*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

Physics of turbulent flow

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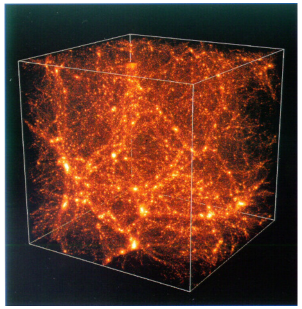
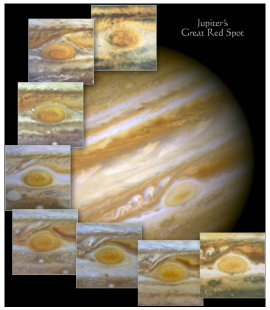
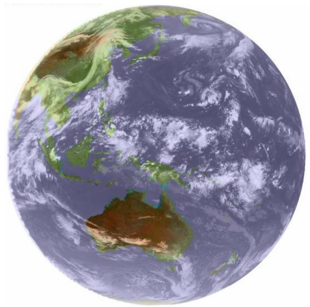
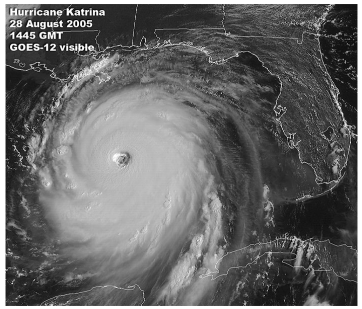
Centrale Lyon – MOD 3rdYear & MSc. – version 16-11-2024

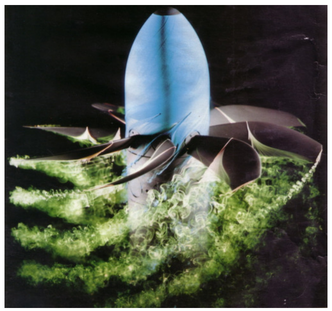
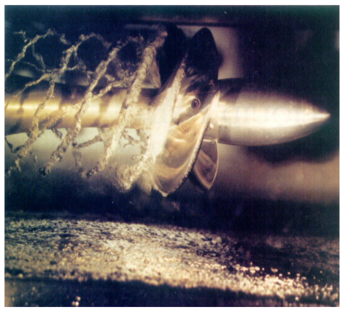
Outline

Dynamics of vorticity






Vorticity vector ω

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \qquad \omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \qquad \underbrace{\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)}_{\text{permutation tensor}}$$

The vorticity is always assumed to be a concentrated (localized) quantity in space, vortex tube or sheet.

The Biot & Savart law allows to express the velocity field induced by a given vorticity distribution.

- For an incompressible velocity field, $\nabla \cdot \boldsymbol{u} = 0$. A vector potential defined by $\boldsymbol{u} = \nabla \times \boldsymbol{A}$ can thus be introduced, associated with the condition $\nabla \cdot \boldsymbol{A} = 0$ (uniqueness)
- This vector potential $\boldsymbol{A}$ satisfies a Poisson equation whose source term is the vorticity vector,

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = -\nabla^2 \boldsymbol{A}$$

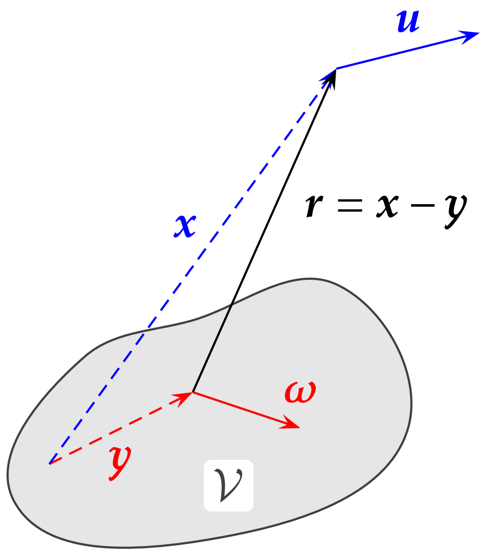
Biot & Savart's law (1820)

From the knowledge of the free-space Green's function, the integral solution is given by

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\boldsymbol{y}$$

The velocity field is then obtained by taking the curl of $\boldsymbol{A}$

$$\boldsymbol{u}(\boldsymbol{x}) = \nabla_x \times \boldsymbol{A} = \frac{1}{4\pi}\nabla_x \times \int_{\mathcal{V}} \frac{\boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\boldsymbol{y} = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{\omega}(\boldsymbol{y})\times(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\,d\boldsymbol{y}$$



