*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

Physics of turbulent flow

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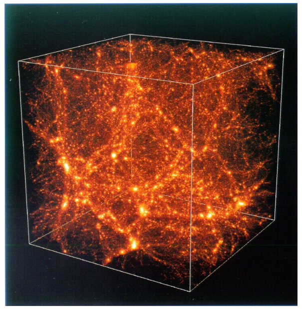
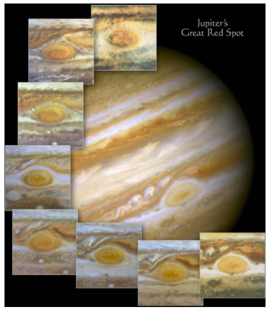
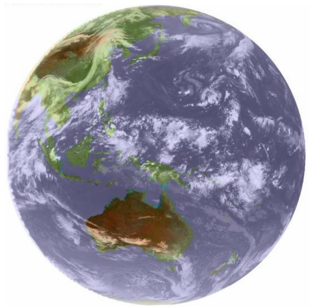
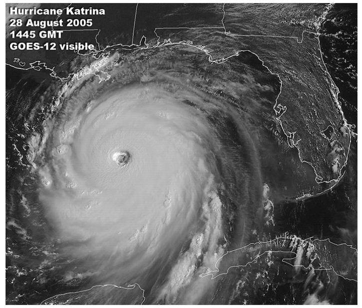
Centrale Lyon – MOD 3rdYear & MSc. – version 15-10-2024

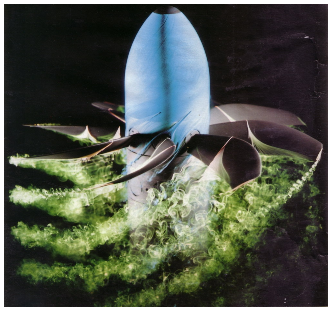
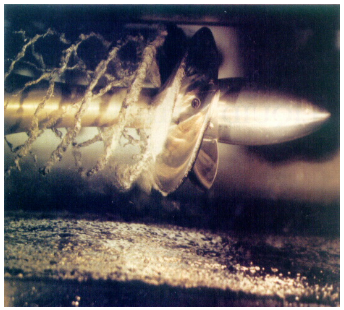
Outline

Wall-bounded turbulent flow

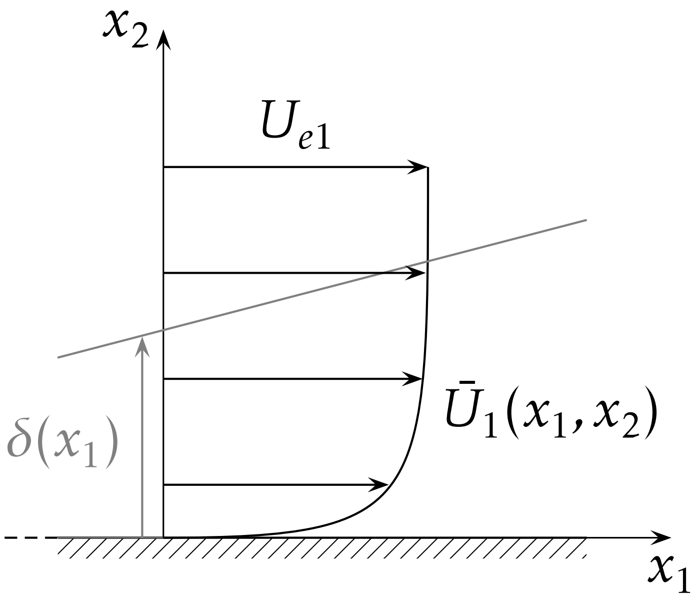



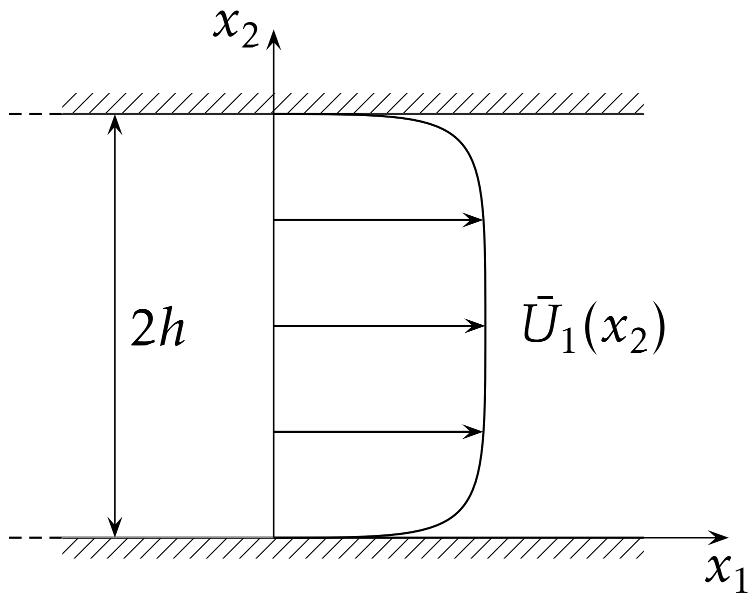





- **Two main classes of wall flows : confined flows & external flows**





flat-plate boundary layer

$$\mathrm{Re}_\delta = \frac{U_{e1}\delta}{\nu}$$

fully turbulent for $\mathrm{Re}_\delta \geq 2800$

channel flow

$$\mathrm{Re}_{2h} = \frac{U_d 2h}{\nu} \quad (U_d \text{ bulk velocity})$$

fully turbulent for $\mathrm{Re}_{2h} \geq 1800$

homogeneous flow along x_1

Fully developed channel flow

Reynolds-Averaged Navier-Stokes equations : $\bar{U}_1 = \bar{U}_1(x_2)$ and $\bar{U}_2 = \bar{U}_3 = 0$.
In addition, the flow is homogenous along x_1

$$\begin{cases} 0 = -\dfrac{\partial \bar{P}}{\partial x_1} - \dfrac{d}{dx_1}\left(\rho \overline{u_1'^2}\right) + \dfrac{d}{dx_2}\left(\mu \dfrac{d\bar{U}_1}{dx_2} - \rho \overline{u_1' u_2'}\right) & \text{(i)} \\ 0 = -\dfrac{\partial \bar{P}}{\partial x_2} - \dfrac{d}{dx_1}\left(\rho \overline{u_1' u_2'}\right) - \dfrac{d}{dx_2}\left(\rho \overline{u_2' u_2'}\right) & \text{(ii)} \end{cases}$$

By integration of Eq. (ii) from the wall ($x_2 = 0$) to a current point x_2, one obtains

$$\bar{P}(x_1, x_2) = \bar{P}_w - \rho \overline{u_2' u_2'}$$

where $\bar{P}_w = \bar{P}(x_1, x_2 = 0)$ is the mean wall pressure (measurable quantity)

- **Fully developed channel flow**

The Navier-Stokes equation (i) can now be rewritten as

$$0 = -\frac{d\bar{P}_w}{dx_1} + \frac{d}{dx_2}\underbrace{\left(-\rho\overline{u_1'u_2'} + \mu\frac{d\bar{U}_1}{dx_2}\right)}_{\bar{\tau}_t(x_2)}$$

$\bar{\tau}_t$ mean total stress applied to the fluid

By integration along the transverse direction again, up to a current point x_2

$$\frac{d\bar{P}_w}{dx_1}x_2 = -\rho\overline{u_1'u_2'} + \mu\frac{d\bar{U}_1}{dx_2} - \bar{\tau}_w \qquad \text{where} \qquad \bar{\tau}_w \equiv \mu\frac{d\bar{U}_1}{dx_2}\bigg|_{x_2=0}$$

where $\bar{\tau}_w$ is the mean shear stress at the wall.

Fully developed channel flow

Introduction of the friction velocity

$$u_\tau \equiv \sqrt{\bar{\tau}_w/\rho}$$

The friction velocity is the characteristic turbulent velocity scale for the turbulent boundary layer near the wall. In particular, $|\overline{u_i'u_j'}| \sim u_\tau^2$

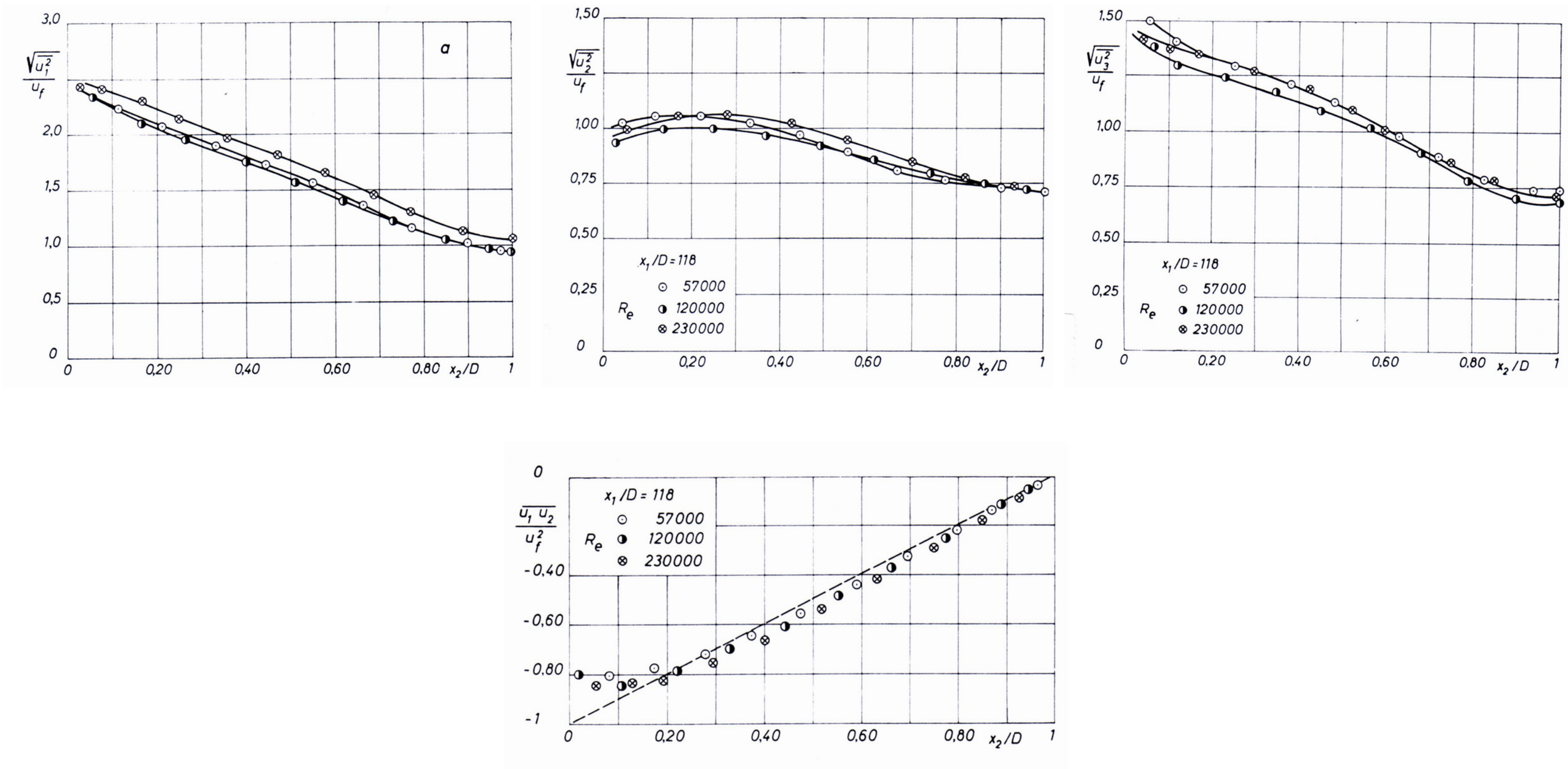
There is a direct link between this friction velocity and the pressure drop. For $x_2 = h$, that is on the symmetry plane of the channel, one has

$$\frac{d\bar{P}_w}{dx_1} h = -\bar{\tau}_w \qquad \Longrightarrow \qquad u_\tau^2 = -\frac{1}{\rho}\frac{d\bar{P}_w}{dx_1} h = \text{ cst for a pipe flow}$$

In the end, the mean velocity $\bar{U}_1$ is governed by

$$u_\tau^2\left(\frac{x_2}{h} - 1\right) - \overline{u_1'u_2'} + \nu\frac{d\bar{U}_1}{dx_2} = 0 \quad \text{or equivalently} \quad \bar{\tau}_t = \bar{\tau}_w\left(1 - \frac{x_2}{h}\right) \qquad (1)$$

