*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

CENTRALE LYON

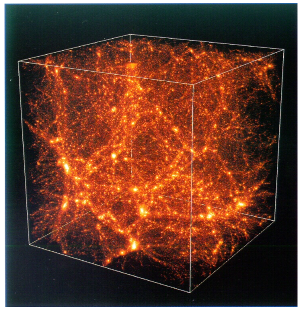
Physics of turbulent flow

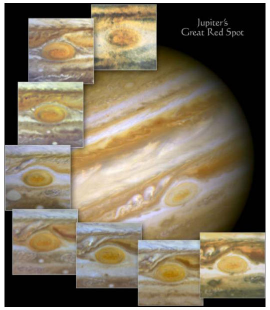
Christophe Bailly

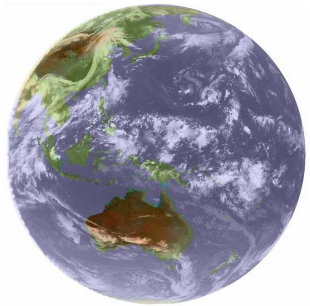
Centrale Lyon – MOD 3rdYear & MSc. – version 04-10-2024

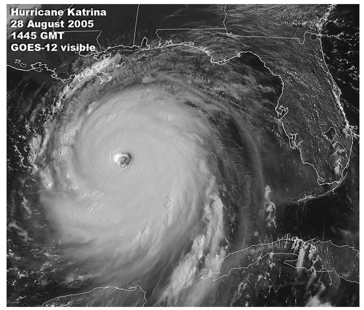
Outline

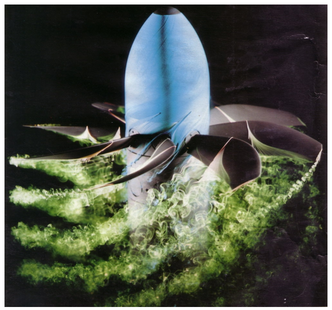
Statistical description of turbulent flow

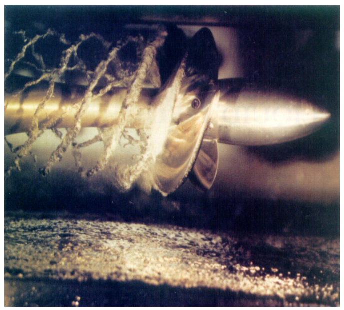












Introduction

The objective of this chapter is to establish the equations governing the mean flow field, and then to provide some hints on the closure of these equations.

For a given variable f, the Reynolds decomposition into mean and fluctuating components is introduced, $f = \bar{F} + f'$. For a stationary turbulence, $\bar{F}(\boldsymbol{x}, t) = \bar{F}(\boldsymbol{x})$, and the mean average can be well estimated by the time average of one realization, as discussed in the previous chapter.

A dual configuration is often considered for homogeneous turbulence. Statistics are independent of space, in particular $\bar{F}(\boldsymbol{x}, t) = \bar{F}(t)$. The ensemble mean is then usually approximated by spatial average,

$$\bar{F}(t) = \frac{1}{V}\int_V f(\boldsymbol{x}', t)d\boldsymbol{x}'$$

Properties of Reynolds decomposition

The statistical mean is a linear operator, which commutes with time and space derivative operators (the so-called rules of Reynolds)

- Centered fluctuating field

  $f \equiv \bar{F} + f' \quad \text{with} \quad \overline{f'} = 0 \qquad (f' = f - \bar{F}, \text{ and } \overline{f'} = \bar{F} - \bar{F} = 0)$

- Product of two variables f and g

  $fg \equiv (\bar{F} + f')(\bar{G} + g') = \bar{F}\bar{G} + \bar{F}g' + f'\bar{G} + f'g'$

  and thus,

  $$\overline{fg} = \bar{F}\,\bar{G} + \bar{F}\,\overline{g'} + \overline{f'}\,\bar{G} + \overline{f'g'} = \bar{F}\,\bar{G} + \overline{f'g'}$$

  $\overline{f'g'}$ is a new second-moment unknown variable

- Philosophy of the Reynolds decomposition, $u_i \equiv \bar{U}_i + u'_i$ with $\overline{u'_i} = 0$

  $\bar{U}_i$ part which can be reasonably calculated

  u'_i part which must be modelled (turbulent fluctuations)

The Reynolds Averaged Navier-Stokes (RANS) equations

For an incompressible flow $\nabla \cdot \boldsymbol{u} = 0$ with constant density $\rho = \text{cst}$ to simplify, the Navier-Stokes equations are given by

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \tau_{ij} = 2\mu s_{ij} \quad s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

By introducing the Reynolds decomposition, and taking the average

$u_i \equiv \bar{U}_i + u_i'$ $\qquad p \equiv \bar{P} + p'$ $\qquad \tau_{ij} \equiv \bar{\tau}_{ij} + \tau_{ij}'$

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0 \qquad \Longrightarrow \qquad \frac{\partial u_i'}{\partial x_i} = 0$$

$$\frac{\partial (\rho \bar{U}_i)}{\partial t} + \frac{\partial (\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho \overline{u_i' u_j'}\right)$$

Reynolds Averaged Navier-Stokes (RANS) equations

$-\rho\overline{u_i'u_j'}$ Reynolds stress tensor (new unknown)

Generally this term is larger than the mean viscous stress tensor except for wall bounded flows, where viscosity effects become preponderant close to the wall.

Total stress seen by the fluid, $\tau_t = \bar{\tau}_{ij} - \rho\overline{u_i'u_j'}$

closure problem for $-\rho\overline{u_i'u_j'}$

- by writting a transport equation for $-\rho\overline{u_i'u_j'}$
- by directly modelling the Reynolds stress tensor

The study of the turbulent kinetic energy balance gives a global view on the energy exchange between the mean field and the turbulent field, and allows to identify the term(s) responsible of the production of this turbulent kinetic energy.

Kinetic energy budget of the mean flow

$$\bar{U}_i \times \left\{ \frac{\partial(\rho \bar{U}_i)}{\partial t} + \frac{\partial(\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho \overline{u_i' u_j'}\right) \right\} \quad \text{and} \quad \frac{\partial \bar{U}_i}{\partial x_i} = 0$$

The final result can be recast as

$$\rho \frac{\bar{D}}{Dt}\left(\frac{\bar{U}_i^2}{2}\right) = \rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} - \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} \underbrace{- \frac{\partial(\bar{U}_i \bar{P})}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{U}_i \bar{\tau}_{ij}) - \frac{\partial}{\partial x_j}(\bar{U}_i \rho \overline{u_i' u_j'})}_{\text{transport terms}}$$

where $\bar{D}/Dt \equiv \partial/\partial t + \bar{\boldsymbol{U}} \cdot \nabla = \partial/\partial t + \bar{U}_j \partial/\partial x_j$ is the material derivative along the mean flow. We recall that

$$\rho \frac{\bar{D}\varphi}{Dt} = \rho\left(\frac{\partial \varphi}{\partial t} + \bar{\boldsymbol{U}} \cdot \nabla \varphi\right) = \frac{\partial(\rho\varphi)}{\partial t} + \nabla \cdot (\rho \varphi \bar{\boldsymbol{U}})$$

Kinetic energy budget of the mean flow (cont.)