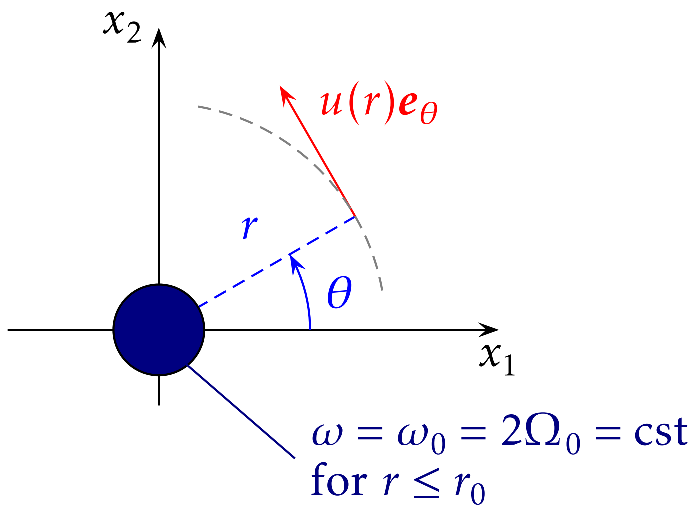
$$\boxed{\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\boldsymbol{\omega}(\boldsymbol{y})\times\boldsymbol{r}}{r^3}\,d\boldsymbol{y}}$$

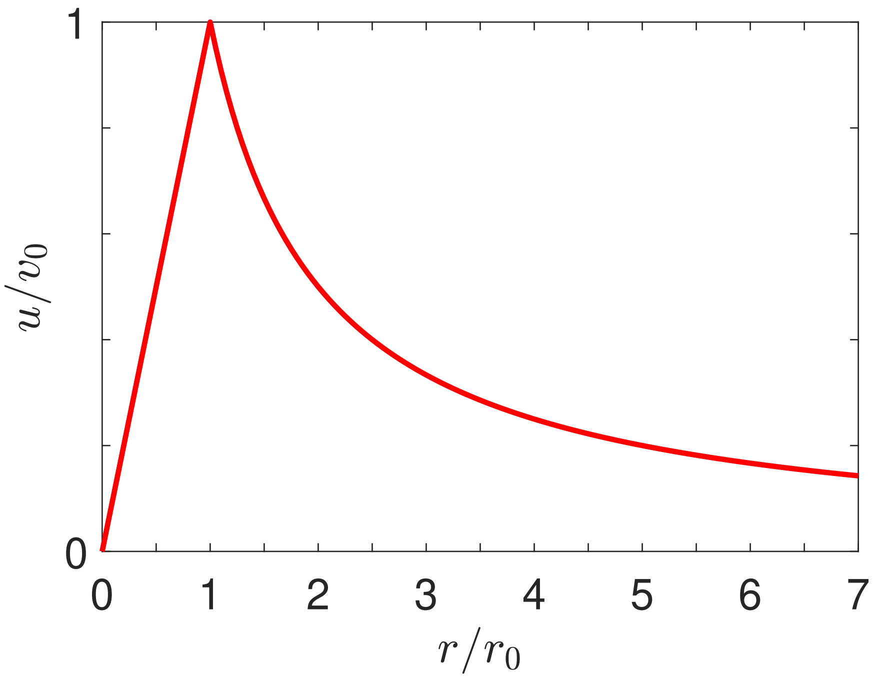
► Nonlocal relationship between the vorticity field $\boldsymbol{\omega}$ and the velocity field $\boldsymbol{u}$

Example of the Rankine vortex (1858) — Rankine (1820-1872)

$$\begin{cases} u(r) = v_0 \dfrac{r}{r_0} = \Omega_0 r & r \leq r_0 \\ u(r) = v_0 \dfrac{r_0}{r} = \Omega_0 r_0 \dfrac{r_0}{r} & r > r_0 \end{cases}$$