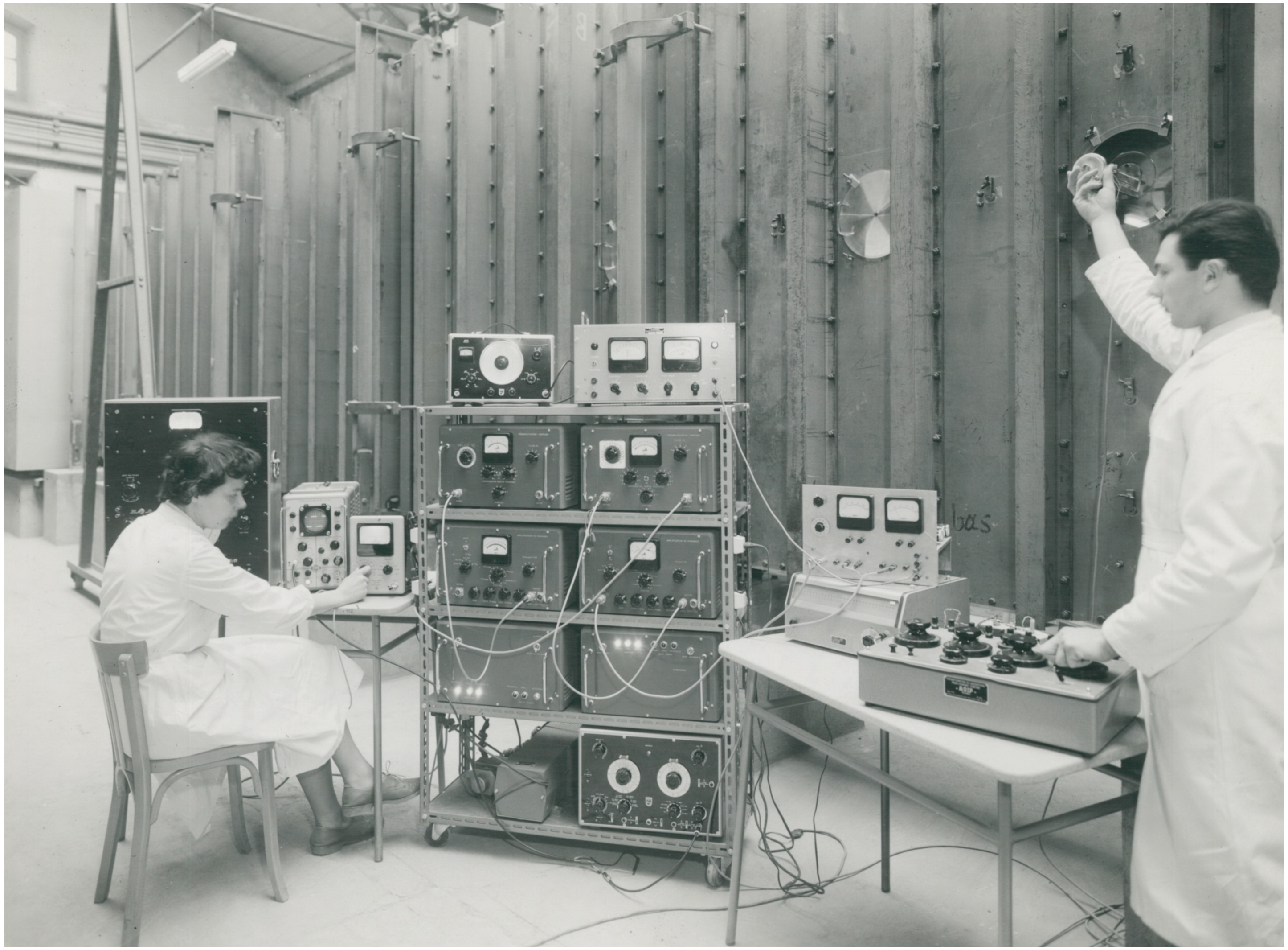
- **Fully developed channel flow**
  Plane channel of width $2h \equiv 2D$ (Comte-Bellot, 1965)



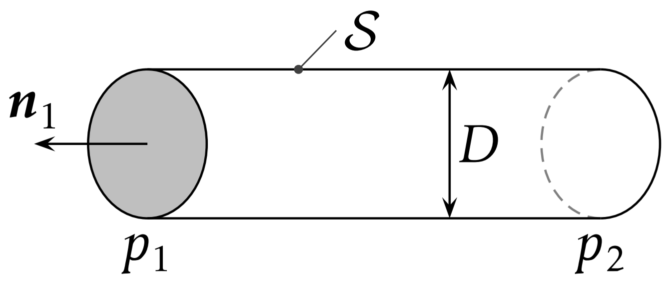
(wall) (axis)

$$5.7 \times 10^4 \leq \mathrm{Re}_h \leq 2.3 \times 10^5 \qquad \mathrm{Re}_\tau = 178, 392, 587$$

- **Fully developed channel flow**
  Geneviève Comte-Bellot (PhD thesis, Grenoble in 1963)



Small exercise : skin-friction coefficient for a circular pipe



1. Identify the following equation,

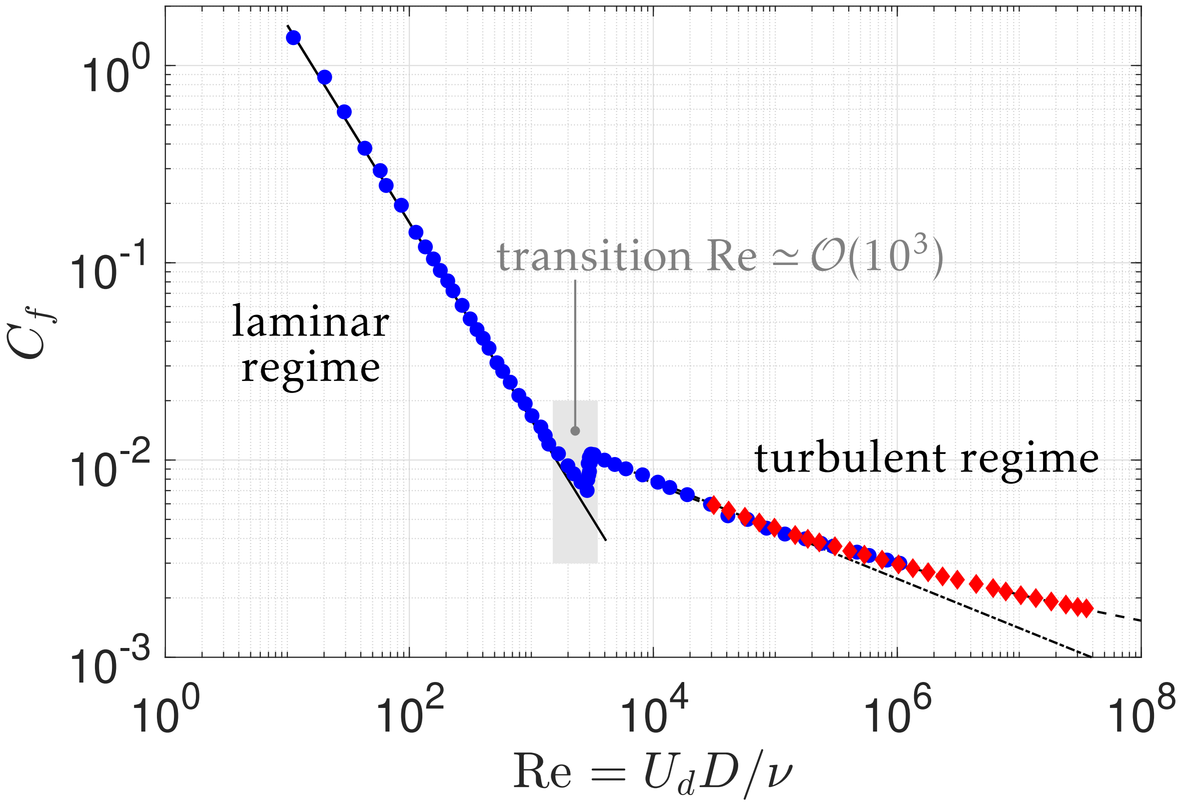
$$\rho \frac{D\boldsymbol{u}}{Dt} = \nabla \cdot \overline{\overline{\sigma}}$$

2. For a pipe of diameter D and length L, write the integral momentum conservation.
3. By introducing the wall shear stress τ_w, and the skin-friction coefficient $C_f = \tau_w/(\rho U_d^2/2)$ where U_d is the bulk velocity, show that the head pressure lost $\Delta p = p_1 - p_2$ can be recast as

$$\Delta p = 4C_f \frac{L}{D} \frac{1}{2} \rho U_d^2$$

4. Now consider in the above Reynolds' decomposition : what should be changed ?

Small exercise : skin-friction coefficient for a circular pipe



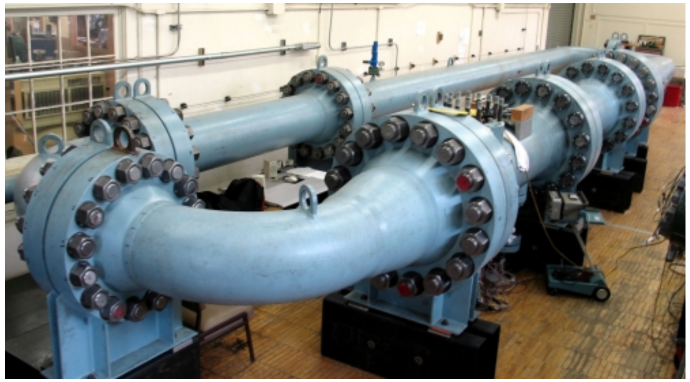
— laminar regime $C_f = 16/\mathrm{Re}$

–·–·– Blasius' relationship, $C_f \simeq 0.0791\,\mathrm{Re}^{-1/4}$

– – – $1/C_f^{1/2} \simeq 3.860 \log_{10}(\mathrm{Re}\, C_f^{1/2}) - 0.088$

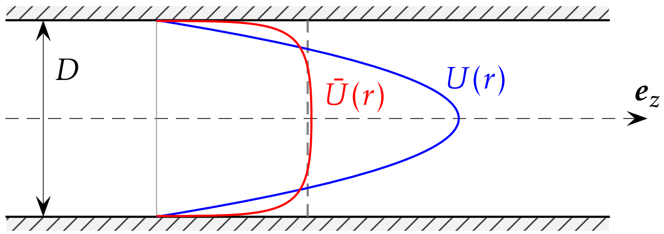
● Oregon facility

♦ Princeton *Superpipe*



McKeon *et al.* (2004) - *Superpipe*, the Reynolds number is increased through the pressure

Laminar versus turbulent regime



- **Turbulent boundary layer equations**

  Prandtl's approximations ($\delta \ll L$) for the RANS equations
  Conservation of mass

$$\frac{\partial \bar{U}_1}{\partial x_1} + \frac{\partial \bar{U}_2}{\partial x_2} = 0 \qquad \Longrightarrow \qquad V \sim \frac{\delta}{L}\, U$$

  Averaged Navier-Stokes equation along x_1 ($\neq$ laminar case)

$$\begin{cases} \bar{U}_1 \dfrac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \dfrac{\partial \bar{U}_1}{\partial x_2} = -\dfrac{1}{\rho} \dfrac{\partial \bar{P}}{\partial x_1} - \dfrac{\partial \overline{u_1'^2}}{\partial x_1} - \dfrac{\partial \overline{u_1' u_2'}}{\partial x_2} + \nu \left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} \right) \bar{U}_1 \\ \sim \dfrac{U^2}{L} \qquad \sim \dfrac{U^2}{L} \qquad \qquad \sim \dfrac{u^2}{L} \qquad \sim \dfrac{u^2}{\delta} \qquad \sim \nu \left(\dfrac{U}{L^2} ; \dfrac{U}{\delta^2} \right) \end{cases}$$

  We impose the balance between the convection along x_1 and the turbulent diffusion along x_2 : a turbulent flow can only be observed if $u \sim \sqrt{\delta/L}\, U$

Turbulent boundary layer equations (cont.)

Averaged Navier-Stokes equation along x_2

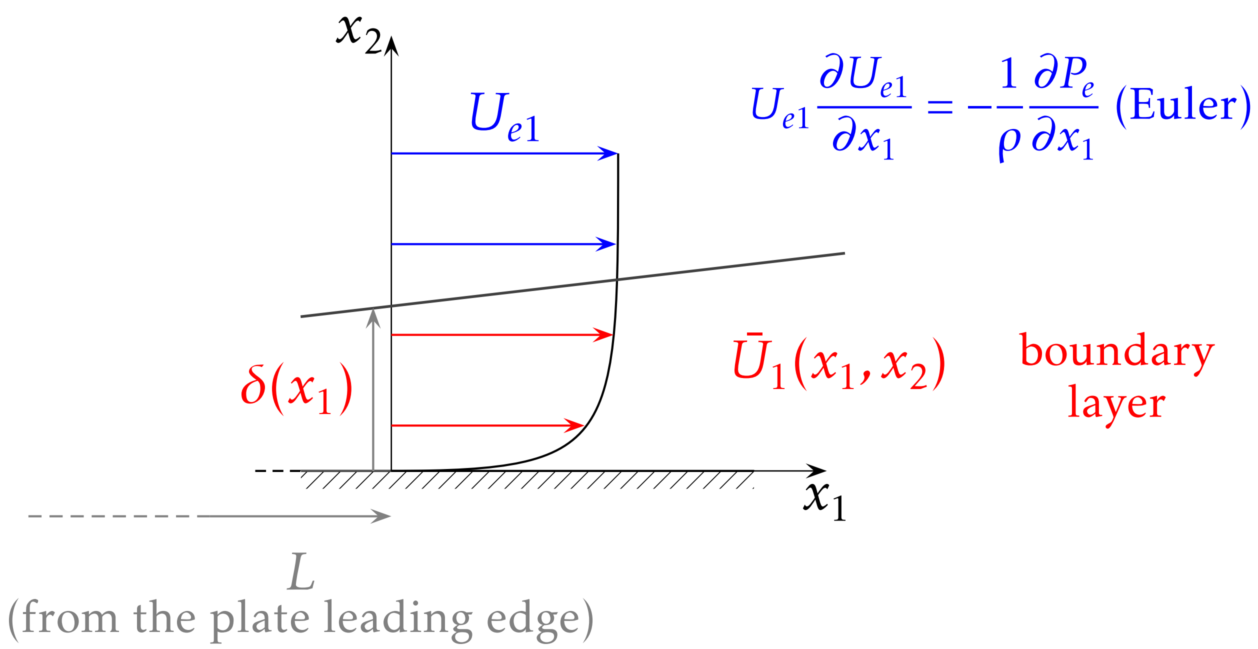
$$\begin{cases} \bar{U}_1 \dfrac{\partial \bar{U}_2}{\partial x_1} + \bar{U}_2 \dfrac{\partial \bar{U}_2}{\partial x_2} = -\dfrac{1}{\rho}\dfrac{\partial \bar{P}}{\partial x_2} - \dfrac{\partial \overline{u_1' u_2'}}{\partial x_1} - \dfrac{\partial \overline{u_2'^2}}{\partial x_2} + \nu \left(\dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} \right) \bar{U}_2 \\ \sim \dfrac{\delta}{L}\dfrac{U^2}{L} \quad \sim \dfrac{\delta}{L}\dfrac{U^2}{L} \qquad\qquad \sim \dfrac{\delta}{L}\dfrac{U^2}{L} \quad \sim \dfrac{\delta}{L}\dfrac{U^2}{\delta} \quad \sim \nu \dfrac{\delta}{L}\dfrac{U}{\delta^2} \sim \dfrac{1}{\mathrm{Re}_\delta}\dfrac{\delta}{L}\dfrac{U^2}{\delta} \end{cases}$$

