1$ (inviscid flow)
• Mean velocity field

Conservation of mass, which provides the transverse velocity scale $V$
(where $\sim$ stands for order of magnitude)

$$\frac{\partial \overline{U}_1}{\partial x_1} + \frac{\partial \overline{U}_2}{\partial x_2} = 0 \quad \frac{\partial \overline{U}_1}{\partial x_1} \sim \frac{\Delta U}{L} \quad \Rightarrow \quad V \sim \frac{l}{L} \Delta U$$

Reynolds-averaged Navier-Stokes equation in the $x_1$ direction

$$\overline{U}_1 \frac{\partial \overline{U}_1}{\partial x_1} + \overline{U}_2 \frac{\partial \overline{U}_1}{\partial x_2} = -\frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_1} - \frac{\partial \overline{u_1'u_1'}}{\partial x_1} - \frac{\partial \overline{u_1'u_2'}}{\partial x_2} + \nu \nabla^2 \overline{U}_1$$

– for the convection terms,

$$\overline{U}_1 \frac{\partial \overline{U}_1}{\partial x_1} + \overline{U}_2 \frac{\partial \overline{U}_1}{\partial x_2} = u_m \frac{\partial \overline{U}_1}{\partial x_1} + (\overline{U}_1 - u_m) \frac{\partial \overline{U}_1}{\partial x_1} + \overline{U}_2 \frac{\partial \overline{U}_1}{\partial x_2}$$

$$\sim \Delta U \frac{\Delta U}{L}$$
Mean velocity field (cont.)

– for the Reynolds stress,
with \( u \sim \Delta U \) for the scale of velocity fluctuations,

\[
\frac{- \frac{\partial u'_1 u'_2}{\partial x_2}}{l} \sim \frac{u^2}{l} \sim \frac{\left( \Delta U \right)^2}{l}
\]

Then, by considering the balance between the two dominant (red) terms

\[
u_m \frac{\Delta U}{L} \sim \frac{(\Delta U)^2}{l} \implies \Delta U \sim \frac{l}{L} u_m
\]

– for the pressure term, using the RANS Eq. in the \( x_2 \) direction

\[
\bar{U}_1 \frac{\partial \bar{U}_2}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_2}{\partial x_2} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_2} - \frac{\partial \bar{u}_1' u'_2}{\partial x_1} - \frac{\partial \bar{u}_2' \bar{u}_2'}{\partial x_2} + \nu \nabla^2 \bar{U}_2
\]

\[
\sim \frac{l \ u_m \Delta U}{L} \sim \frac{u_m \Delta U}{L}
\]
Mean velocity field (cont.)

– for the pressure term (cont.)

\[
\bar{P} + \rho u_2'^2 \simeq \text{cst} = p_\infty \quad \Rightarrow \quad \frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_1} \simeq 0 \quad \text{in the RANS Eq. in the } x_1 \text{ direction}
\]

Finally, the equation governing the mean velocity is

\[
u_m \frac{\partial \bar{U}_1}{\partial x_1} \simeq -\frac{\partial u'_1 u'_2}{\partial x_2}
\]  \(2\)
Self-similar solution

We now look for self-similar solutions of Eq. (2)

Definition of the mixing layer width \( \delta(x_1) = x_{2}^{0.9} - x_{2}^{0.1} \)
where \( x_{2}^{\alpha} \) is the transverse location such that \( \overline{U}_1 = U_1 + \alpha \Delta U \)

From this definition, we have \( \overline{U}_1(\pm \delta/2) = u_m \pm 0.4 \Delta U \)

Self-similar variable \( \eta = \frac{x_2 - \bar{x}_2}{\delta} \)

where \( \bar{x}_2(x_1) \) is usually the line along which \( \overline{U}_2 = 0 \) (the flow is not symmetric about \( x_2 = 0 \) for various reasons)

\[
\begin{cases}
\overline{U}_1 = u_m + \Delta U f(\eta) \\
u'_1u'_2 = (\Delta U)^2 g(\eta)
\end{cases}
\]