Transport terms are terms of the form $\nabla \cdot \overline{\boldsymbol{F}}$, with $\bar{F}_j = \bar{U}_i \rho \overline{u_i' u_j'}$ for instance. From the divergence theorem,

$$\int_{\mathcal{V}} \nabla \cdot \overline{\boldsymbol{F}}\, dv = \int_{\mathcal{S}} \overline{\boldsymbol{F}} \cdot \boldsymbol{n}\, ds \to 0$$

if $\bar{F}$ tends to zero on the control surface $\mathcal{S}$. In general, these terms act to homogenise $\overline{\boldsymbol{F}}$ inside the volume $\mathcal{V}$.

By integration over a control volume including the turbulent region of the flow, the kinetic energy budget is reduced to

$$\int_{\mathcal{V}} \rho \frac{\bar{D}}{Dt}\left(\frac{\bar{U}_i^2}{2}\right) ds \quad = \quad \int_{\mathcal{V}} \rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} ds \quad - \quad \int_{\mathcal{V}} \bar{\tau}_{ij} \frac{\partial \bar{U}_i}{\partial x_j} ds$$

variation of the kinetic energy inside $\mathcal{V}$	in general, transfer to the turbulent field	dissipation of the kinetic energy by viscous effects

Kinetic energy budget of the fluctuating field

To derive the transport equation on $\rho\overline{u_i'^2}/2$, we first consider the equation for the fluctuating velocity u_i', obtained by substraction between

$$\text{and} \qquad \begin{cases} \dfrac{\partial(\rho u_i)}{\partial t} + \dfrac{\partial}{\partial x_j}(\rho u_i u_j) = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial \tau_{ij}}{\partial x_j} \\ \dfrac{\partial(\rho \bar{U}_i)}{\partial t} + \dfrac{\partial(\rho \bar{U}_i \bar{U}_j)}{\partial x_j} = -\dfrac{\partial \bar{P}}{\partial x_i} + \dfrac{\partial}{\partial x_j}(\bar{\tau}_{ij} - \rho\overline{u_i' u_j'}) \end{cases}$$

which provides

$$\frac{\partial(\rho u_i')}{\partial t} + \frac{\partial}{\partial x_k}\left[\rho(u_i'\bar{U}_k + \bar{U}_i u_k' + u_i' u_k')\right] = -\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\rho\overline{u_i' u_k'} + \tau_{ik}'\right)$$

That equation is then multiplied by u_i' and statistically averaged, by remembering that $\partial u_i'/\partial x_i = 0$. One obtains,

Kinetic energy budget of the fluctuating field (cont.)

$$\rho \frac{\bar{D}k_t}{Dt} = -\rho \overline{u_i' u_k'} \frac{\partial \bar{U}_i}{\partial x_k} - \overline{\tau_{ik}' \frac{\partial u_i'}{\partial x_k}} \underbrace{- \frac{1}{2} \frac{\partial}{\partial x_k} \overline{\rho u_i' u_i' u_k'} - \overline{u_i' \frac{\partial p'}{\partial x_i}} + \frac{\partial}{\partial x_k} \overline{u_i' \tau_{ik}'}}_{\text{transport terms}}$$

$$k_t \equiv \frac{1}{2} \overline{u_i' u_i'} = \frac{\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}}{2} \quad \text{(mean) turbulent kinetic energy} \quad (\mathrm{m^2.s^{-2}})$$

Homogeneous turbulence case : statistical properties of turbulence are independent of the observer position $\boldsymbol{x}$, leading to

$$\rho \frac{\bar{D}k_t}{Dt} = -\rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} - \overline{\tau_{ik}' \frac{\partial u_i'}{\partial x_k}}$$

$\rho \frac{\bar{D}k_t}{Dt}$	$-\rho \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j}$	$\overline{\tau_{ik}' \frac{\partial u_i'}{\partial x_k}}$
variation of the kinetic energy along the mean flow	energy transfer between the mean and turbulent fields	dissipation of the turbulent kinetic energy by viscous effects

Kinetic energy budget of the fluctuating field (cont.)

Dissipation rate of k_t

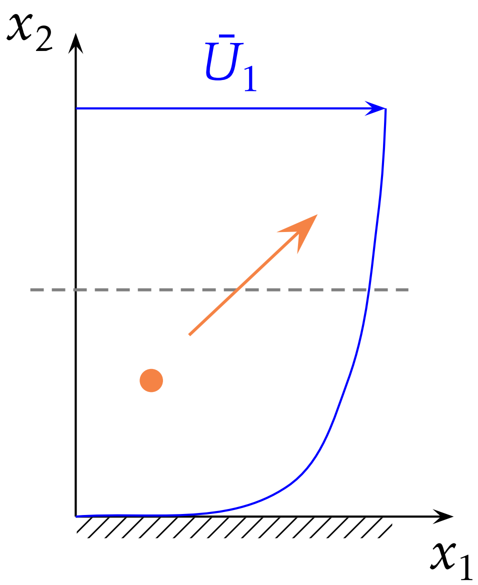
$$\rho\epsilon \equiv \overline{\tau'_{ik}\frac{\partial u'_i}{\partial x_k}} = 2\mu\,\overline{s'^2_{ij}} = \frac{1}{2}\mu\overline{\left(\frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i}\right)^2} \geq 0 \qquad (\epsilon \sim \mathrm{m}^2.\mathrm{s}^{-3})$$

Homogeneous turbulence case

$$\rho\frac{\bar{D}k_t}{Dt} = -\rho\overline{u'_iu'_k}\,\frac{\partial \bar{U}_i}{\partial x_k} - \overline{\tau'_{ik}\frac{\partial u'_i}{\partial x_k}}$$

$$= \underbrace{-\rho\overline{u'_iu'_k}\,\frac{\partial \bar{U}_i}{\partial x_k}}_{?} - \rho\epsilon \qquad\qquad \mathcal{P} \equiv -\rho\overline{u'_iu'_j}\,\frac{\partial \bar{U}_i}{\partial x_j}$$

Heuristic interpretation of the term $\mathcal{P}$



$$\begin{cases} u_2' > 0 \\ u_1' < 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