All the terms are smaller by a factor δ/L (refer also to the laminar boundary layer). In addition, the pressure term must balance the dominant red term. By integration in the transverse direction x_2, one gets $\bar{P} + \rho \overline{u_2'^2} = \text{cst}$ across the boundary layer.

Turbulent boundary layer equations (cont.)

The mean pressure gradient is imposed by the external flow (through wall curvature for instance)

$$\bar{P} + \rho\overline{u_2'^2} = P_e = \bar{P}_w$$



- **Turbulent boundary layer equations (cont.)**

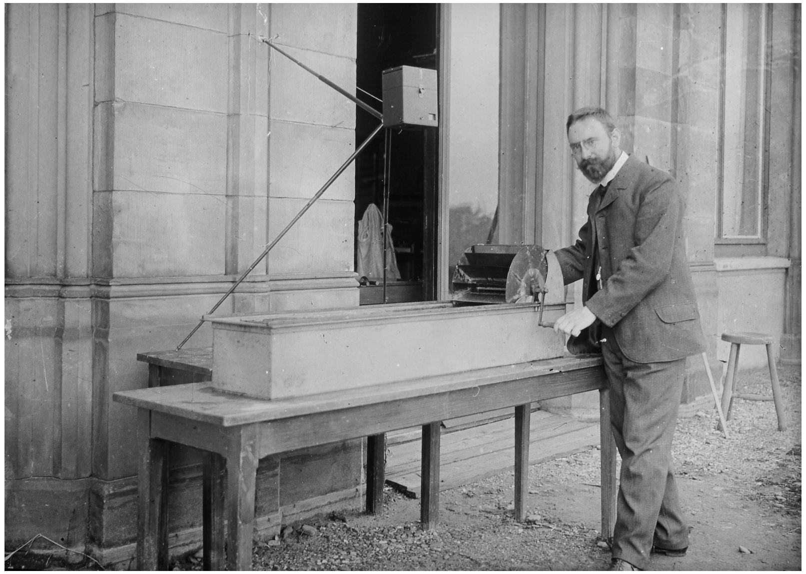
$$\begin{cases} \bar{U}_1 \dfrac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \dfrac{\partial \bar{U}_1}{\partial x_2} = -\dfrac{1}{\rho}\dfrac{dP_e}{dx_1} + \dfrac{\partial}{\partial x_2}\left(\nu \dfrac{\partial \bar{U}_1}{\partial x_2} - \overline{u_1' u_2'} \right) & \text{(i)} \\ \bar{P}(x_1, x_2) = P_e - \rho \overline{u_2'^2} & \text{(ii)} \end{cases}$$

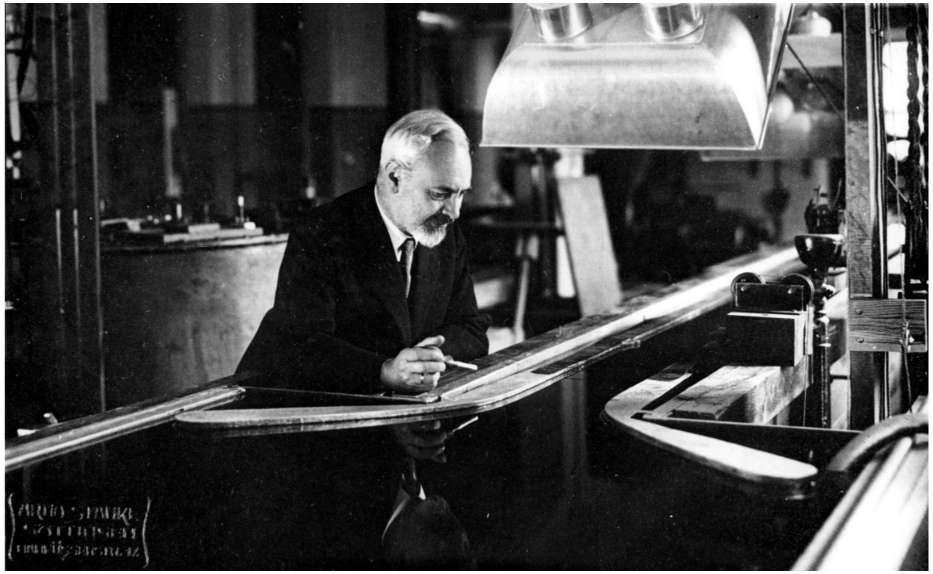
Compared with pipe and channel flows, there is a continuous growth of the boundary layer, and the flow is thus never homogeneous along the x_1 direction (but slowly variable). In addition, the mean pressure gradient is imposed by the external flow.

In what follows, a zero-pressure-gradient (ZPG) boundary layer is assumed,

$$\frac{dP_e}{dx_1} = 0 \qquad \text{(uniform external mean flow, } U_{e1} = \text{cst)}$$

Turbulent boundary layer equations : Ludwig Prandtl (1875-1953)





Ludwig Prandtl with his water tunnel in 1903
(for flow visualization of large structures
using particle tracers)

and in the mid to late 1930s

A voyage through Turbulence
edited by, P. A. Davidson, Y Kaneda, H.K. Moffatt & K.R. Sreenivasan
(Cambridge University Press, 2011)

Anderson Jr, D.J., 2005, *Physics Today*, **58**(12), 42–48.

Small exercise : unsteady free stream velocity

The following unsteady external velocity U_{e1} is imposed for a flow past a flat plate, $U_{e1} = u_\infty(1 - a\tilde{x}_1) + u_\infty a\tilde{x}_1 \sin(\omega t)$ where $0 \leq \tilde{x}_1 \leq 1$ is a normalized distance, and $a > 0$ a dimensionless control parameter.

1. Discuss briefly the expression of U_{e1}
2. Calculate the pressure gradient dp_e/dx_1 associated with the unsteady free stream, and its mean value $d\bar{P}_e/dx_1$ over one oscillation period
3. Examine the two cases $f = 0$ and $f \neq 0$

Zero-pressure-gradient boundary layer

For a boundary layer (as also for wake flows), a velocity defect $U_{e1} - \bar{U}_1$ is usually introduced : this quantity is bounded in $x_2 = 0$ and in $x_2 \to \infty$(δ in practice).
The rearrangement of the mass conservation equation leads to,

$$\frac{\partial}{\partial x_1}(\bar{U}_1 U_{e1}) + \frac{\partial}{\partial x_2}(\bar{U}_2 U_{e1}) = 0 \qquad \text{(iii)}$$

By integration in the transverse direction of the Navier-Stokes Eqs. (i) + (iii)

$$\int_0^\infty \frac{\partial}{\partial x_1}\bar{U}_1(\bar{U}_1 - U_{e1})dx_2 + \left[\bar{U}_2(\bar{U}_1 - U_{e1})\right]_0^\infty = \left[-\overline{u_1'u_2'} + \nu\frac{\partial \bar{U}_1}{\partial x_2}\right]_0^\infty$$

$$U_{e1}^2 \frac{\partial}{\partial x_1}\int_0^\infty \frac{\bar{U}_1}{U_{e1}}\left(\frac{\bar{U}_1}{U_{e1}} - 1\right)dx_2 + 0 = 0 - u_\tau^2$$

$$u_\tau^2 = U_{e1}^2 \frac{d\delta_\theta}{dx_1} \qquad \text{with} \qquad \delta_\theta \equiv \int_0^\infty \frac{\bar{U}_1}{U_{e1}}\left(1 - \frac{\bar{U}_1}{U_{e1}}\right)dx_2$$

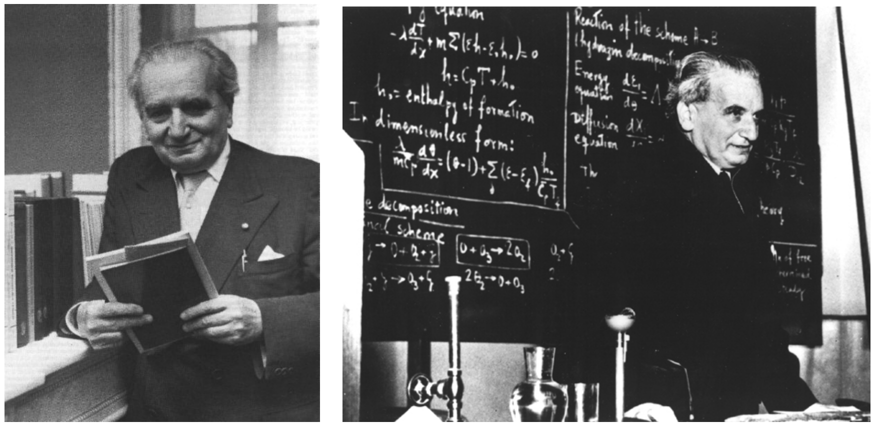
δ_θ is the momentum thickness of the boundary layer

Zero-pressure-gradient boundary layer

Friction velocity u_τ and local skin-friction coefficient C_f

$$u_\tau^2 = U_{e1}^2 \frac{d\delta_\theta}{dx_1} \qquad\qquad C_f \equiv \frac{\rho u_\tau^2}{\frac{1}{2}\rho U_{e1}^2} = 2\,\frac{d\delta_\theta}{dx_1}$$

The friction velocity u_τ is a function of x_1 (but slow variable) in a boundary layer (≠ established flow in pipe)

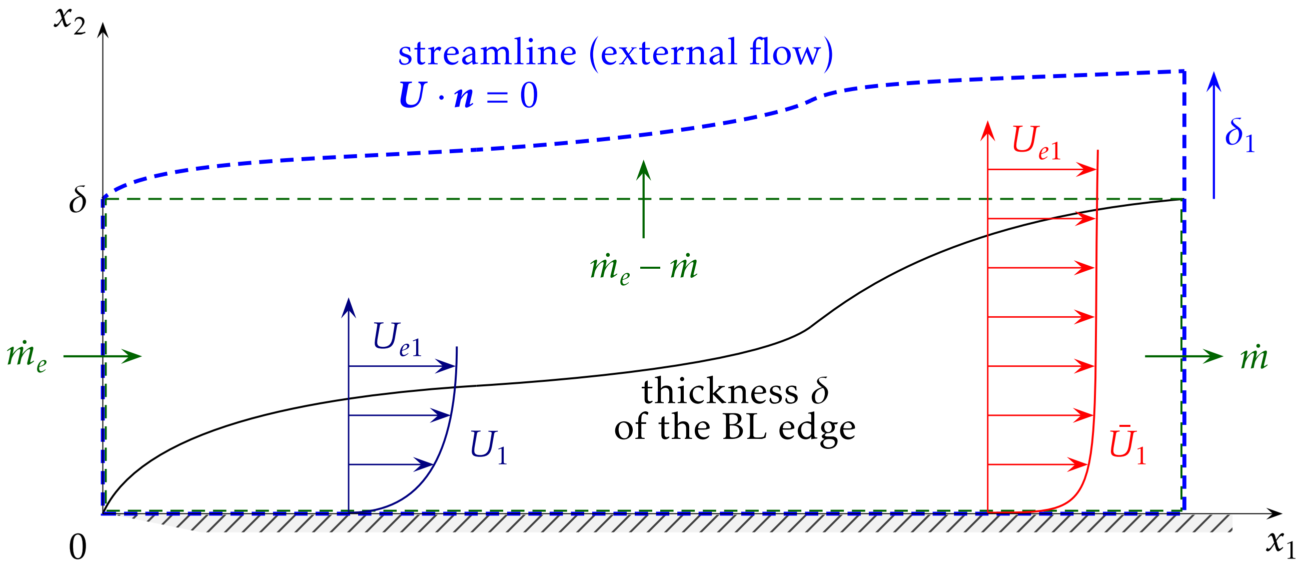


Theodore von Kármán (1881-1963)

General expression for the momentum-integral equation (Gruschwitz, 1931)

$$u_\tau^2 = \frac{d(U_{e1}^2\delta_\theta)}{dx_1} + \delta^\star \underbrace{U_{e1}\frac{dU_{e1}}{dx_1}}_{=-(1/\rho)\partial_{x_1}P_e}$$

Zero-pressure-gradient boundary layer : interpretation of δ_θ ?