$$(v_0 = \Omega_0 r_0 = \omega_0 r_0 / 2)$$



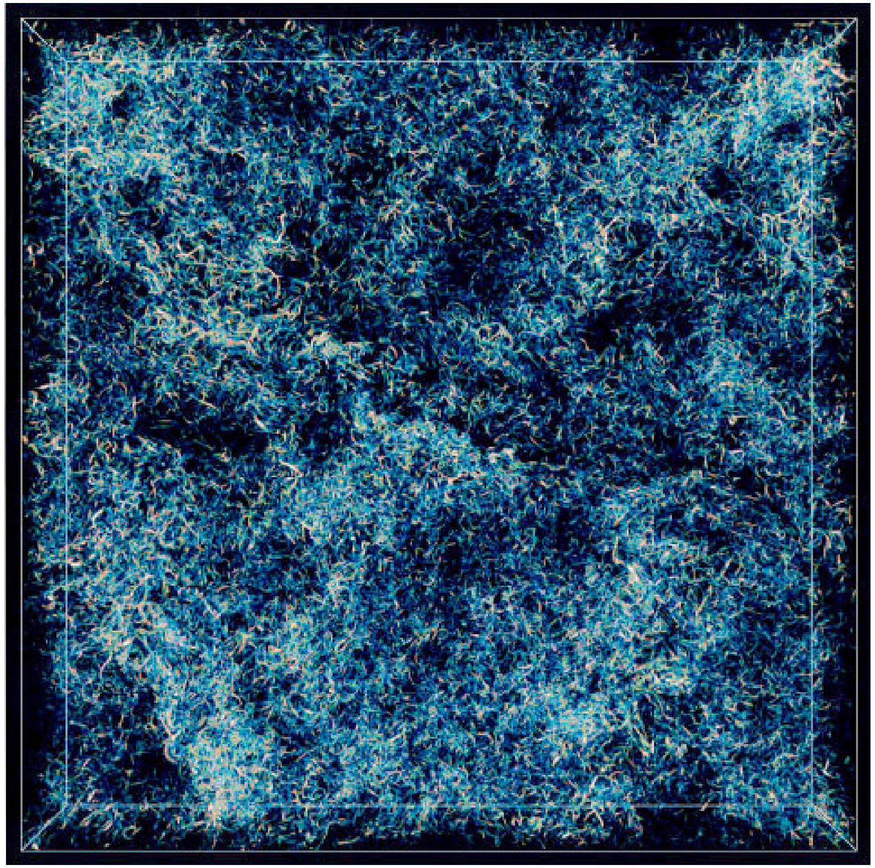


Solid body motion inside the vortex itself, *i.e.* for $r \leq r_0$ in the vortical region

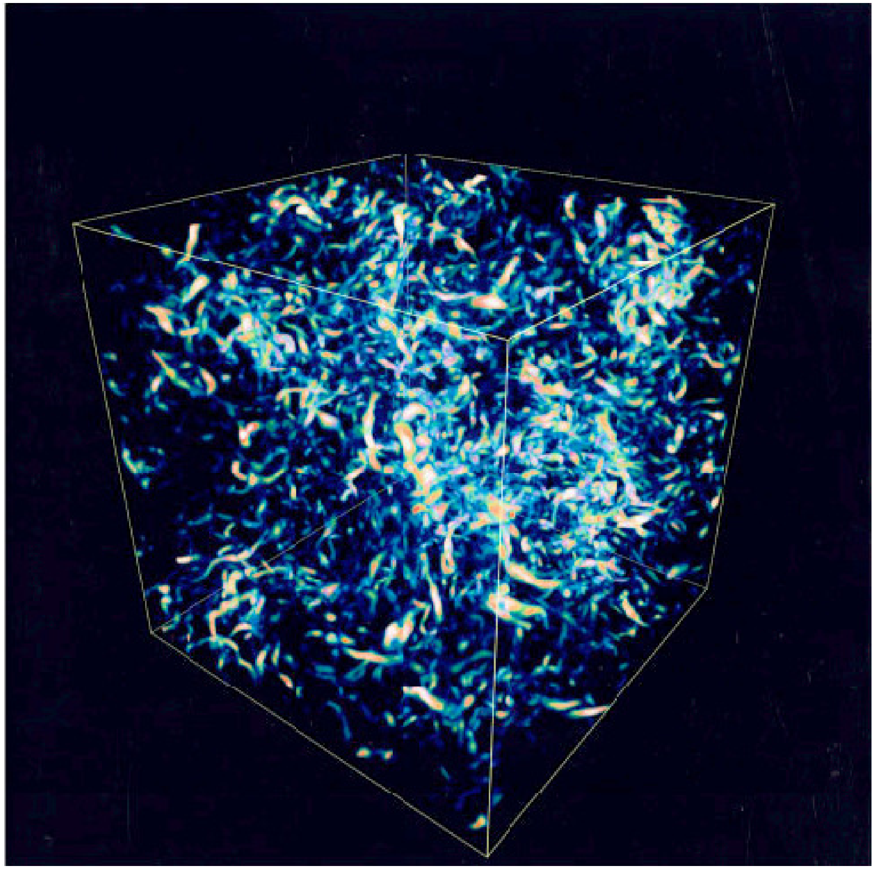
Irrotational flow outside, for $r > r_0$: the localized circular patch of vorticity produces a velocity field away from the vortical region