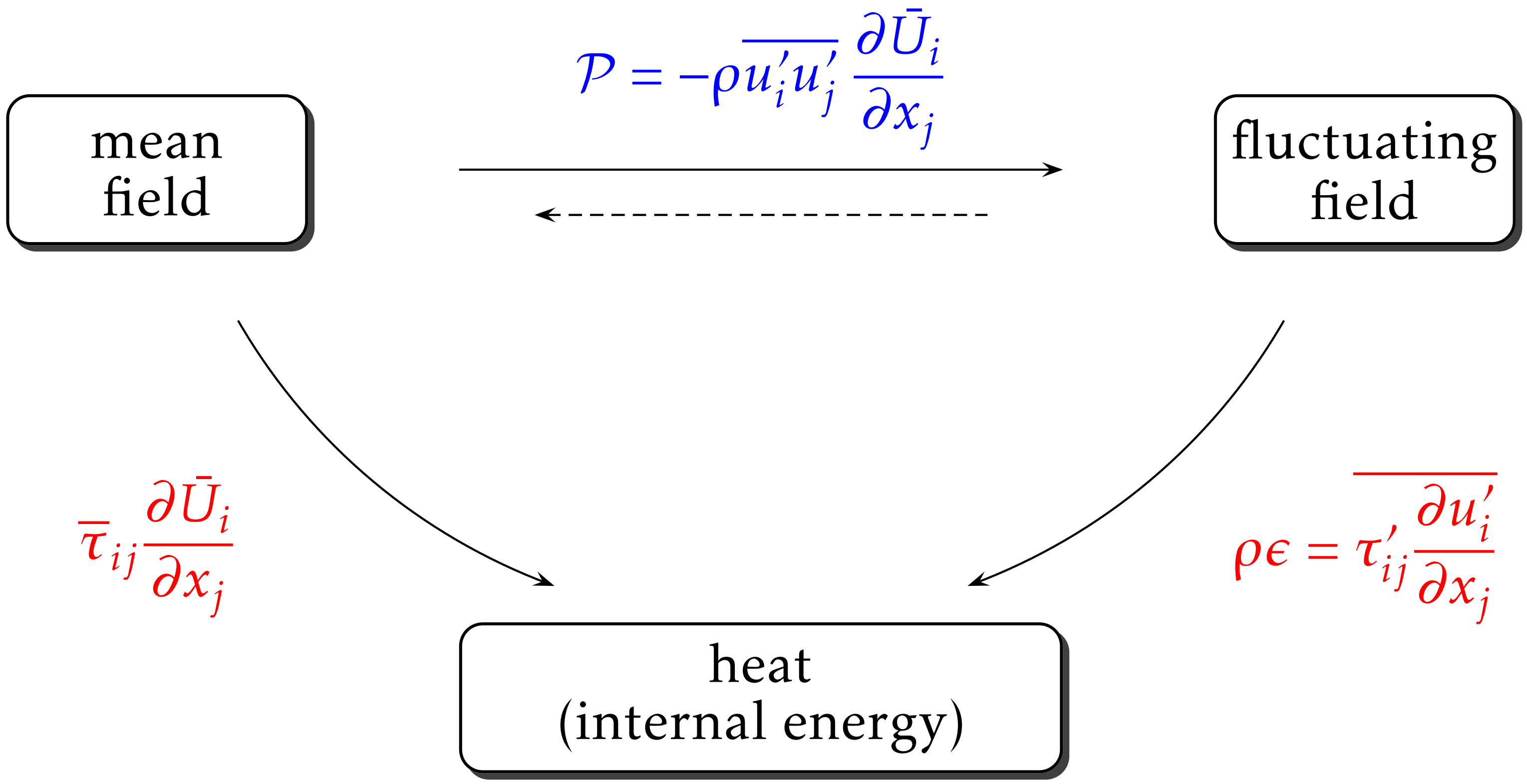
$$\begin{cases} u_2' < 0 \\ u_1' > 0 \end{cases} \qquad \overline{u_1' u_2'} < 0$$

Therefore, a positive production term is expected

$$\mathcal{P} \simeq -\rho \overline{u_1' u_2'} \frac{d\bar{U}_1}{dx_2} > 0$$

The term $\mathcal{P}$ is generally a production term for the turbulent kinetic energy k_t.

- **Transfers between the mean flow and the turbulent field**



$$T = \bar{T} + \theta' \qquad a = \lambda/(\rho c_p)$$

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial (\bar{U}_j \bar{T})}{\partial x_j} = -\frac{\partial}{\partial x_j}\left(-a\frac{\partial \bar{T}}{\partial x_j} + \overline{\theta u_j'}\right) + \frac{1}{\rho c_p}\left(\bar{\tau}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau_{ij}' \frac{\partial u_i'}{\partial x_j}}\right)$$

Small exercise : fluctuating irrotational field

The necessary rotational feature of a turbulent velocity field has been emphasized by Corrsin & Kistler (1954)

1. Remind the definition of an irrotational flow
2. By considering the following quantity

$$\overline{u_i'\left(\frac{\partial u_i'}{\partial x_j}-\frac{\partial u_j'}{\partial x_i}\right)}$$

   demonstrate that

$$\frac{\partial}{\partial x_i}\overline{u_i'u_j'}=\frac{\partial k_t}{\partial x_j}$$

3. Deduce the form of the RANS Equation for the case of a fluctuating irrotational velocity field, and comment carefully your result.
4. Is the Reynolds tensor diagonal for an irrotational flow ?

Turbulent viscosity concept for the Reynolds tensor Boussinesq model (1877)

- By analogy with the definition of the viscous tensor $\overline{\overline{\boldsymbol{\tau}}}$, the Reynolds stress tensor $-\rho\overline{u_i'u_j'}$ is modelled by

$$-\rho\overline{u_i'u_j'} = 2\mu_t\bar{S}_{ij} - \frac{2}{3}\rho k_t\delta_{ij} = \mu_t\left(\frac{\partial\bar{U}_i}{\partial x_j} + \frac{\partial\bar{U}_j}{\partial x_i}\right) - \frac{2}{3}\rho k_t\delta_{ij}$$

turbulent viscosity $\mu_t = \mu_t(\boldsymbol{x}, t)$: intrinsic property of the turbulent flow, and not of the fluid as the molecular viscosity.

- There is still a closure problem since the expression of μ_t is not defined (6 unknowns $\overline{u_i'u_j'}$ $\rightarrow$ 1 unknown μ_t).

- A consequence of the turbulent-viscosity hypothesis is that $\mathcal{P} = 2\mu_t\overline{S}_{ij}^{2} \geq 0$ by construction : always a positive energy transfer towards the turbulent field.

Application to free shear flows (jet, wake, mixing layer)

Mean velocity field $(\bar{U}_1, \bar{U}_2)$, and slowly variable flow along the x_1 direction, that is $\partial/\partial x_1 \ll \partial/\partial x_2$: quasi-homogeneous flow in the x_1 direction

Averaged Navier-Stokes equation along the x_1 axis

$$\frac{\partial(\bar{U}_1\bar{U}_1)}{\partial x_1} + \frac{\partial(\bar{U}_1\bar{U}_2)}{\partial x_2} = -\frac{\partial \bar{P}}{\partial x_1} + \frac{\partial}{\partial x_1}(\bar{\tau}_{11} - \rho\overline{u_1'u_1'}) + \frac{\partial}{\partial x_2}\left(\bar{\tau}_{12} - \rho\overline{u_1'u_2'}\right)$$

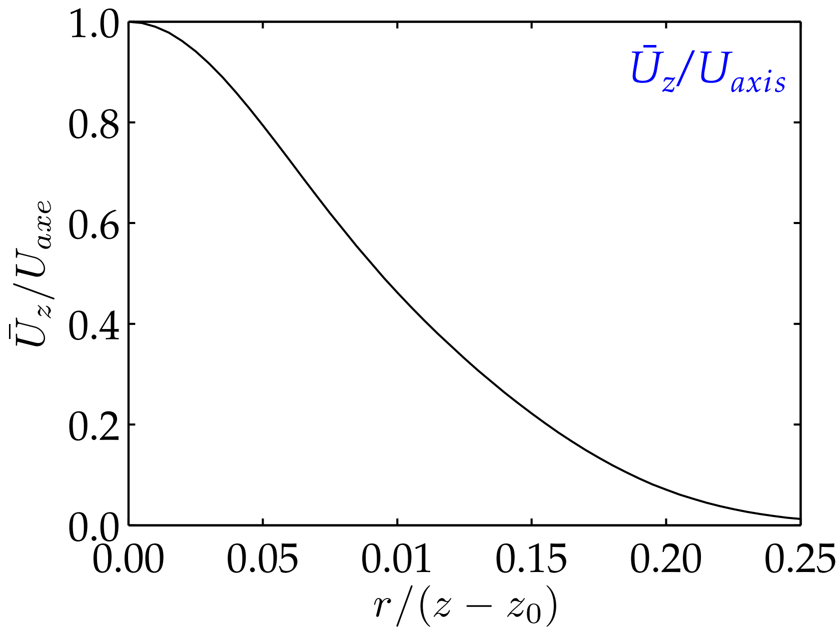
The total stress τ_t acting on the fluid reads