Using the green control volume,

$$\dot{m}_e - \dot{m} = \int_0^\delta (\rho U_{e1} - \rho \bar{U}_1)\,dx_2 = \rho U_{e1}\delta_1$$

Using the blue control volume,

$$\rho U_{e1}^2\delta - \int_0^\delta \rho \bar{U}_1^2 dx_2 - \rho U_{e1}^2\delta_1 = \rho U_{e1}^2 \int_0^\delta \frac{\bar{U}_1}{U_{e1}}\left(1 - \frac{\bar{U}_1}{U_{e1}}\right)dx_2 = \rho U_{e1}^2 \delta_\theta$$

Integral momentum conservation (cst pressure)

$$\frac{d}{dt}\int_{\mathcal{V}} \rho\boldsymbol{u} = \boldsymbol{0} = \int_{\mathcal{V}} \rho\frac{D\boldsymbol{u}}{Dt} - \int_{\mathcal{S}} \rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n})\,ds = \int_{\mathcal{S}} \overline{\overline{\boldsymbol{\tau}}}\cdot\boldsymbol{n}\,ds - \int_{\mathcal{S}} \rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n})\,ds$$

Wall force acting on the wall, $\boldsymbol{F}_{f\to w} = \rho U_{e1}^2 \delta_\theta \boldsymbol{e}_1$

Mean velocity of a zero-pressure-gradient boundary layer

From the Navier-Stokes Eq. (i), by integration in the normal direction to the wall up to a given point x_2

$$\int_0^{x_2} \rho\left(\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2}\right) dx_2 = \bar{\tau}_t(x_2) - \bar{\tau}_w \qquad \bar{\tau}_t(x_2) \equiv -\rho \overline{u_1' u_2'} + \mu \frac{\partial \bar{U}_1}{\partial x_2} \tag{2}$$

Simplistic assumption : the left-hand side is approximated by a linear term as follows,

$$\int_0^{x_2} \rho\left(\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2}\right) dx_2 \simeq -\frac{x_2}{\delta}\tau_w \qquad \boxed{\bar{\tau}_t(x_2) \simeq \tau_w\left(1 - \frac{x_2}{\delta}\right)}$$

As a result, the mean velocity $\bar{U}_1$ is governed by the same equation than for the channel/pipe (by noting $\delta \equiv h$) except that the friction velocity is now a function of x_1, $u_\tau = u_\tau(x_1)$.

(refer to the exercises for further discussion)

Mean velocity profile : the viscous sublayer

Very close to the wall, $x_2/\delta \ll 1$, turbulence cannot develop and the viscous stress dominates the total stress $\bar{\tau}_t$ (at the wall $u_i' = 0$),

$$\bar{\tau}_t \simeq \mu \frac{\partial \bar{U}_1}{\partial x_2} \qquad \text{and} \qquad \bar{\tau}_w = \rho u_\tau^2 \qquad \text{in the viscous sublayer}$$

Consequently, a linear evolution of the mean velocity $\bar{U}_1$ is predicted, as in the case of the Couette flow,

$$\frac{\bar{U}_1}{u_\tau} = \frac{x_2 u_\tau}{\nu}$$

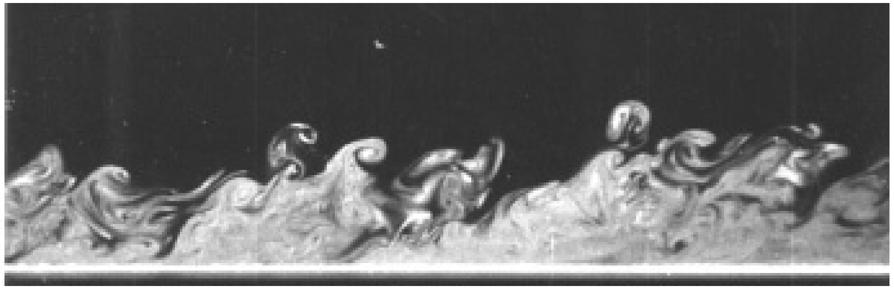
Introduction of wall units to form dimensionless variables

$$\bar{U}_1^+ \equiv \frac{\bar{U}_1}{u_\tau} \qquad x_2^+ \equiv \frac{x_2 u_\tau}{\nu} = \frac{x_2}{l_v} \qquad \text{with} \qquad l_v = \frac{\nu}{u_\tau} \equiv \text{wall unit length}$$

In the viscous sublayer, $\boxed{\bar{U}_1^+ = x_2^+}$

- **Mean velocity profile : the viscous sublayer (cont.)**

  - The viscous length scale l_v and the friction velocity u_τ are the two appropriate scales for describing flow in the near-wall region : inner scales of the boundary layer



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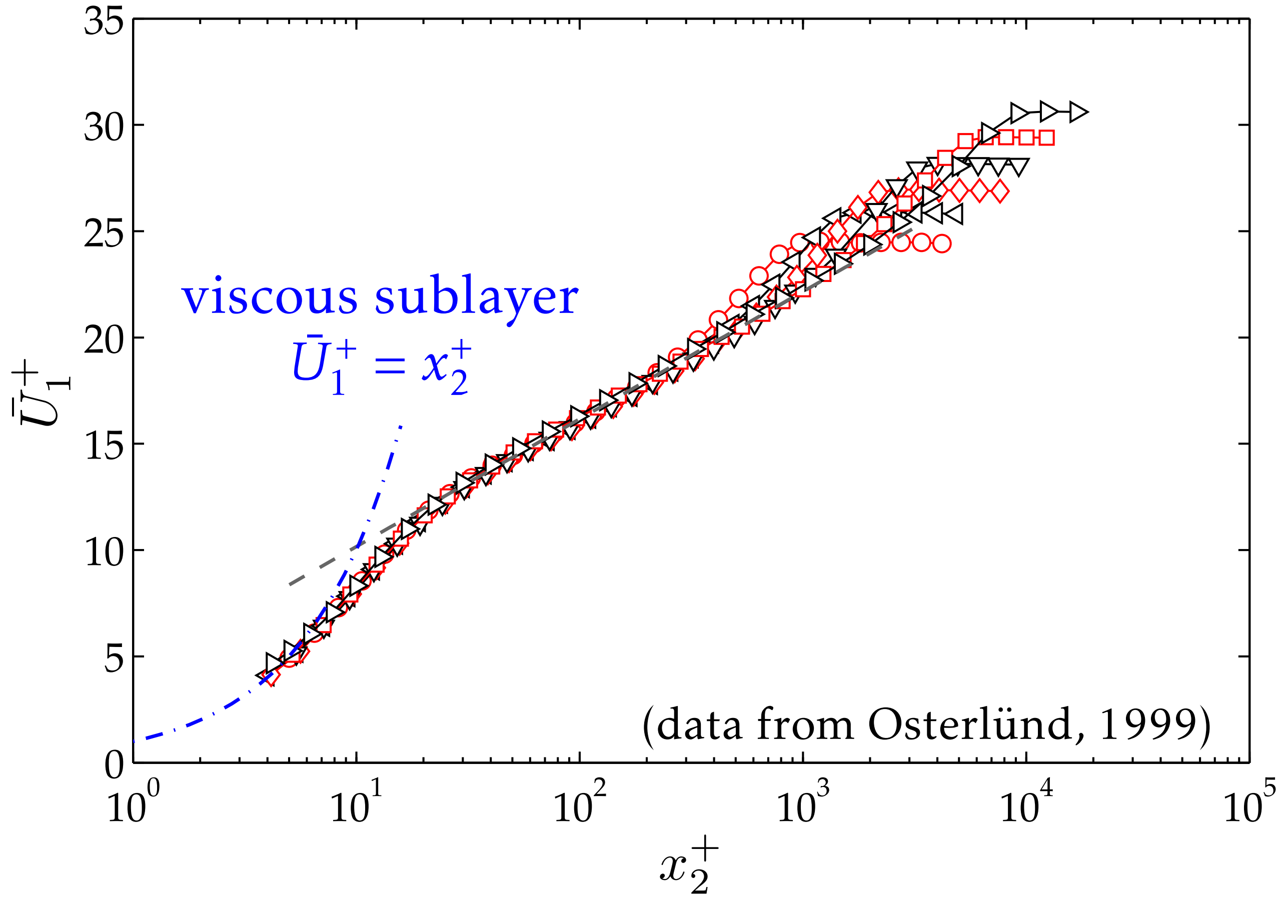
$$\mathrm{Re}_\theta \simeq 1000 \quad U_{e1} = 2.1\ \mathrm{m.s^{-1}}$$
$$\delta \simeq 7\ \mathrm{cm} \quad u_\tau \simeq 0.1\ \mathrm{m.s^{-1}}$$
$$x_1 \simeq 3\ \mathrm{m} \quad \text{(air flow)}$$

$$x_2^+ = \frac{u_\tau x_2}{\nu} = 1 \quad \Longrightarrow \quad x_2 = l_v = 0.15\ \mathrm{mm}$$

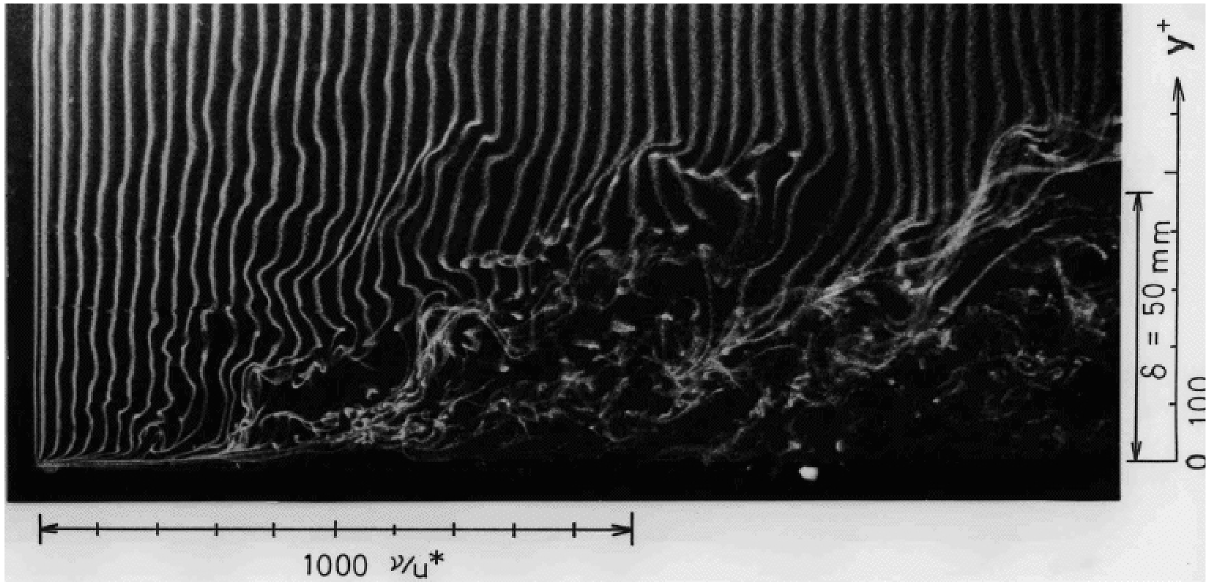
Mean velocity profile : the viscous sublayer (cont.)