Vorticity distribution in a turbulence box

in a slab of $1024^2 \times 128$



in the inertial range



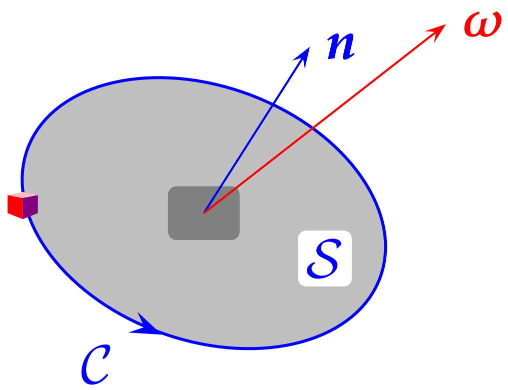
Porter, Woodward & Pouquet, *Phys. Fluids*, 1998

Kelvin's circulation theorem (1869)

For an inviscid flow submitted to conservative body forces, the circulation around a material closed curve $\mathcal{C}$ is governed by

$$\frac{D\Gamma_{\mathcal{C}}}{Dt} = \frac{D}{Dt}\oint \boldsymbol{U}\cdot d\boldsymbol{l} = \int_{\mathcal{S}} \frac{1}{\rho^2}\nabla\rho\times\nabla p\cdot\boldsymbol{n}\,ds = 0 \qquad \text{for barotropic flows, } \rho=\rho(p)$$

Note that constant density, isothermal, and isentropic flows are barotropic. As a result, the material circulation $\Gamma_{\mathcal{C}}$ is preserved,



$$\frac{D\Gamma_{\mathcal{C}}}{Dt} = 0$$

$$\Gamma_c = \oint_{\mathcal{C}} \boldsymbol{U}\cdot d\boldsymbol{l} = \int_{\mathcal{S}} \boldsymbol{\omega}\cdot\boldsymbol{n}\,ds = \text{cst}$$

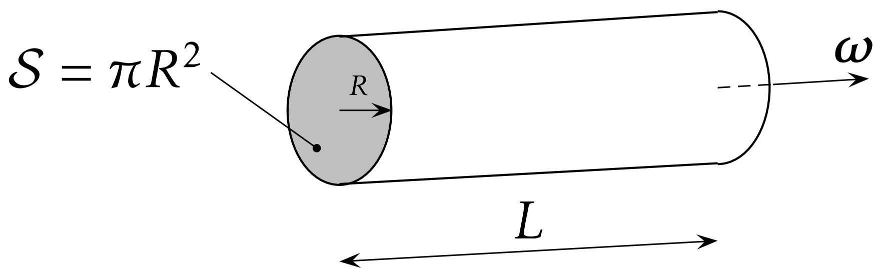
Introduction to vortex stretching

A consequence of Kelvin's circulation theorem

$$\frac{d\Gamma}{dt} = \frac{d}{dt}\left[\oint_{\mathcal{C}} \boldsymbol{u}\cdot d\boldsymbol{l}\right] = \frac{d}{dt}\int_{\mathcal{S}} (\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}\, d\mathcal{S} = \frac{d}{dt}\int_{\mathcal{S}} \boldsymbol{\omega}\cdot\boldsymbol{n}\, d\mathcal{S} = 0$$

is that the vorticity flux crossing the material surface $\mathcal{S}$ is also an invariant.

Consider an elementary homogeneous vortex tube of length L, radius R and vorticity ω,



$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{u}\cdot d\boldsymbol{l} = \int_{\mathcal{S}} \boldsymbol{\omega}\cdot d\boldsymbol{x} = \pi R^2\omega$$

Introduction to vortex strechting (cont.)

For this elementary vortex,

- conservation of circulation Γ, $R^2\omega = \text{cst}$
- conservation of mass, $\rho\pi R^2 L \sim R^2 L = \text{cst}$