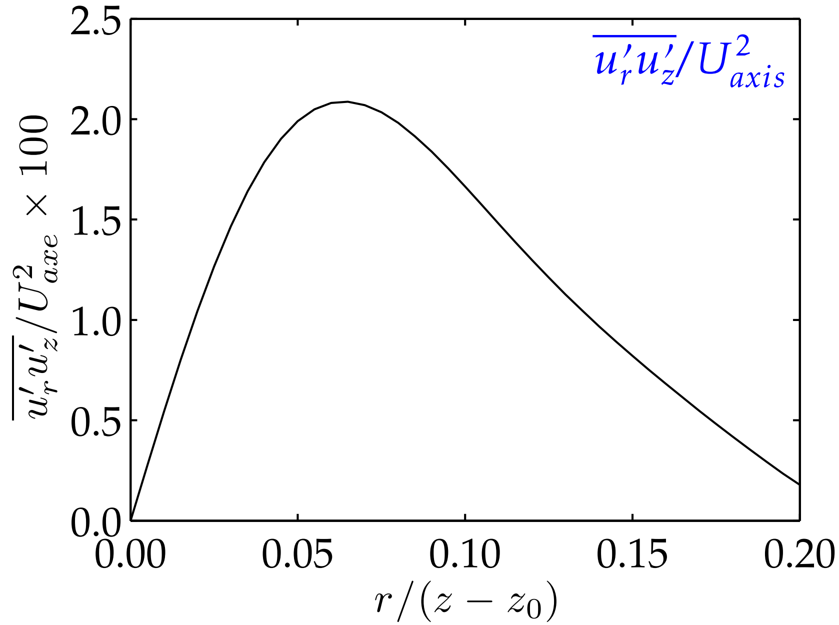
$$\tau_t = \bar{\tau}_{12} - \rho\overline{u_1'u_2'} = \mu\frac{d\bar{U}_1}{dx_2} + \mu_t\frac{d\bar{U}_1}{dx_2} = (\mu + \mu_t)\frac{d\bar{U}_1}{dx_2}$$

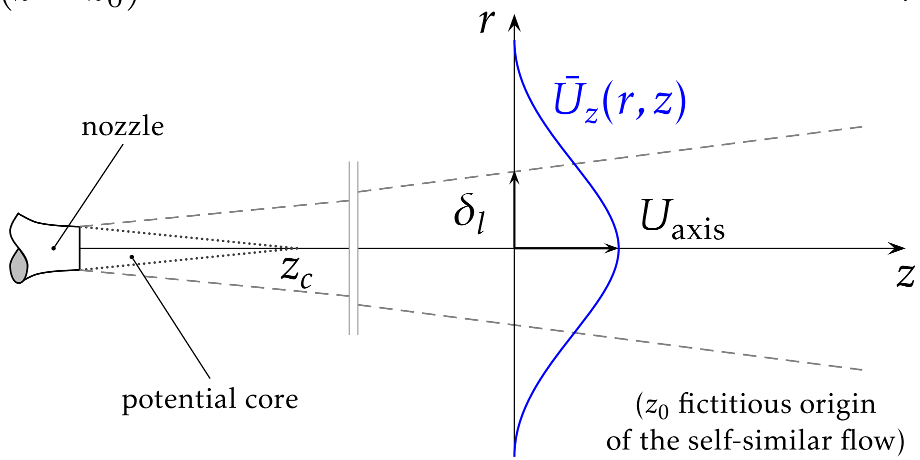
The total stress τ_t is therefore null when the mean velocity profile $\bar{U}_1$ reaches a local extremum. Furthermore, it is also expected that $\mu \ll \mu_t$

Illustration for a subsonic round jet

M = 0.16 and $\mathrm{Re}_D = 9.5 \times 10^4$ (from Hussein, Capp & George, 1994)

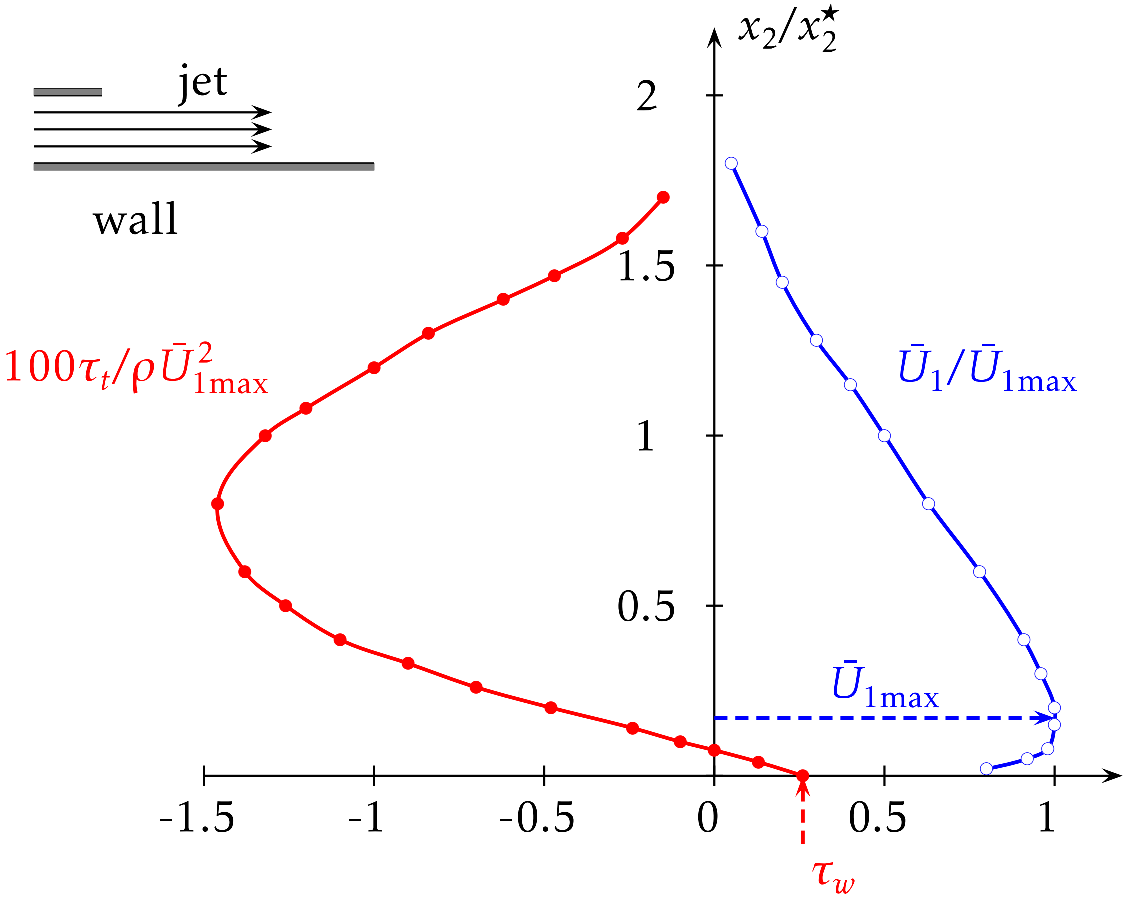






Two famous counter-examples (asymmetrical mean flow)