$Re_{\delta_{0.95}}$	1.7×10^4	2.8×10^4	4.3×10^4	6.9×10^4	1.1×10^5	1.9×10^5
$Re^+_{\delta_{0.95}}$	684	1092	1594	2462	3944	6147
	○	◁	◊	▽	□	▷

Two illustrations of the disparity in scales

Turbulent boundary layer along a flat plate : particle tracing in water, hydrogen bubble method, $U_\infty = 20.4\ \text{cm.s}^{-1}, \text{Re}_{\delta_\theta} = 990$
from *Visualized flow*, Japan Soc. Mech. Eng. (1988)

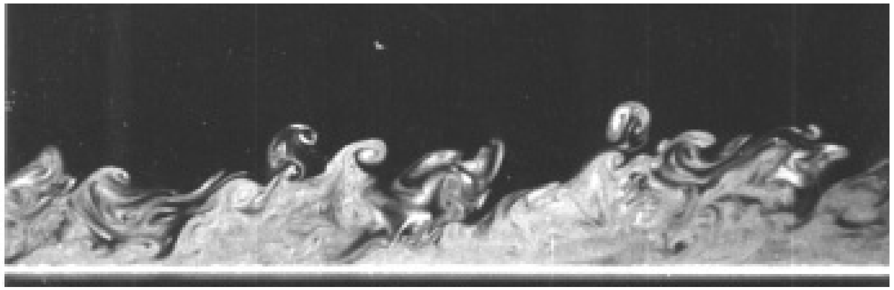


Spatially developing turbulent boundary layer on a flat plate
from Lee, Kwon, Hutchins & Monty (University of Melbourne)

Mean velocity profile : the logarithmic law

We have two characteristic length scales in a boundary layer : δ and $l_v = \nu/u_\tau$,

$$x_2^+ = \frac{x_2 u_\tau}{\nu} = \mathrm{Re}^+ \times \frac{x_2}{\delta} \qquad \boxed{\mathrm{Re}^+ \equiv \frac{u_\tau \delta}{\nu} = \delta^+} \quad \text{Karman number}$$



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$$\mathrm{Re}_\theta \simeq 1000 \quad U_{e1} = 2.1\ \mathrm{m.s}^{-1}$$
$$\delta \simeq 7\ \mathrm{cm} \quad u_\tau \simeq 0.1\ \mathrm{m.s}^{-1}$$
$$x_1 \simeq 3\ \mathrm{m} \quad \text{(air flow)}$$
$$l_v = 0.15\ \mathrm{mm}\ (x_2^+ = 1) \qquad \mathrm{Re}^+ \simeq 467$$

It is thus possible to satisfy $x_2^+ \gg 1$, $\mathrm{Re}^+ \gg 1$, but also $x_2/\delta \ll 1$

As an illustration, one has for this flow,

$$x_2^+ = 30 \qquad \frac{x_2}{\delta} = \frac{x_2^+}{\mathrm{Re}^+} \simeq 6 \times 10^{-2} \ll 1$$

- **Mean velocity profile : the logarithmic law (cont.)**

Dimensional analysis

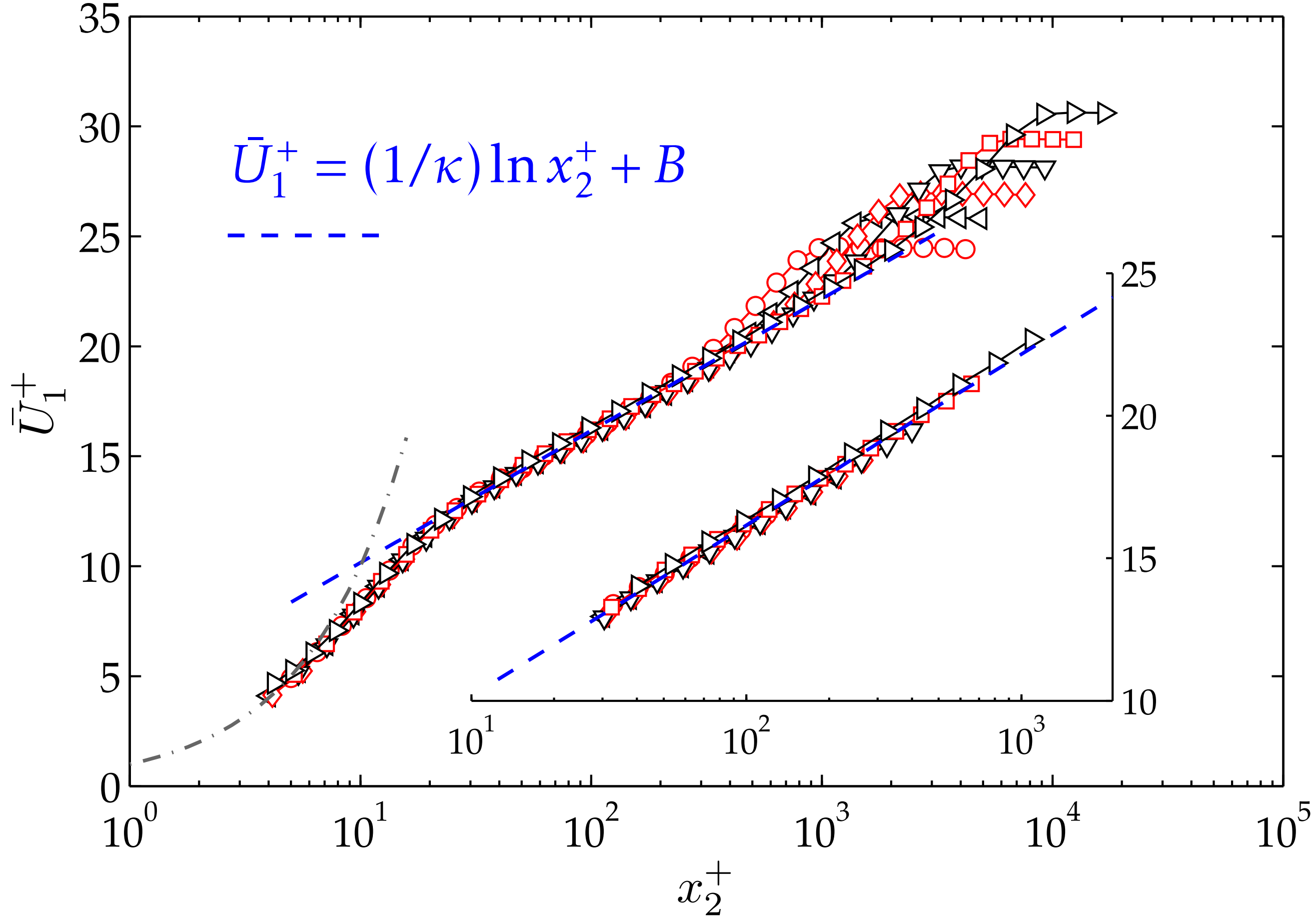
$$\frac{\bar{U}_1}{u_\tau} = f\left(\frac{x_2 u_\tau}{\nu}, \frac{x_2}{\delta}\right) \quad \Longrightarrow \quad \begin{cases} \dfrac{\bar{U}_1}{u_\tau} = f_1\left(\dfrac{u_\tau x_2}{\nu}\right) & \text{in the inner layer} \\ \dfrac{U_{e1} - \bar{U}_1}{u_\tau} = f_2\left(\dfrac{x_2}{\delta}\right) & \text{in the outer layer} \end{cases}$$

By imposing the continuity of the velocity $\bar{U}_1$ and of its derivative $\partial\bar{U}_1/\partial x_2$

$$\begin{cases} \dfrac{\bar{U}_1}{u_\tau} = \dfrac{1}{\kappa}\ln\left(\dfrac{u_\tau x_2}{\nu}\right) + B \\ \dfrac{U_{e1} - \bar{U}_1}{u_\tau} = -\dfrac{1}{\kappa}\ln\left(\dfrac{x_2}{\delta}\right) + A \end{cases} \quad \text{with} \quad \frac{U_{e1}}{u_\tau} = \ln(\mathrm{Re}^+) + A + B$$

where κ is the von Kármán constant : does not seem to be a universal constant, even for canonical flows ! $0.38 \leq \kappa \leq 0.41$

- **Mean velocity profile : the logarithmic law (inner scales)**



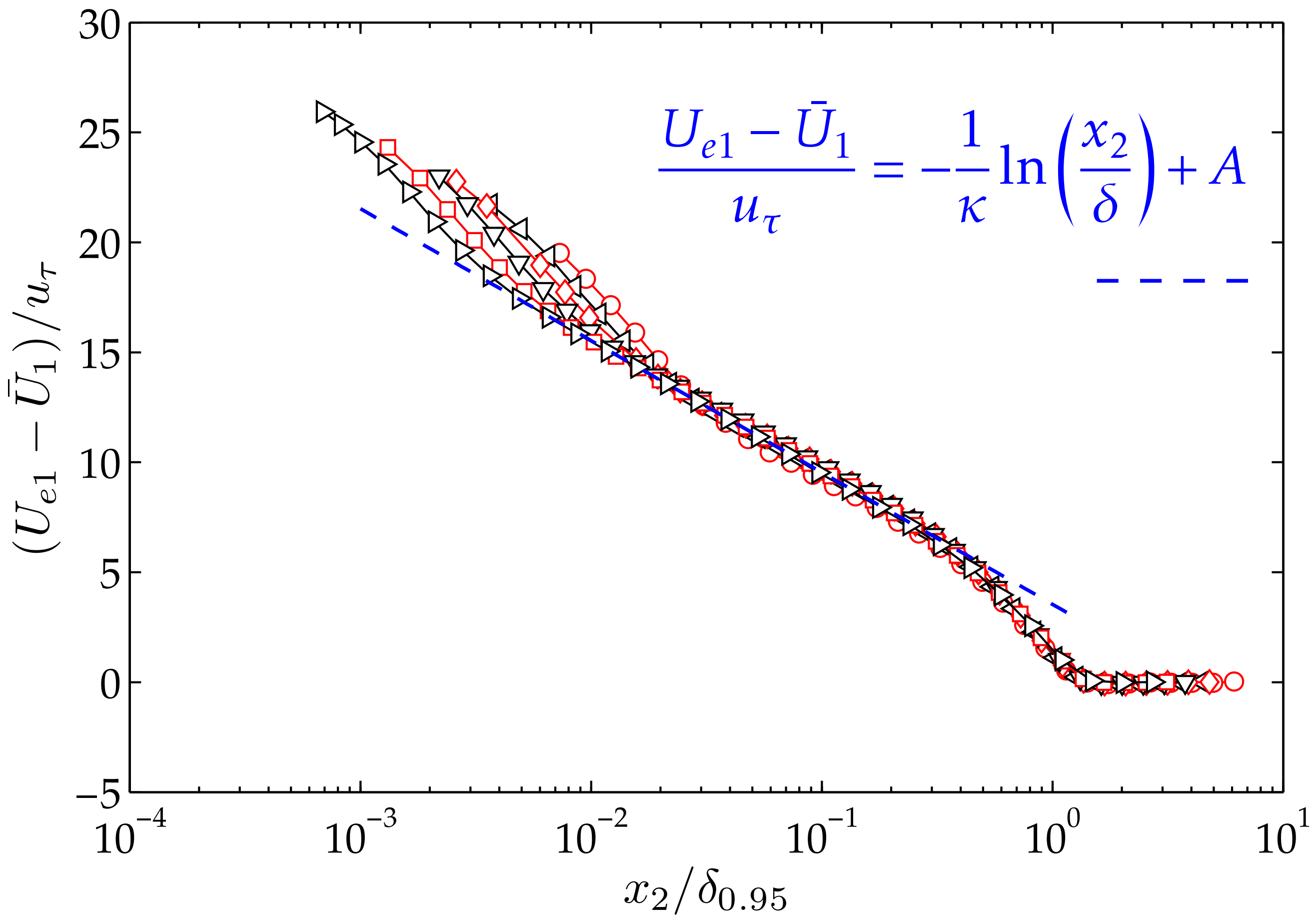
For a zero-pressure-gradient boundary layer,

$\kappa \simeq 0.384 \quad B \simeq 4.17$

log-law $\quad x_2^+ \geq 30$ & $x_2/\delta \leq 0.20$

(data from Osterlünd, 1999)

Mean velocity profile : the logarithmic law (outer scales, wake law)

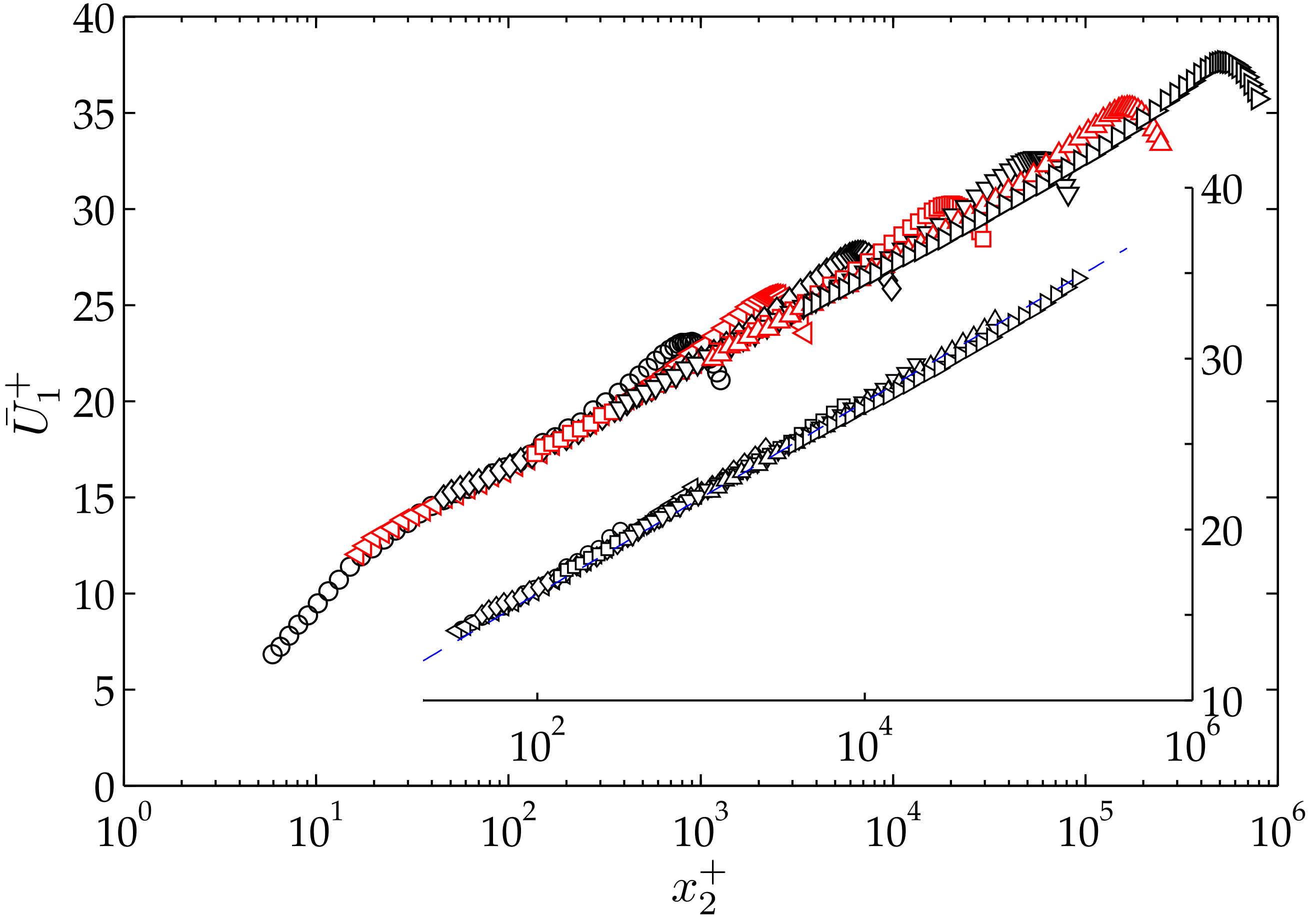


(data from Osterlünd, 1999)