and an estimate of the kinetic energy $\mathcal{E}_c$ is given by

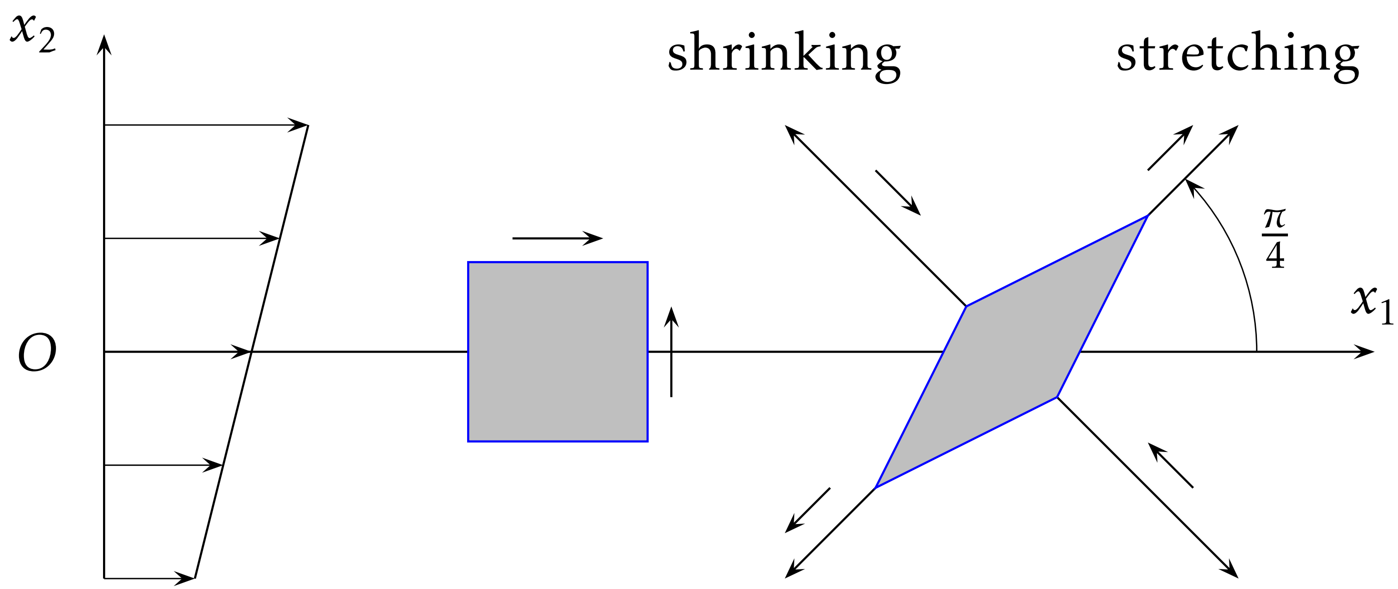
$$\mathcal{E}_c = \rho\pi R^2 L \frac{R^2\omega^2}{2} \sim \underbrace{R^2 L\, R^2\omega}_{\text{cst}}\ \omega \quad \Longrightarrow \quad \mathcal{E}_c \sim \omega \sim \frac{1}{R^2} \sim L$$

The kinetic energy is directly proportional to the vortex length. The increase in kinetic energy for the vortex - and consequently for the turbulent velocity field, is associated with vortex stretching. It's an important basic mechanism to interprete the behaviour of turbulent flow.

In other words, during the stretching process in one direction, the kinetic energy in the perpendicular plane increases whereas the length scales decrease.

Introduction to vortex strechting (cont.)

Principal axes of the deformation tensor for shear flow
$\bar{U}_1 = Sx_2$ et $\bar{U}_2 = \bar{U}_3 = 0$



$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right) = \frac{\overline{S}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \bar{S}^d_{ij} = \frac{\overline{S}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Helmholtz's equation

The Helmholtz equation is the transport equation for the vorticity vector, obtained by taking the curl of the Navier-Stokes equation

$$\nabla \times \left\{ \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{u} \right\}$$

Using the following vectorial identities

$$\nabla \times (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) = \nabla \times \left[\nabla \left(\frac{\boldsymbol{u}^2}{2} \right) + \boldsymbol{\omega} \times \boldsymbol{u} \right] = \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u})$$

and moreover $\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{u} \nabla \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \boldsymbol{\omega} \nabla \cdot \boldsymbol{u}$
since $\nabla \cdot \boldsymbol{\omega} \equiv 0$ (solenoidal vorticity field) and $\nabla \cdot \boldsymbol{u} = 0$ (incompressible flow)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} - \nabla \times \left(\frac{1}{\rho} \nabla p \right) + \nu \nabla^2 \boldsymbol{\omega}$$

Helmholtz's equation (cont.)