- channel flow with smooth and rough surfaces (Hanjalić & Launder, 1972)
- plane wall jet (Mathieu, 1967)



Plane wall jet (nozzle exit)

$\mathrm{Re} \simeq 2 \times 10^4$

$$\tau_t = \mu \frac{d\bar{U}_1}{dx_2} - \rho \overline{u_1' u_2'}$$

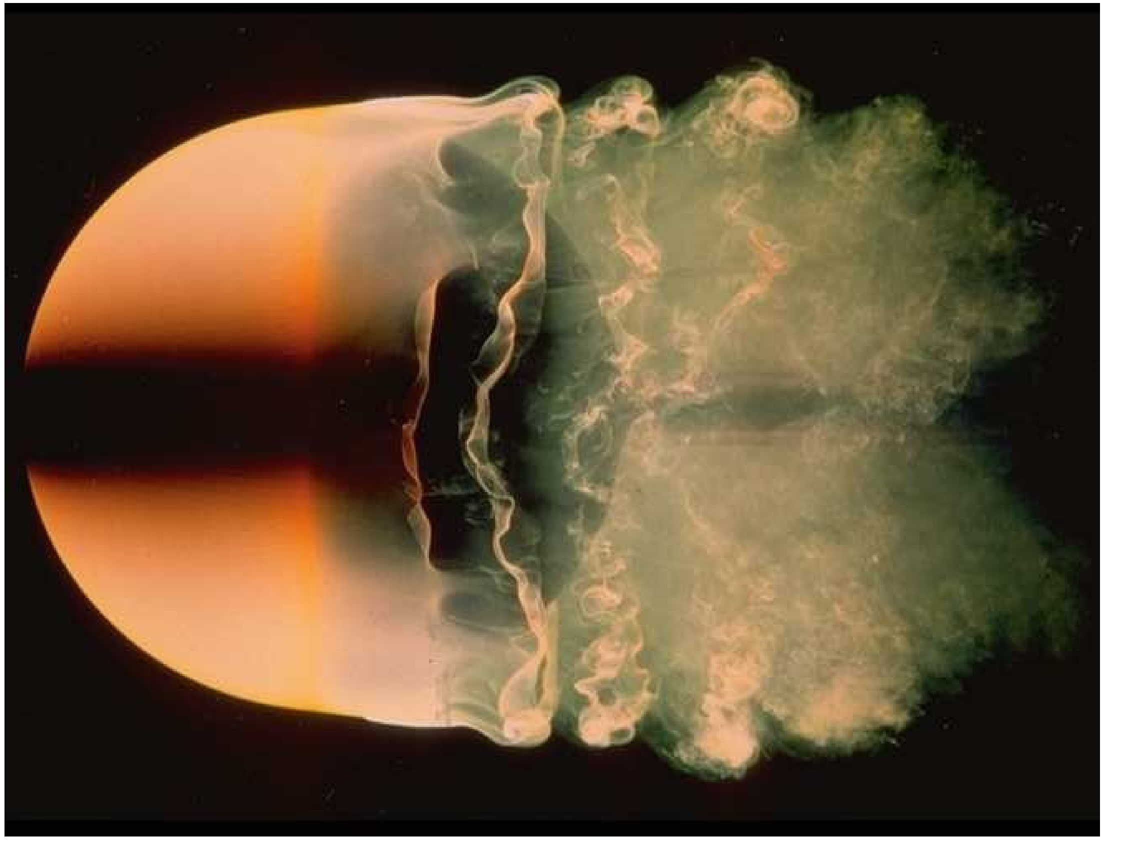
Some practical consequences

The total shear is usually much higher in turbulent régime :

- Reattachment of a turbulent boundary layer after detachment
  (and possible relaminarisation)

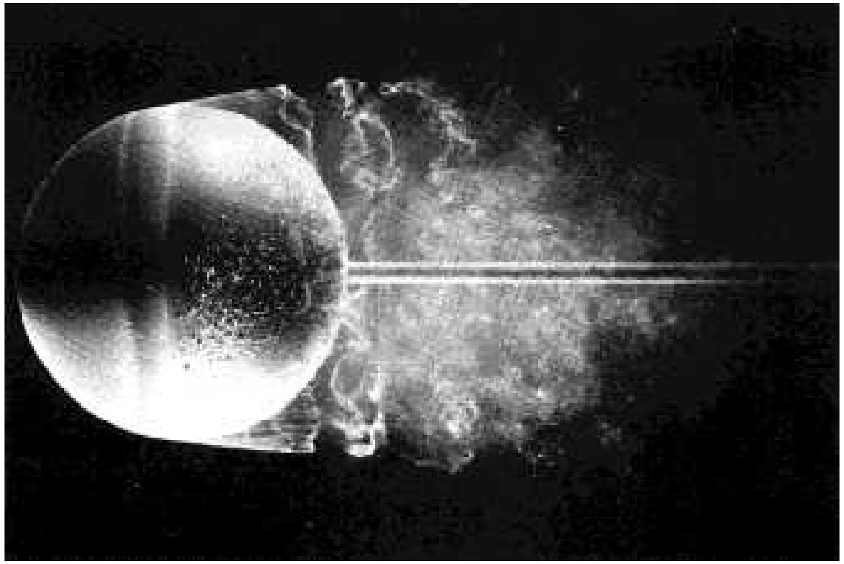
- Reduction of flow separation regions : the drag crisis phenomenon.
  The boundary layer separation point is moved downstream along a bluff body, with a reduction of the total drag with respect to the laminar regime : the increase in friction induced by turbulence is compensated by the reduction of the pressure drag, induced by the turbulent wake.

Flow past a sphere

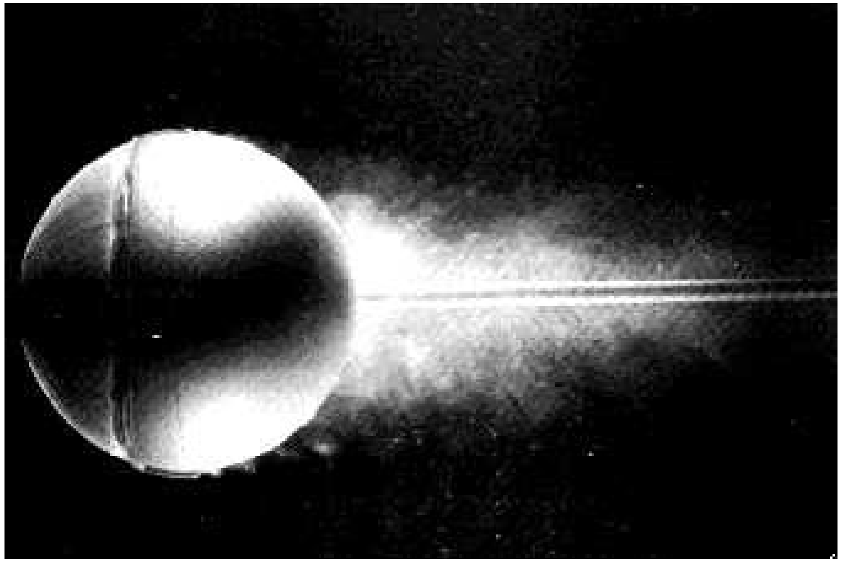


ONERA / DAFE, water tunnel, $\mathrm{Re}_D = 10^3$

Flow past a sphere



$Re_D = 1.5 \times 10^4$



$Re_D = 3.0 \times 10^4$ with a trip wire

ONERA, Werle (1980) in *An album of fluid motion*, Van Dyke (1982)