For a zero-pressure-gradient boundary layer,

$\kappa \simeq 0.384 \quad A \simeq 3.54$

- **Mean velocity profiles in a turbulent pipe flow**
  Zagarola & Smits (1998, Princeton *Superpipe* facility)

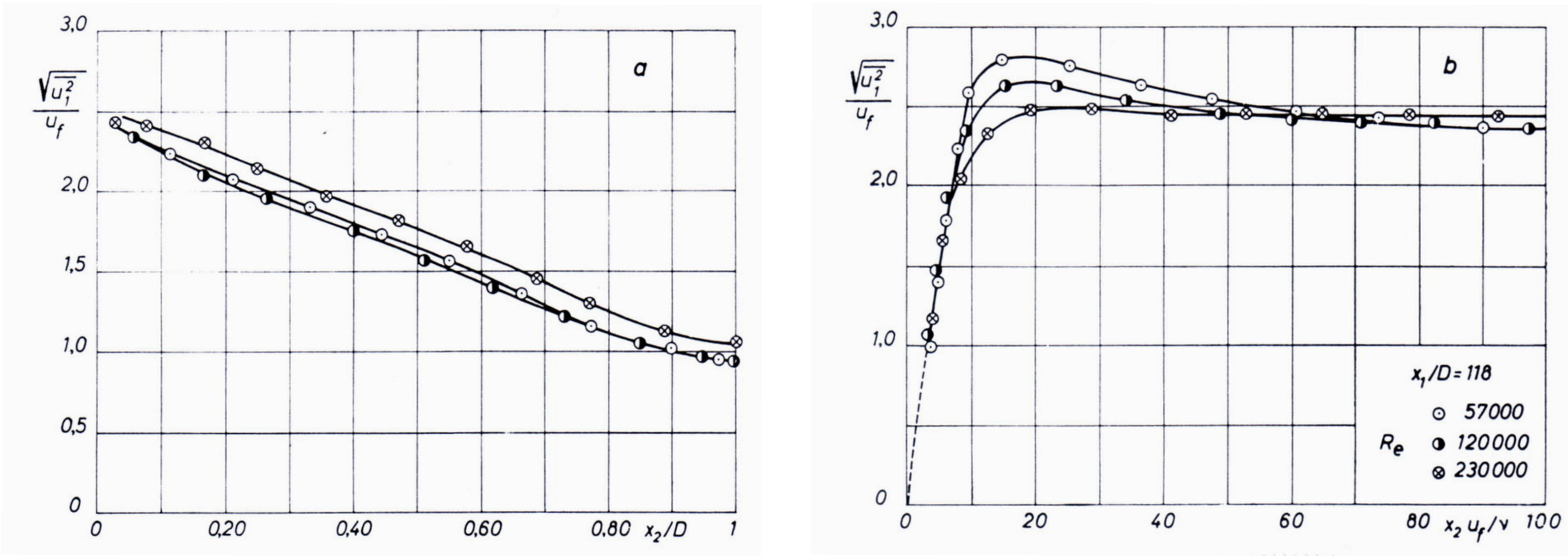


$\mathrm{Re}_D = 3.2 \times 10^3$ ○
9.9×10^4 ◁
3.1×10^5 ◇
1.0×10^6 □
3.1×10^6 ▽
1.0×10^7 △
3.5×10^7 ▷

log-law
$\kappa \simeq 0.41 \quad B \simeq 5.0$

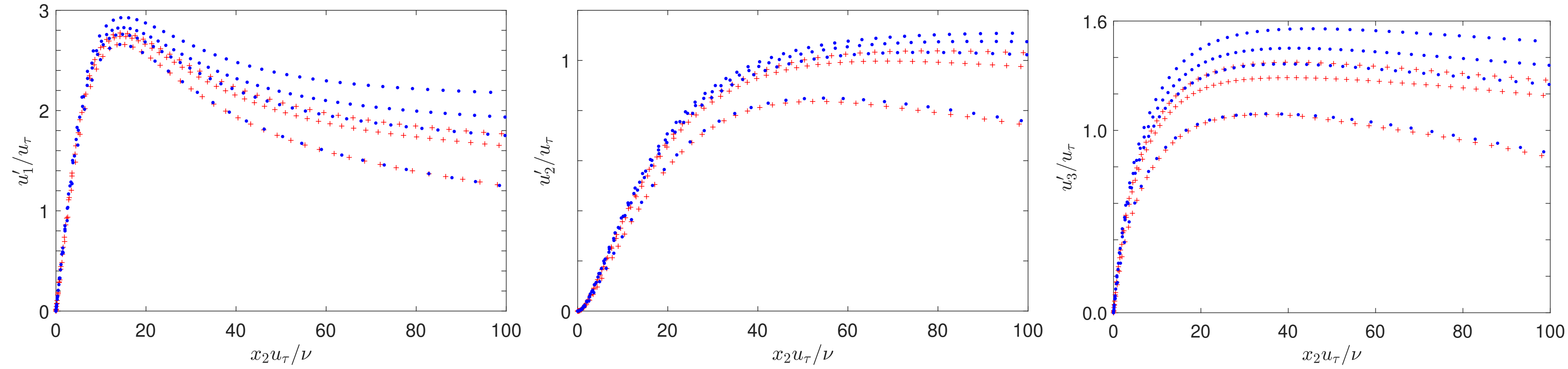
Fully developed channel flow : experiments

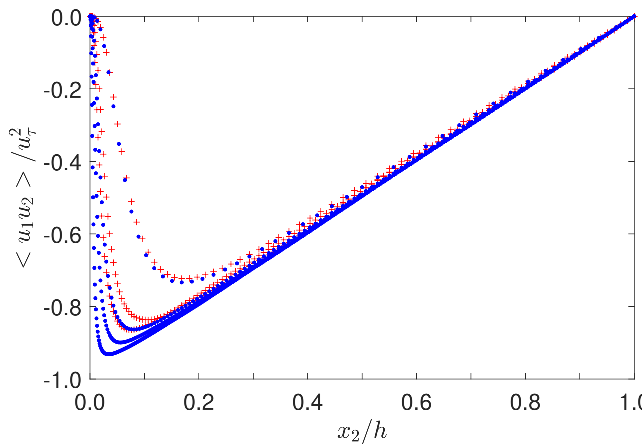
Comte-Bellot, G. (1965)



Plane channel of width $2h \equiv 2D$ $\quad 5.7 \times 10^4 \leq \mathrm{Re}_h \leq 2.3 \times 10^5$

Fully developed channel flow : Direct Numerical Simulation (DNS)





Moser, Kim & Mansour (1999)

\+ $Re^+ = 180$, $Re^+ = 395$, $Re^+ = 590$

Hoyas & Jiménez (2006)

• $Re^+ = 180$, $Re^+ = 550$, $Re^+ = 950$, $Re^+ = 2000$

...

Balance between production and dissipation in the log-law

For an observer located in the log-law region of a boundary layer, an almost perfect balance is found between **production and dissipation** of the turbulent kinetic energy k_t, that is

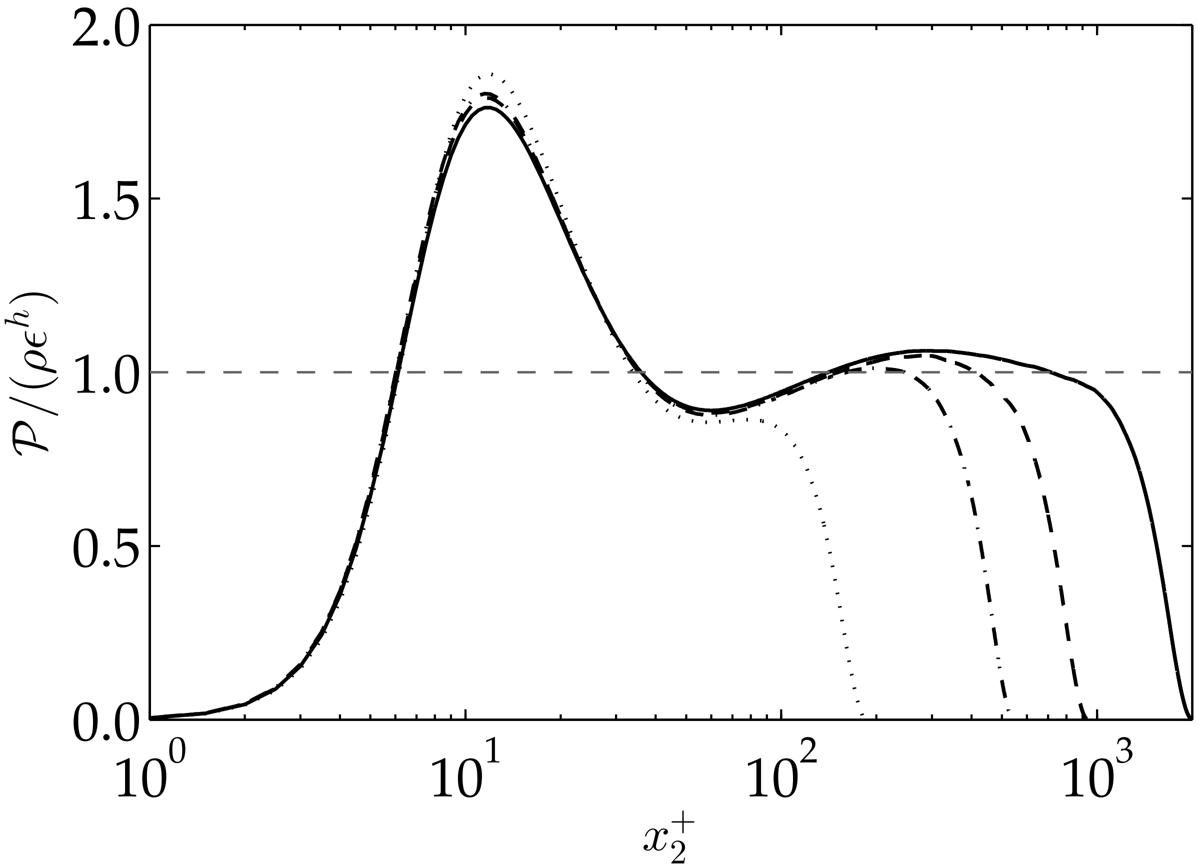
$$\boxed{\mathcal{P} \equiv -\rho \overline{u_1' u_2'} \frac{d\bar{U}_1}{dx_2} \quad \simeq \quad \rho\epsilon} \qquad \text{inside the log-law}$$

This result is the starting point of various developments for turbulence models, even if there is no formal demonstration.

Turbulent kinetic energy budget in a channel flow

Ratio of $\mathcal{P}/(\rho\epsilon^h)$ for $\mathrm{Re}^+ = 180, 550, 950, 2000$ (DNS by Hoyas & Jiménez, 2006)

We can observe the increase of the equilibrium region with the increase of the Reynolds number



A first example of turbulence model : mixing length model

We first investigate the near wall region, in assuming that $x_2/\delta \ll 1$, to derive the mixing length model by Prandtl (1925), and the governing equation for the mean velocity $\bar{U}_1$ (also valid for a channel flow with $h = \delta$)

$$\bar{\tau}_t(x_2) = -\rho\overline{u_1'u_2'} + \mu\frac{d\bar{U}_1}{dx_2} \simeq \bar{\tau}_w \qquad \text{or also} \qquad -\overline{u_1'u_2'}^{+} + \frac{d\bar{U}_1^+}{dx_2^+} \simeq 1$$

The turbulent viscosity concept (introduced in Chapter 2) leads to

$$-\overline{u_1'u_2'}^{+} = -\frac{\overline{u_1'u_2'}}{u_\tau^2} = \nu_t^+\frac{d\bar{U}_1^+}{dx_2^+} \qquad \nu_t^+ \equiv \frac{\nu_t}{\nu}$$

where the turbulent viscosity is dimensionally the product of a velocity scale u' by a length scale l_m, that is $\nu_t \sim u' \times l_m$.

(by analogy with the molecular motion for a perfect gas : ν is roughly the product of the speed of sound by the free mean path)

Mixing length model (cont.)