Assuming a barotropic flow, that is a flow whose pressure is a function of density only $p = p(\rho)$, one has for the pressure term

$$\nabla \times \left(\frac{1}{\rho}\nabla p\right) = \nabla\left(\frac{1}{\rho}\right) \times \nabla p + \frac{1}{\rho}\nabla \times (\nabla p) = -\frac{1}{\rho^2}\nabla\rho \times \nabla p = 0$$

The transport equation for vorticity reads

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}$$

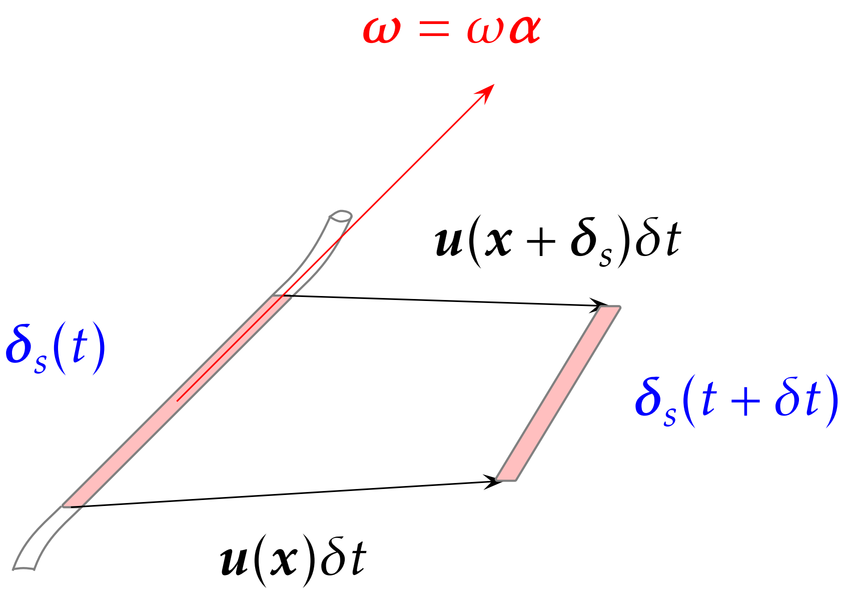
$$\left\{\begin{matrix}\text{convection}\\ \text{of } \boldsymbol{\omega}\end{matrix}\right\} = \left\{\begin{matrix}\text{3-D effect}\\ \text{(source term)}\end{matrix}\right\} + \left\{\begin{matrix}\text{viscous}\\ \text{diffusion}\end{matrix}\right\}$$

Hermann von Helmholtz
(1821 - 1894)

The evolution of vorticity is directly linked to the term associated with 3-D effect : this term is zero for a two-dimensional flow, $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} \equiv 0$

► 2-D flow represents a specific/particular configuration ...

Interpretation of Helmholtz's equation



Deformation of an elementary tube (filament) of vorticity

$$\frac{\boldsymbol{\delta}_s(t+dt)-\boldsymbol{\delta}_s(t)}{\delta t} = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\delta}_s)-\boldsymbol{u}(\boldsymbol{x})$$

$$\frac{d\boldsymbol{\delta}_s(t)}{dt} = \boldsymbol{\delta}_s\cdot\nabla\boldsymbol{u}$$

Length of the elementary tube

$\tilde{\delta}_s = \|\boldsymbol{\delta}_s\| = \boldsymbol{\delta}_s\cdot\boldsymbol{\alpha} \qquad \boldsymbol{\alpha}^2 = 1$

$$\frac{D\tilde{\delta}_s}{Dt} = \boldsymbol{\alpha}\cdot\,(\boldsymbol{\delta}_s\cdot\nabla\boldsymbol{u})$$

$$= \frac{\omega_i}{\omega}\left(\tilde{\delta}_s\,\frac{\omega_j}{\omega}\,\frac{\partial u_i}{\partial x_j}\right)$$

$$= \frac{\omega_i\omega_j}{\omega^2}\,\frac{\partial u_i}{\partial x_j}\,\tilde{\delta}_s$$

Interpretation of Helmholtz's equation (cont.)

Furthermore, by neglecting the viscous term in Helmholtz's 'equation and taking the scalar product with $\boldsymbol{\omega}$, we obtain