Sphere : critical Reynolds number $Re_D^c \simeq 3 \times 10^5$

Flow around a bluff body

Drag crisis – critical Reynolds number for which the flow pattern changes, leaving a narrower turbulent wake : the boundary layer on the front surface becomes turbulent

RANS equation

$$\rho\left(\frac{\partial \bar{\boldsymbol{U}}}{\partial t} + \bar{\boldsymbol{U}} \cdot \nabla \bar{\boldsymbol{U}}\right) = -\nabla \bar{P} + \nabla \cdot (\overline{\overline{\boldsymbol{R}}} + \overline{\overline{\boldsymbol{\tau}}})$$

$\overline{\overline{\boldsymbol{R}}}$ Reynolds stress tensor

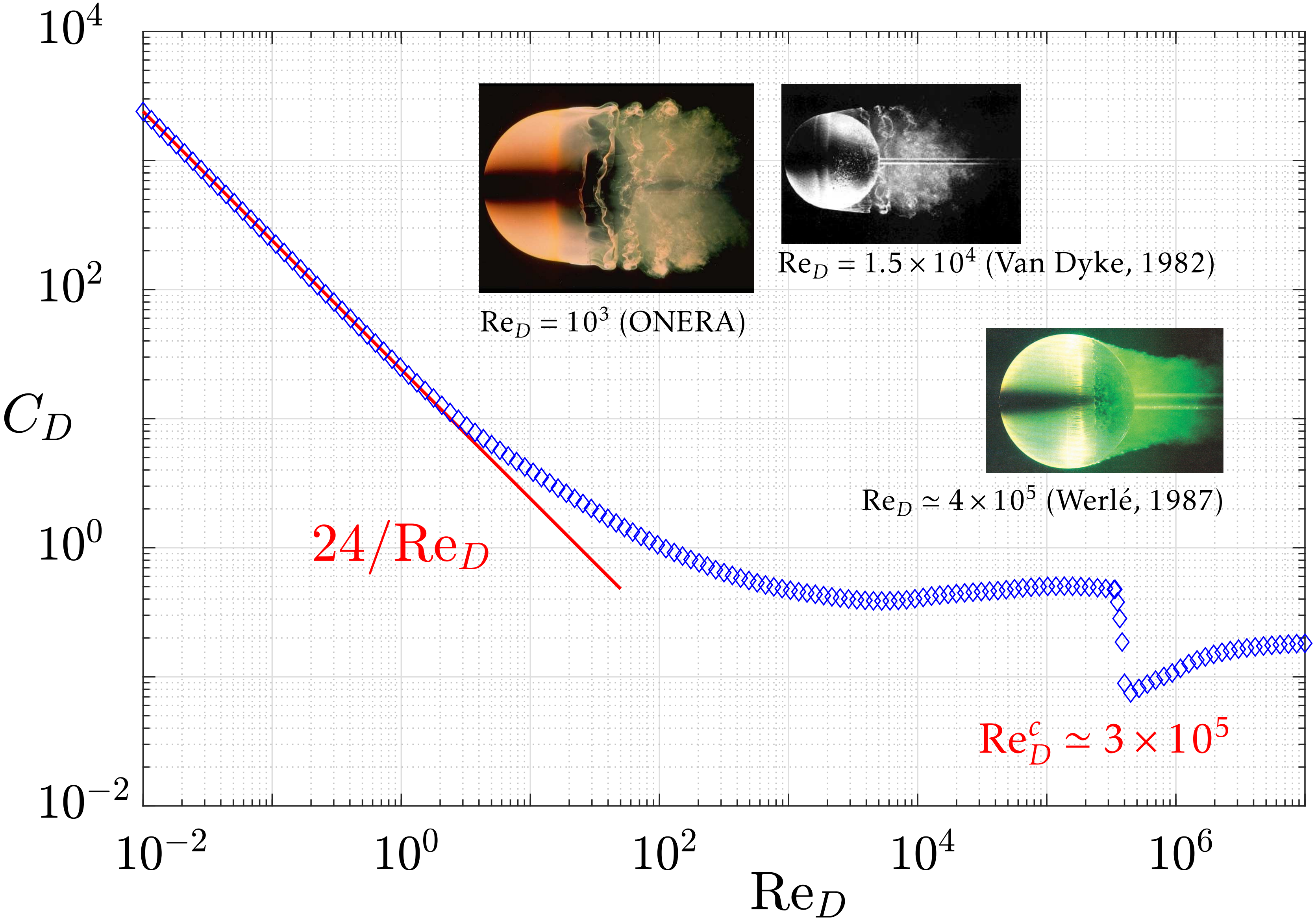
$R_{ij} \equiv -\rho \overline{u_i' u_j'}$

$$\boldsymbol{F}_{\text{flow} \to \text{body}} \equiv \overline{\boldsymbol{F}} = \int_{\mathcal{S}_w} -\bar{P}\boldsymbol{n}\, ds + \int_{\mathcal{S}_w} \overline{\overline{\boldsymbol{\tau}}} \cdot \boldsymbol{n}\, ds \qquad \text{(remembering that } \overline{\overline{\boldsymbol{R}}} = 0 \text{ on } \mathcal{S}_w\text{)}$$

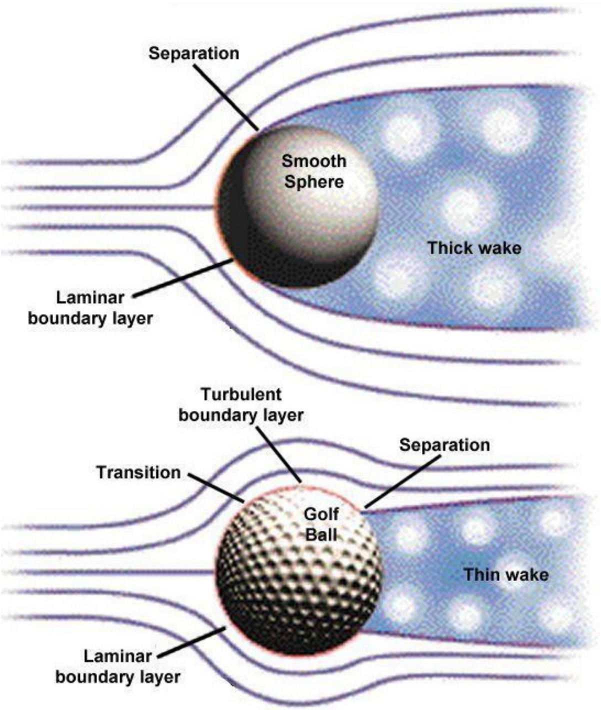
Mean drag force $F_D = \boldsymbol{F} \cdot \boldsymbol{e}_\infty$ = pressure drag (form drag) + skin friction drag

A streamlined body looks like an airfoil at small angles of attack (narrow wake), whereas a bluff body looks like a sphere, or an airfoil at large angles of attack. For streamlined bodies, frictional drag is the dominant term. For a bluff body, drag is dominated by the pressure term

- **Drag coefficient for a smooth sphere** (adapted from Clift, Grace, & Weber, 1978)