We expect that the nondimensionalized quantities \( f \) and \( g \) are functions of \( \eta \) only
Plane mixing layer


\[ U_1 = U_E = 8 \text{ m.s}^{-1}, \ U_2 = 0 \]

(the flow spreads preferentially into the low-speed stream)
Shear layer at a stream confluence

Kaskaskia River - Copper Slough confluence in East Central Illinois

(Rhoads & Sukhodolov, 2004, Water Resources Research)
Identification of vortical structures

(When is a region of vorticity a vortex?)

2-D 2-C PIV snapshot, $u - 0.85U_\infty$, colored by vorticity magnitude $\omega_2$, from Salze et al. (2015)
Deformation of a fluid particle

Taylor series for the velocity inside the fluid particle at \( x_p \) (at a given time \( t \))

\[
\mathbf{u}(x) = \mathbf{u}(x_p) + \nabla \mathbf{u}(x_p) \cdot (x - x_p) + \cdots
\]

\[
\mathbf{u}_i(x) = \mathbf{u}_i(x_p) + \left. \frac{\partial \mathbf{u}_i}{\partial x_j} \right|_{x_p} (x_j - x_p) + \cdots
\]

\[
\frac{\partial \mathbf{u}_i}{\partial x_j} = \frac{1}{2} \left( \frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial \mathbf{u}_i}{\partial x_j} - \frac{\partial \mathbf{u}_j}{\partial x_i} \right)
\]

\[
\begin{align*}
D_{ij} & \quad \text{symmetric part of } \nabla \mathbf{u} \\
W_{ij} & \quad \text{antisymmetric part of } \nabla \mathbf{u}
\end{align*}
\]

\[
W_{ij} \text{ is associated with the rotation of the fluid particle}
\]

\[
\overline{W} = \begin{pmatrix}
0 & -\Omega_3 & \Omega_2 \\
\Omega_3 & 0 & -\Omega_1 \\
-\Omega_2 & \Omega_1 & 0
\end{pmatrix} \quad \Omega \equiv \frac{1}{2} \nabla \times \mathbf{u}
\]
**Definition**

The *vorticity vector* is defined as \( \omega = \nabla \times \mathbf{u} \).

When \( \omega = 0 \), absence of vorticity, the flow is *irrotational*.

The vorticity is twice the *angular velocity* \( \Omega \) of the solid-body rotation motion of the fluid particle (see slide 79):

\[
\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_P) + \mathbf{D}(\mathbf{x}_P) \cdot (\mathbf{x} - \mathbf{x}_P) + \frac{\Omega(\mathbf{x}_P) \times (\mathbf{x} - \mathbf{x}_P)}{2} \quad \text{as } \mathbf{x} \to \mathbf{x}_P \\
\Omega = \frac{1}{2} \omega
\]

Decomposition of fluid-particle motion in a shear flow:
Example of the Rankine vortex (1858)

\[
\begin{align*}
  u(r) &= v_0 \frac{r}{r_0} = \Omega_0 r \quad r \leq r_0 \\
  u(r) &= v_0 \frac{r_0}{r} = \Omega_0 r_0 \frac{r_0}{r} \quad r > r_0
\end{align*}
\]

\(v_0 = \Omega_0 r_0 = \omega_0 r_0/2\)

Solid body motion inside the vortex itself, i.e. for \(r \leq r_0\) in the vortical region

Irrotational flow outside, for \(r > r_0\): the localized circular patch of vorticity produces a velocity field away from the vortical region
Example of the Rankine vortex (cont.)

Pressure field?