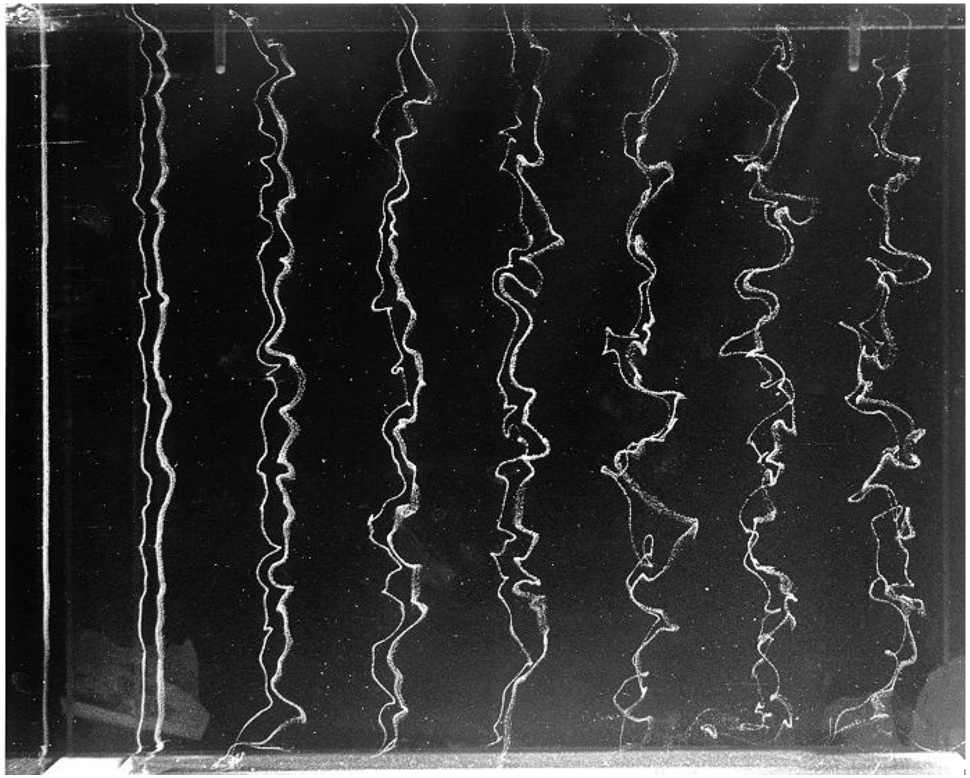
$$\boldsymbol{\omega}\cdot\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}) \quad \text{that is} \quad \frac{D}{Dt}\left(\frac{\omega^2}{2}\right) = \omega_i\omega_j\frac{\partial u_i}{\partial x_j}$$

By identification with the previous equation, it can be deduced that

$$\frac{1}{\omega^2}\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) = \frac{1}{\tilde{\delta}_s}\frac{D\tilde{\delta}_s}{Dt} \quad \text{and by integration,} \quad \frac{\omega}{\tilde{\delta}_s} = \text{cst}$$

The length of an elementary tube vortex is thus proportional to vorticity ω. We find the conclusion already obtained with the dimensional analysis, to highlight vortex stretching mechanism and the increase in turbulent fluctuations. In addition, the term associated with the lengthening of vortex tubes corresponds to the term of 3-D effect in the transport equation of vorticity.

Interpretation of Helmholtz's equation (cont.)



Growth of material lines in isotropic turbulence $\mathrm{Re}_D = 1360$ (based on the grid rod diameter)

Corrsin & Karweit, 1969, *J. Fluid Mech.*, **39**(1)

The increase in vortex intensity, and thus in turbulent fluctuations, is accompanied by stretching of vorticity filaments, and by the increase of distance between fluid particles : the origin of sensitivity to initial conditions ...

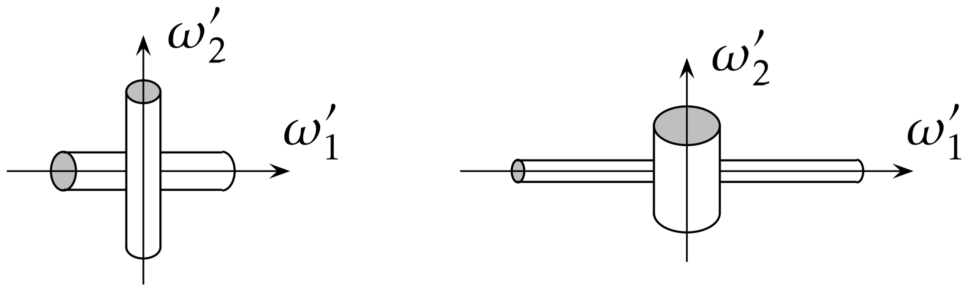
- **An illustration of the lengthening of vortex filament**
  (from Tennekes & Lumley, 1972, chap. 8)