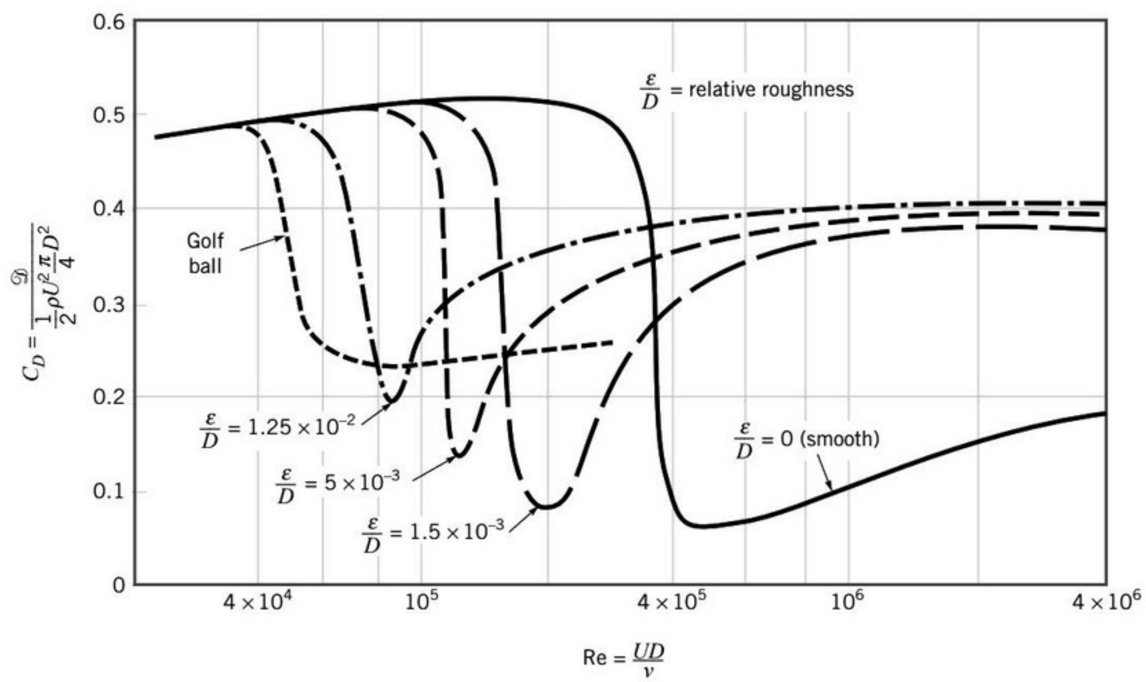
Why golf balls have dimples?



Moin & Kim, 1997, *Scientific American*

Drag coefficient of spheres with varying surface roughness. The drag crisis or sudden drop in drag as Reynolds number increases occurs when the boundary layer transitions to turbulence upstream of separation

$D = 4.3$ cm, $U \simeq 67$ m.s^{-1}, $\mathrm{Re} \simeq 1.9 \times 10^5$ (professional golfer)



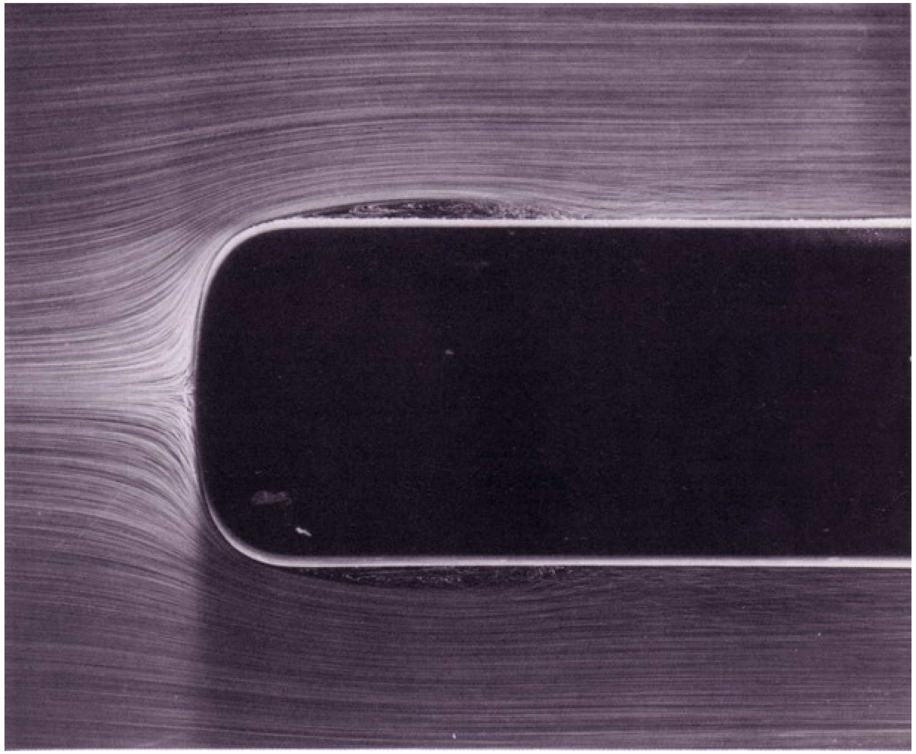
Munson *et al.*, 2014, *Fundamentals of fluid mechanics*

Vortex generators for delaying boundary layer separation

Beechcraft Baron
(twin-engined piston aircraft)