In an algebraic model (*aka* a zero-equation model), the evolution of the mixing length l_m is imposed by the user. For a boundary layer, a linear evolution in the normal direction to the wall is assumed, that is $l_m^+ = \alpha x_2^+$

The velocity scale u' is then obtained by assuming that the frequency of the mean flow is imposed to the turbulent motion (through the production term). This frequency matching leads to

$$\frac{u'^+}{l_m^+} = \frac{d\bar{U}_1^+}{dx_2^+}$$

As a result, the turbulent viscosity and the Reynolds stress component are given in wall unit by

$$\nu_t^+ = (l_m^+)^2 \left| \frac{d\bar{U}_1^+}{dx_2^+} \right| \qquad \text{and} \qquad -\overline{u_1' u_2'}^+ = \underbrace{(l_m^+)^2 \left| \frac{d\bar{U}_1^+}{dx_2^+} \right|}_{\nu_t^+} \frac{d\bar{U}_1^+}{dx_2^+}$$

Mixing length model (cont.)

The governing equation for the mean velocity can thus be recast as follows with our assumptions

$$(\alpha x_2^+)^2 \left(\frac{d\bar{U}_1^+}{dx_2^+}\right)^2 + \frac{d\bar{U}_1^+}{dx_2^+} - 1 = 0$$

The mean velocity gradient $d\bar{U}_1^+/dx_2^+$ satisfies a quadratic equation. The relevant solution is given by

$$\frac{d\bar{U}_1^+}{dx_2^+} = \frac{-1 + \sqrt{1 + 4\,(\alpha x_2^+)^2}}{2\,(\alpha x_2^+)^2} \geq 0$$

For $x_2^+ \to 0, \quad \dfrac{d\bar{U}_1^+}{dx_2^+} \to 1$

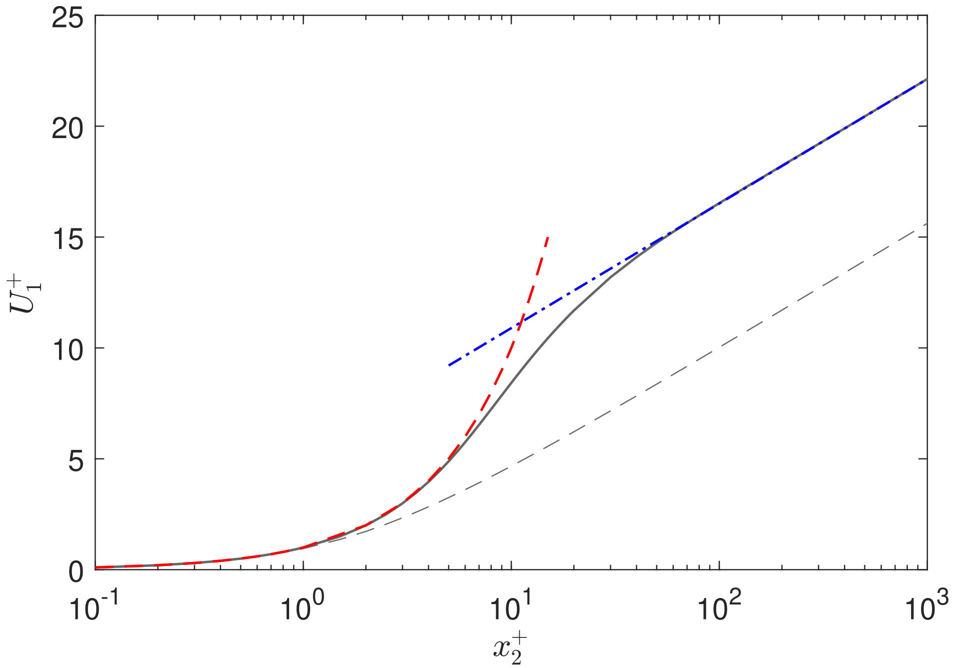
One finds $\bar{U}_1^+ = x_2^+$, that is the velocity law expected in the viscous sublayer

For $x_2^+ \to \infty, \quad \dfrac{d\bar{U}_1^+}{dx_2^+} \to \dfrac{1}{\alpha x_2^+}$

A log-law is found for $\bar{U}_1^+$, and by identification, $\alpha = \kappa$ $(x_2/\delta \ll 1)$

Mixing length model (cont.)

However, the previous model has one flaw, and thus requires a correction proposed by Van Driest (see next small classe)



- - - $l_m^+ = \kappa x_2^+$

—— $l_m^+ = \kappa x_2^+ (1 - e^{-x_2^+/A_0^+})$ $\quad A_0^+ = 26$

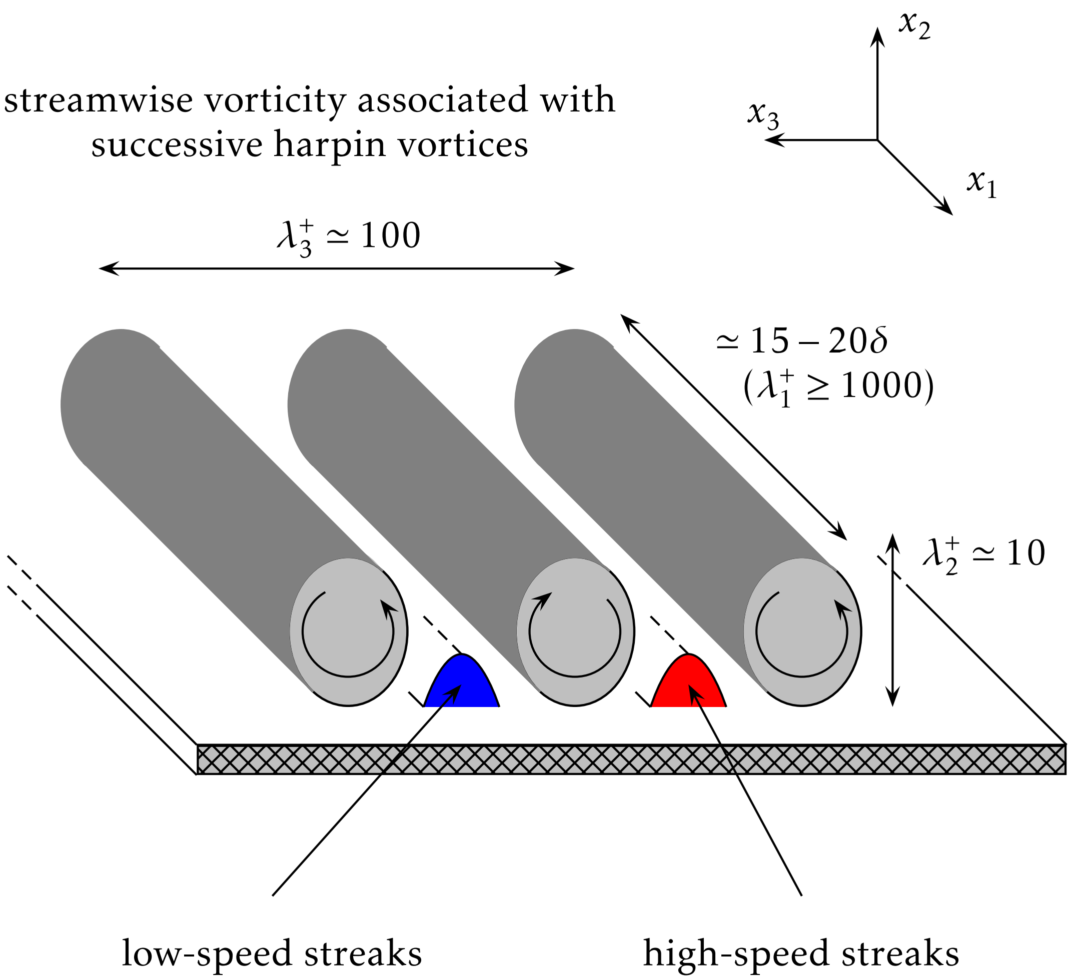
- - - $U_1^+ = x_2^+$ viscous sublayer

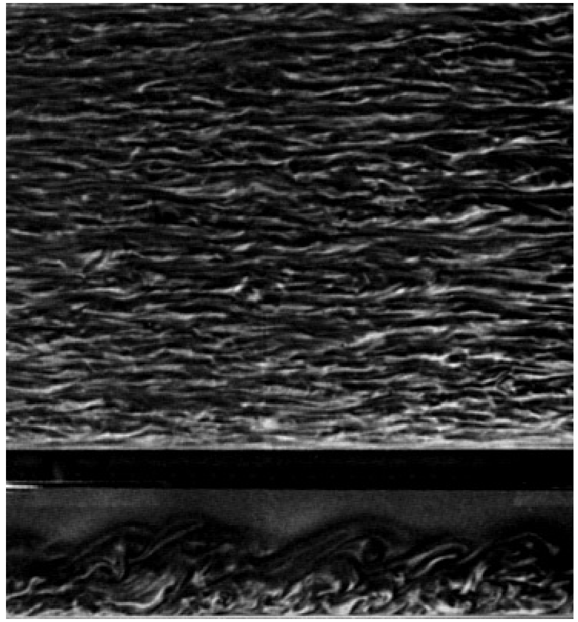
-·-·- $U_1^+ = (1/\kappa)\ln(x_2^+) + B$

Using Van Driest damping function, $-\overline{u_1' u_2'}^+ \sim {x_2^+}^4$ as $x_2^+ \to 0$, rather than $-\overline{u_1' u_2'}^+ \sim {x_2^+}^2$ using $l_m^+ = \kappa x_2^+$.

However,from the governing equations, it can be shown that $-\overline{u_1' u_2'}^+ \sim {x_2^+}^3$!

The buffer layer : streaks and harpin (horseshoe) vortices

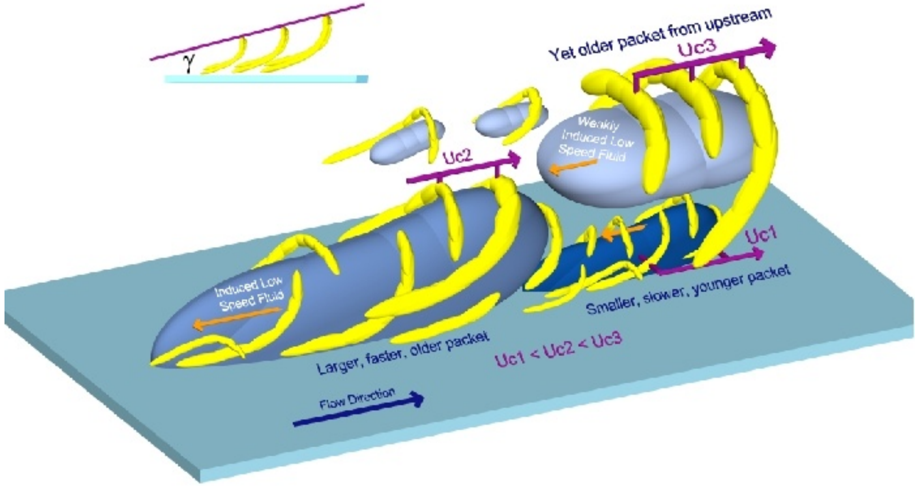


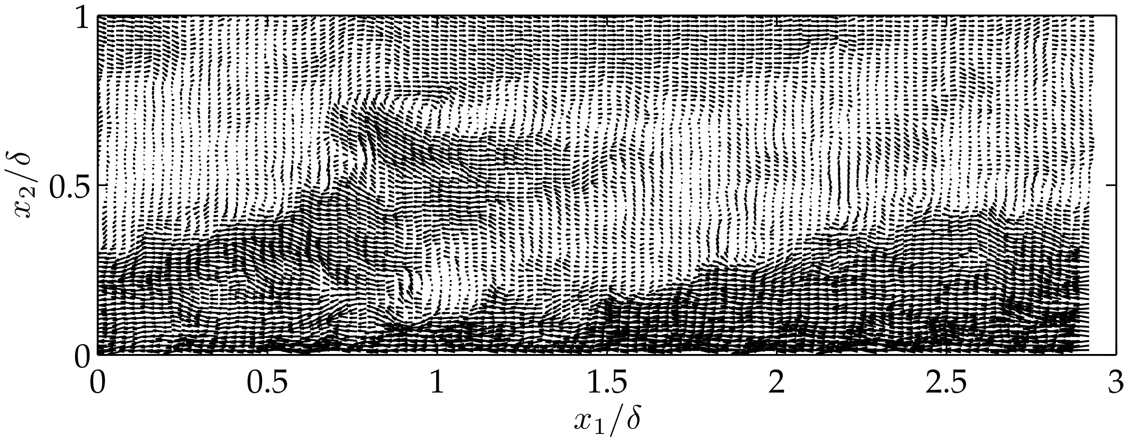


Cantwell, Coles & Dimotakis (1978)

Visualization of sublayer streaks from a suspension of aluminium particules (water, $U_\infty = 15$ cm.s^{-1})