Inviscid steady flow, Euler’s equation

\[
\frac{1}{\rho} \frac{\partial p}{\partial r} = -\frac{u_\theta^2}{r}
\]

\[
\begin{cases}
  p = p_\infty - \frac{\rho v_0^2}{2} \left( 2 - \frac{r^2}{r_0^2} \right) & \text{if } r \leq r_0 \\
  p = p_\infty - \frac{\rho v_0^2 r_0^2}{2 r^2} & \text{if } r > r_0
\end{cases}
\]

More generally, can vortex structures be identified with local pressure minimum?
Identification of vortices in turbulent flow

Equation for the pressure (incompressible flow, \( \rho = \text{cst} \)), by taking the divergence of the Navier-Stokes equation

\[
\frac{1}{\rho} \nabla^2 p = \frac{\partial u_i}{\partial u_j} \frac{\partial u_j}{\partial u_i}
\]

Using the previous decomposition of the velocity gradient tensor

\[
\frac{\partial u_i}{\partial u_j} \frac{\partial u_j}{\partial u_i} = (D_{ij} + W_{ij})(D_{ij} - W_{ij}) = D_{ij}D_{ij} - W_{ij}W_{ij} = D^2 - W^2
\]

A vortex may then be defined by a concentrated flow region dominated by \( W \), and thus \( \nabla^2 p > 0 \) (positive curvature)

\[
\frac{1}{\rho} \nabla^2 p = W^2 - D^2
\]
Identification of vortices (cont.)

The source term of Poisson’s equation for the pressure is one of the three invariants (invariant, that is independent of the orientation of the coordinate system) of the velocity gradient tensor \( A_{ij} \equiv \partial u_i / \partial x_j = (\nabla u)_{ij} \)

The three invariants of a second-order tensor are given by

\[
\begin{cases}
    P = \text{tr}(\overline{A}) = A_{ii} \\
    Q = \frac{1}{2} \left[ \text{tr}^2(\overline{A}) - \text{tr}(\overline{A}^2) \right] = \frac{1}{2} \left[ (A_{ii})^2 - A_{ij}A_{ji} \right] \\
    R = \text{det}(\overline{A}) = \frac{1}{6} \text{tr}^3(\overline{A}) - \frac{1}{2} \text{tr}(\overline{A}) \text{tr}^2(\overline{A}) + \frac{1}{3} \text{tr}(\overline{A}^3) 
\end{cases}
\]

The incompressibility condition \( \nabla \cdot u = 0 \) leads to \( P = 0 \) and to \( Q = -A_{ik}A_{kj}/2 \).
Hence, the pressure equation reads

\[
\frac{1}{\rho} \nabla^2 p = W^2 - D^2 = 2Q
\]

Vortical structures are thus expected to be identified for high positive values of the invariant \( Q \), \( Q \)-criterion
Identification of vortices (cont.)

Total drag breakdown (Fiabane, PhD thesis, 2011, ENSTA-PSA), and iso-surfaces of $Q$-criterion colored with velocity magnitude
Identification of vortices (cont.)

The eigenvalues $\lambda_i$ of $A$ are the roots of the characteristic equation

$$\lambda^3 - P\lambda^2 + Q\lambda - R = 0,$$

with $P = \lambda_1 + \lambda_2 + \lambda_3$, $Q = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3$ and $R = \lambda_1 \lambda_2 \lambda_3$

For incompressible flow, $P = \nabla \cdot \mathbf{u} = 0$
and the characteristic equation reads $\lambda^3 + Q\lambda - R = 0$

In introducing the discriminant $\Delta = Q^3/27 + R^2/4$, one finds three real values $\lambda_i$ for $\Delta < 0$, or two complex conjugate values $\lambda_{1,2} = \sigma \pm i\omega$ and one real value $\lambda_3$ for $\Delta > 0$.

\[ Q^* = -3(R^2/4)^{1/3} \quad Q^* = 3(R^2/4)^{1/3} \]
Incompressible 2-D flow
Stream function \( \psi = ax_1^2 + bx_2^2 \) and velocity field \( (\partial \psi/\partial x_2, -\partial \psi/\partial x_1) \)
Local topologies for incompressible flows