Boeing-777-3ZG-ER
http://www.airliners.net/

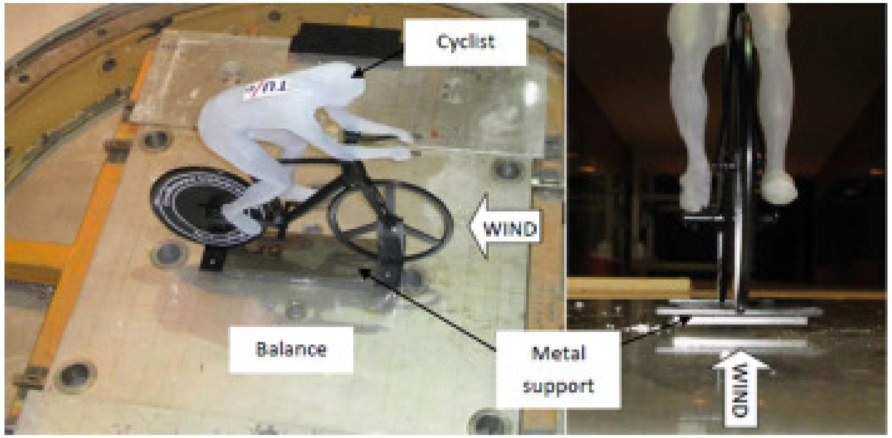
Boundary layer separation

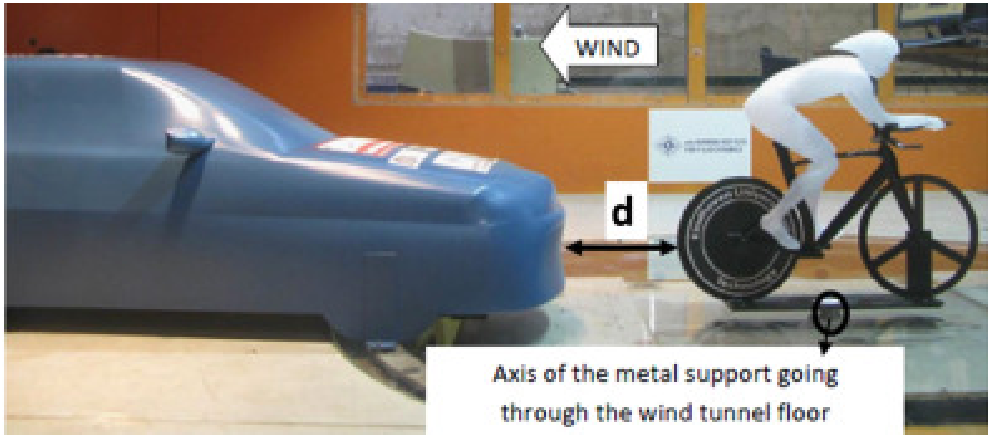


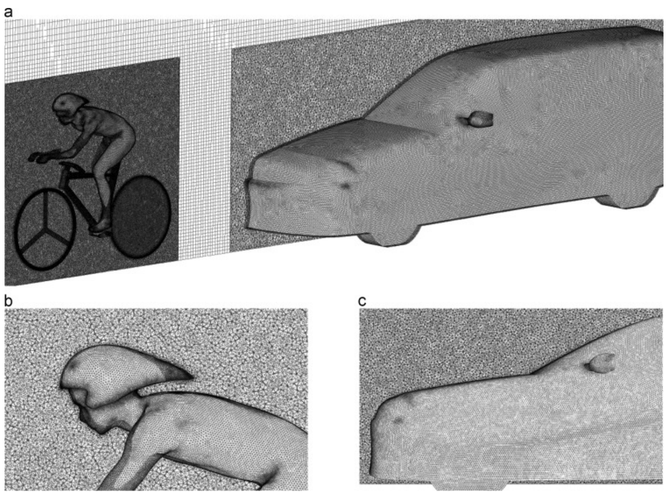
Separation of the laminar boundary layer on a body of revolution (Rankine ogive, $\mathrm{Re}_D = 6000$). The boundary layer becomes quickly turbulent and then reattaches to the surface, enclosing a short thin region of recirculation flow (visualization by air bubbles in water)

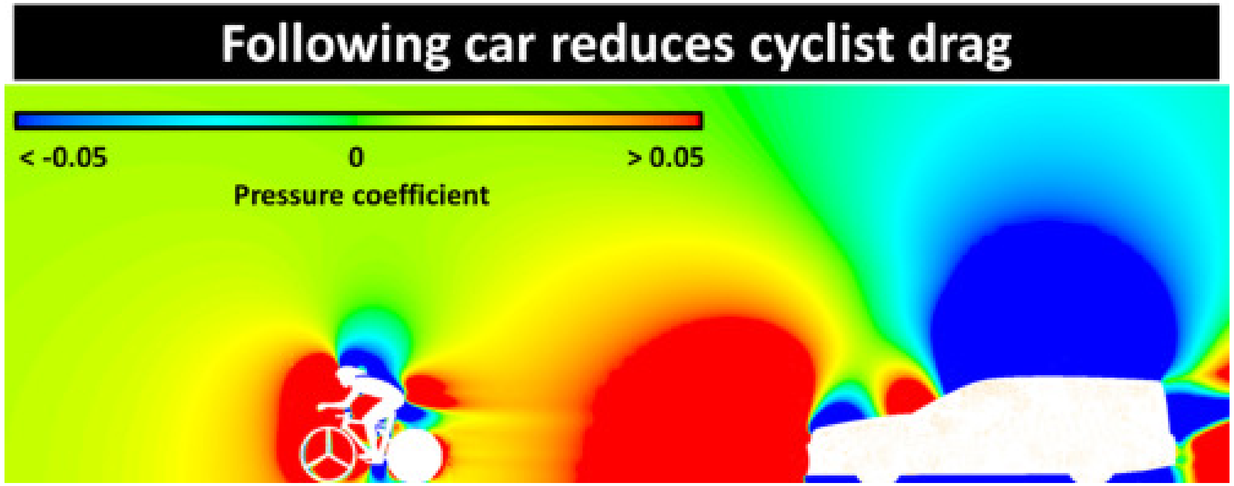
Werlé (ONERA) in Van Dyke (1982, fig. 33)

- **Elite cyclist : reduction of drag ...** (Blocken & Toparlar, *J. Wing. Eng. Ind. Aerodyn.*, 2015)
  ... when a cyclist is followed by a car



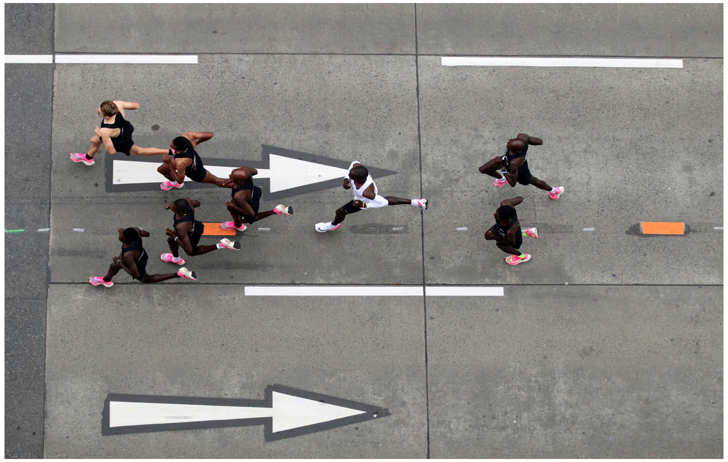






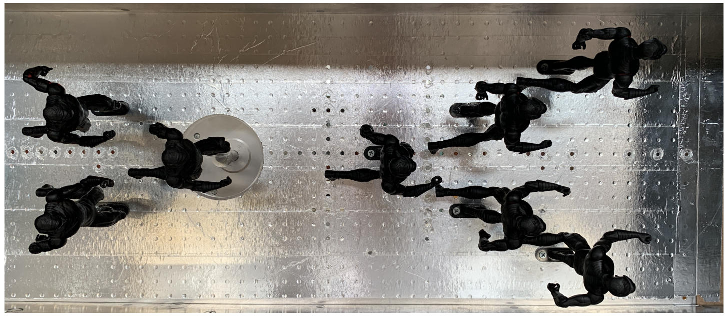
For a 50 km individual time trial : $3 \leq d \leq 10$ m $\Longrightarrow$ 1 mm $\rightarrow$ 4 s time reduction !
Recommendation for UCI, $d \geq 30$ m

Run a new marathon record in under two hours



Drafting formation used to reduce air resistance by positioning other pacers around the top runner

Mannequins mounted around the main runner (fixed on the load cell support) in wind tunnel to replicate the formation used by Eliud Kipchoge (drag reduced by 50%)



Elite runner Eliud Kipchoge became the first person to run a marathon in under two hours in Vienna (INEOS 1 :59 Challenge on 12th Oct. 2019, unofficial race in 1 :59 :40). He is assisted by seven pacers, five forming an inverted arrow in front of him and two others behind him

Swordfish-shaped arrangement of seven pacers that lowered the air resistance on the top runner by about 60% compared with a solo runner

The identified swordfish-shaped configuration, a skinny diamond in front of the top runner and two pacers in the back, would save roughly four minutes off of a marathon time. Eliud Kipchoge could reduced his time by an additional 40 seconds

Massimo Marro, Jack Leckert, Ethan Rollier, Pietro Salizzoni and Christophe Bailly
Wind tunnel evaluation of novel drafting formations for an elite marathon runner,
Proc. Roy Soc. A, **479**, 2023