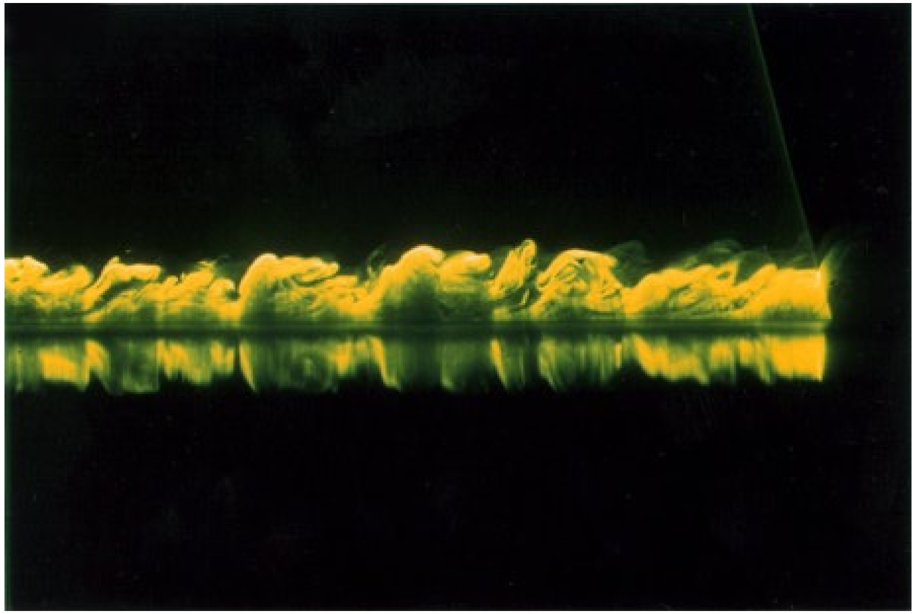
- **The buffer layer : streaks and harpin (horseshoe) vortices**
  Conceptual view from Adrian, Meinhart & Tomkins (2000)





PIV measurements
$Re_{\delta_\theta} = 7705$
plot of $\boldsymbol{u} - 0.87\boldsymbol{U}_{e1}$

The buffer layer : streaks and harpin (horseshoe) vortices

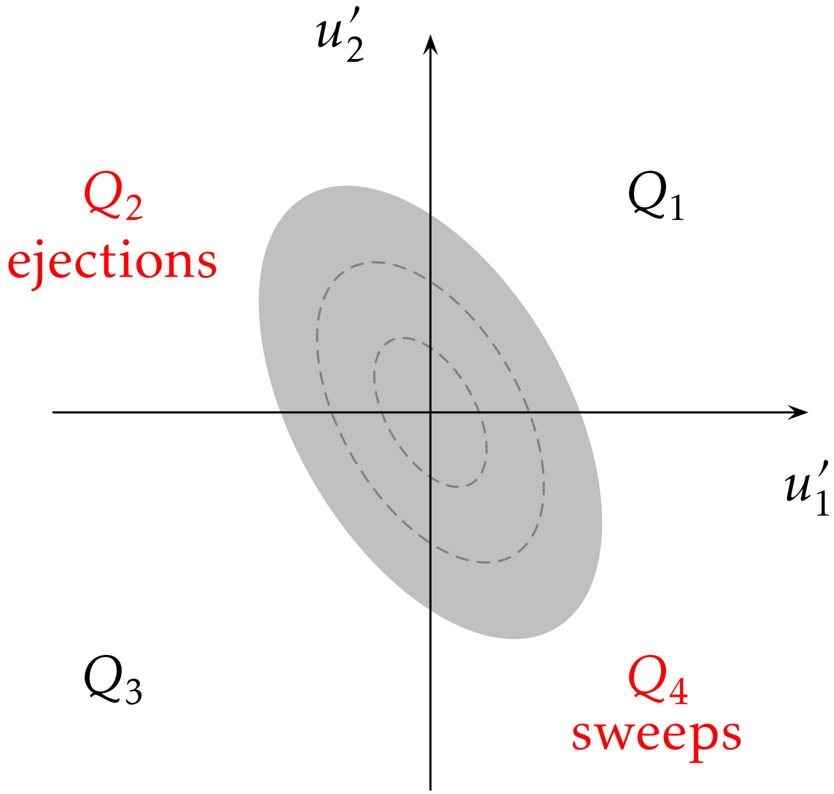


Side view of large eddies in a turbulent boundary layer
by laser-induced fluorescence

Gad-el-Hak, University of Notre-Dame, USA

http://www.efluids.com

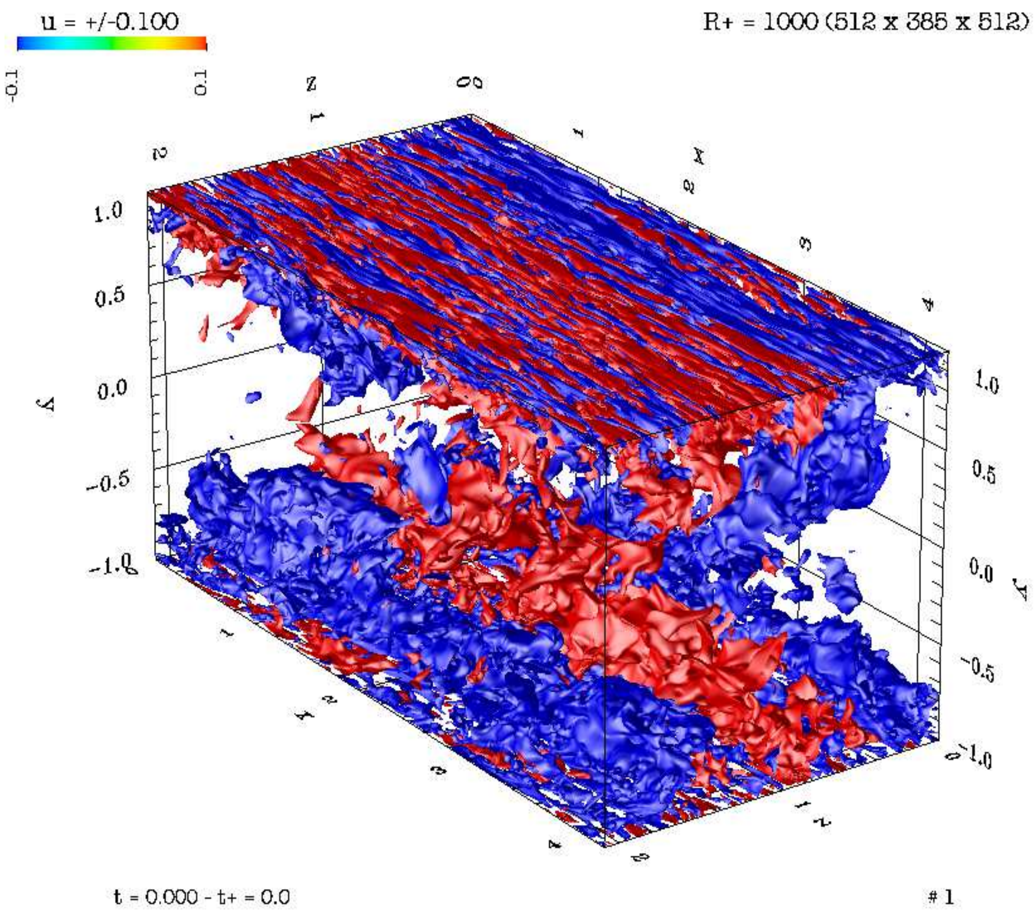
The buffer layer



- Drag generating events fall in the second and fourth quadrant, positive turbulent production

$$\mathcal{P} \simeq -\rho \overline{u_1' u_2'} \frac{\partial \bar{U}_1}{\partial x_2}$$

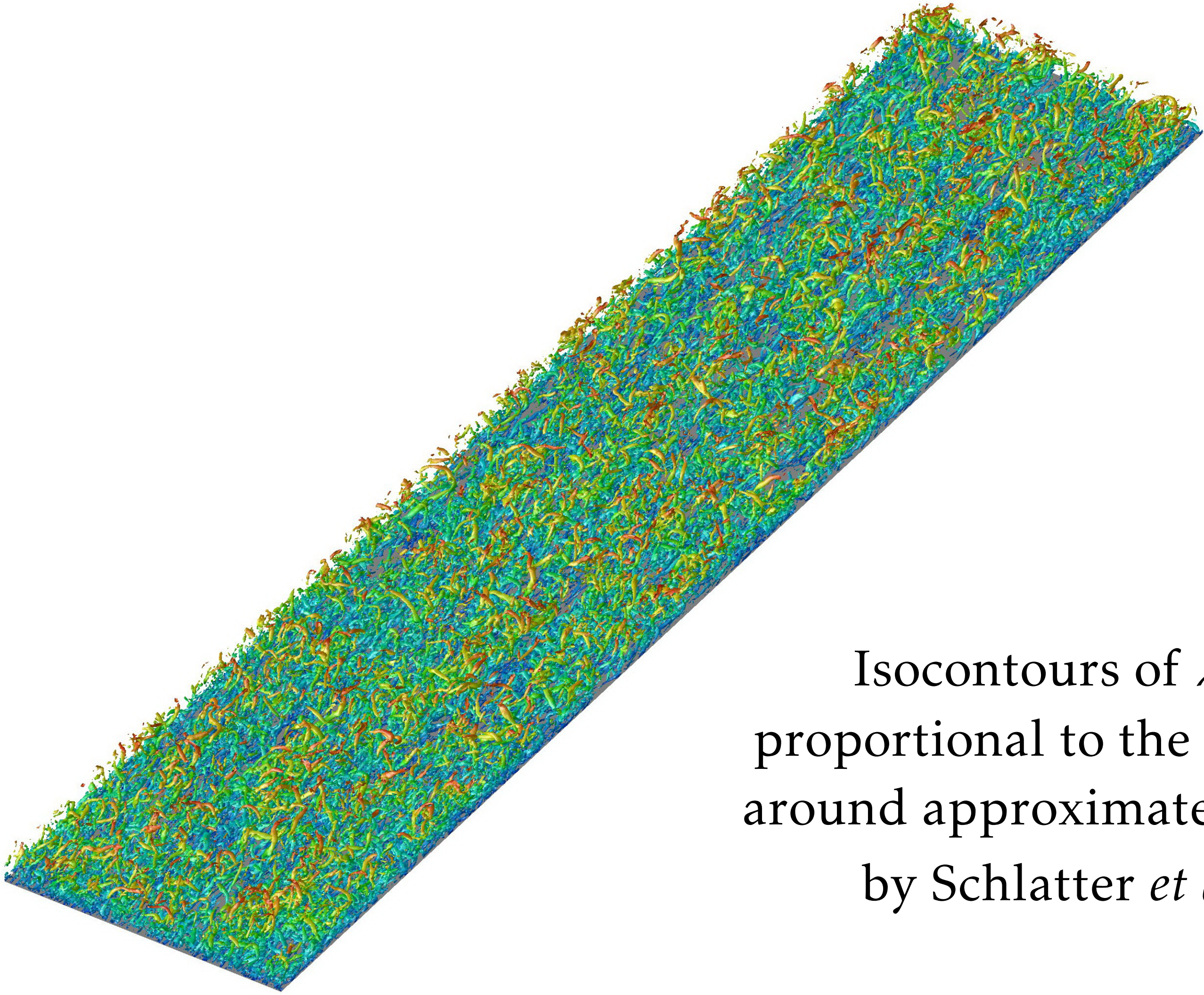
DNS of a plane channel flow



Iso-surfaces of the streamwise fluctating velocity
(red $u'/U_c = 0.12$, blue $u'/U_c = -0.12$)

- Laadhari, *Phys. Fluids* (2007)
  $n_{\text{dof}} = 512 \times 385 \times 512 \simeq 101 \times 10^6$
  $\text{Re}_h = 20100$, $\text{Re}^+ = 1000$
  $(\text{Re}^+)^{3/4}/n_y \simeq 0.46$
  cost$\sim \text{Re}^{+3} \sim 10^9$
  IBM SP4 / CINES

- **DNS of a boundary layer over a flate plate**



Isocontours of λ_2 (colour proportional to the wall distance) around approximately $Re_{\delta_\theta} = 1400$ by Schlatter *et al.* (2009)

http://www.mech.kth.se/~pschlatt/DATA/

Turbulent boundary layer with pressure gradient

Dimensionless parameter β

$$\beta = \frac{\delta_1}{\rho u_\tau^2}\frac{dP_e}{dx_1} = -\frac{\delta_c}{u_\tau}\frac{dU_{e1}}{dx_1} \sim \frac{\tau_{\mathrm{bl}}}{\tau_e} \qquad \delta_c = \int_0^\infty \frac{U_{e1}-U_1}{u_\tau}dx_2 = \delta_1 \frac{U_{e1}}{u_\tau}$$

- time scale of the boundary layer $\tau_{\mathrm{bl}} \sim \delta_c/u_\tau$
- time scale of the external flow $\tau_e \sim (dU_{e1}/dx_1)^{-1}$

Coles (1956)

$$f_2 = A\cos^2\left(\frac{\pi x_2}{2\delta}\right) - \frac{1}{\kappa}\ln\left(\frac{x_2}{\delta}\right) \qquad \frac{U_{e1}}{u_\tau} = \frac{1}{\kappa}\ln(\mathrm{Re}^+) + A + B$$

$A \simeq 2.5$ zero pressure gradient
$A < 2.5$ favorable gradient
$A > 2.5$ adverse gradient

Small exercise : key scales for the log-law of a boundary layer

1. Determine the general expression of the Kolmogorov length scale l_η by considering the dissipation ϵ and the Reynolds number for viscous scales.

2. Show that in the logarithmic region of the mean velocity profile of a turbulent boundary layer, the Kolmogorov scale is approximated by the expression
$l_\eta^+ \simeq (\kappa x_2^+)1/4$

3. Recall also the expression of the mixing length l_m^+ and of the turbulent viscosity ν_t^+ ?