  Mean flow for which gradients are aligned with the frame axes

$$\frac{\partial \bar{U}_i}{\partial x_j} = \begin{pmatrix} \overline{S} & 0 \\ 0 & -\overline{S} \end{pmatrix}$$

$$\bar{U}_1 = \overline{S} x_1 \quad \bar{U}_2 = -\overline{S} x_2 \quad \bar{\Omega}_i \equiv 0$$

pure strain flow



  Helmholtz's Eq. linearized around this mean flow (inviscid flow to simplify algebra, not an issue because the viscous terms are linear)

$$\frac{\partial \boldsymbol{\omega}'}{\partial t} + \bar{\boldsymbol{U}} \cdot \nabla \boldsymbol{\omega}' = \boldsymbol{\omega}' \cdot \nabla \bar{\boldsymbol{U}}$$

$$\frac{\bar{D}\omega_1'}{\bar{D}t} = +\overline{S}\omega_1' \qquad \frac{\bar{D}\omega_2'}{\bar{D}t} = -\overline{S}\omega_2' \qquad \frac{\bar{D}}{\bar{D}t} \equiv \frac{\partial}{\partial t} + \bar{U}_j \frac{\partial}{\partial x_j}$$

An illustration of the lengthening of vortex filament (cont.)

By integration along the mean flow, or with the following formal change of variables $\xi_1 = x_1 e^{-\overline{S}t}$, $\xi_2 = x_2 e^{\overline{S}t}$ and $\tau = t$, one gets

$$\omega_1' = \omega_0 e^{\overline{S}t} \qquad \omega_2' = \omega_0 e^{-\overline{S}t}$$

The vorticity component ω_1' is thus stretched faster than the component ω_2' through a nonlinear processus, and finally vorticity fluctuations increase as

$$\omega_1'^2 + \omega_2'^2 = 2\omega_0^2 \cosh(2\overline{S}t)$$

- **Bradshaw's tree diagram (1971) illustrating of the concept of energy cascade** originally introduced by Richardson (1926)

direction of vortex streching

anisotropy $\quad x_3$

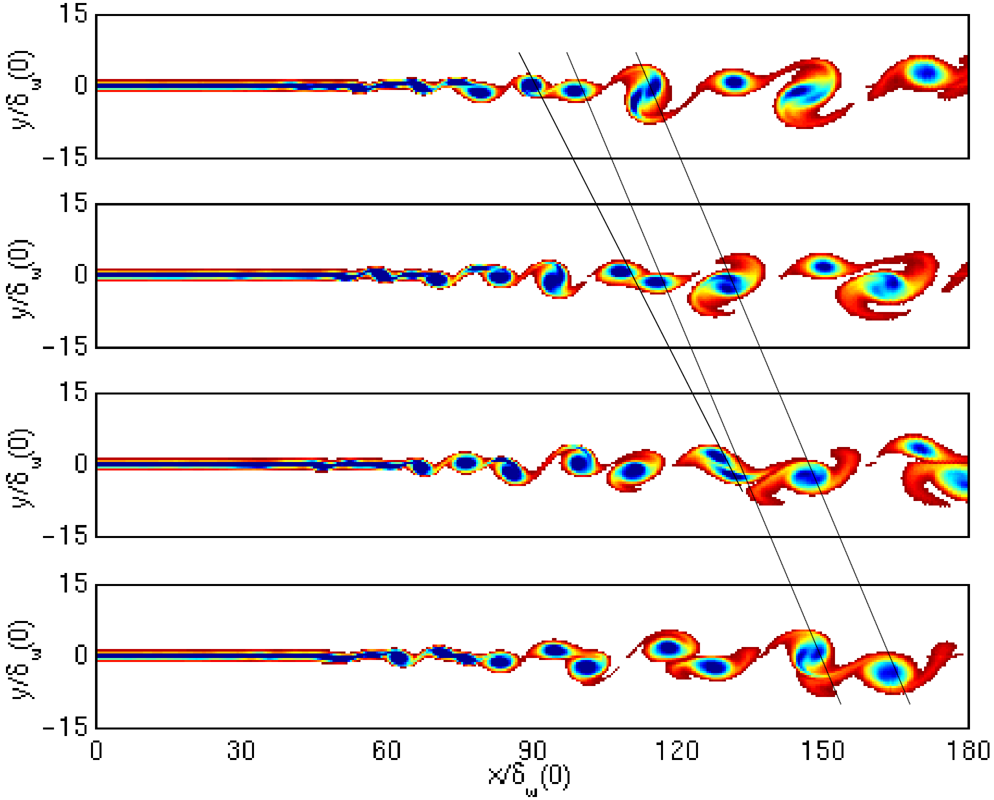
$\overbrace{x_1 \qquad\qquad x_2}$

$\overbrace{x_2 \quad x_3} \qquad \overbrace{x_1 \quad x_3}$

$\overbrace{x_3 \quad x_1} \; \overbrace{x_1 \quad x_2} \; \overbrace{x_2 \quad x_3} \; \overbrace{x_1 \quad x_2}$

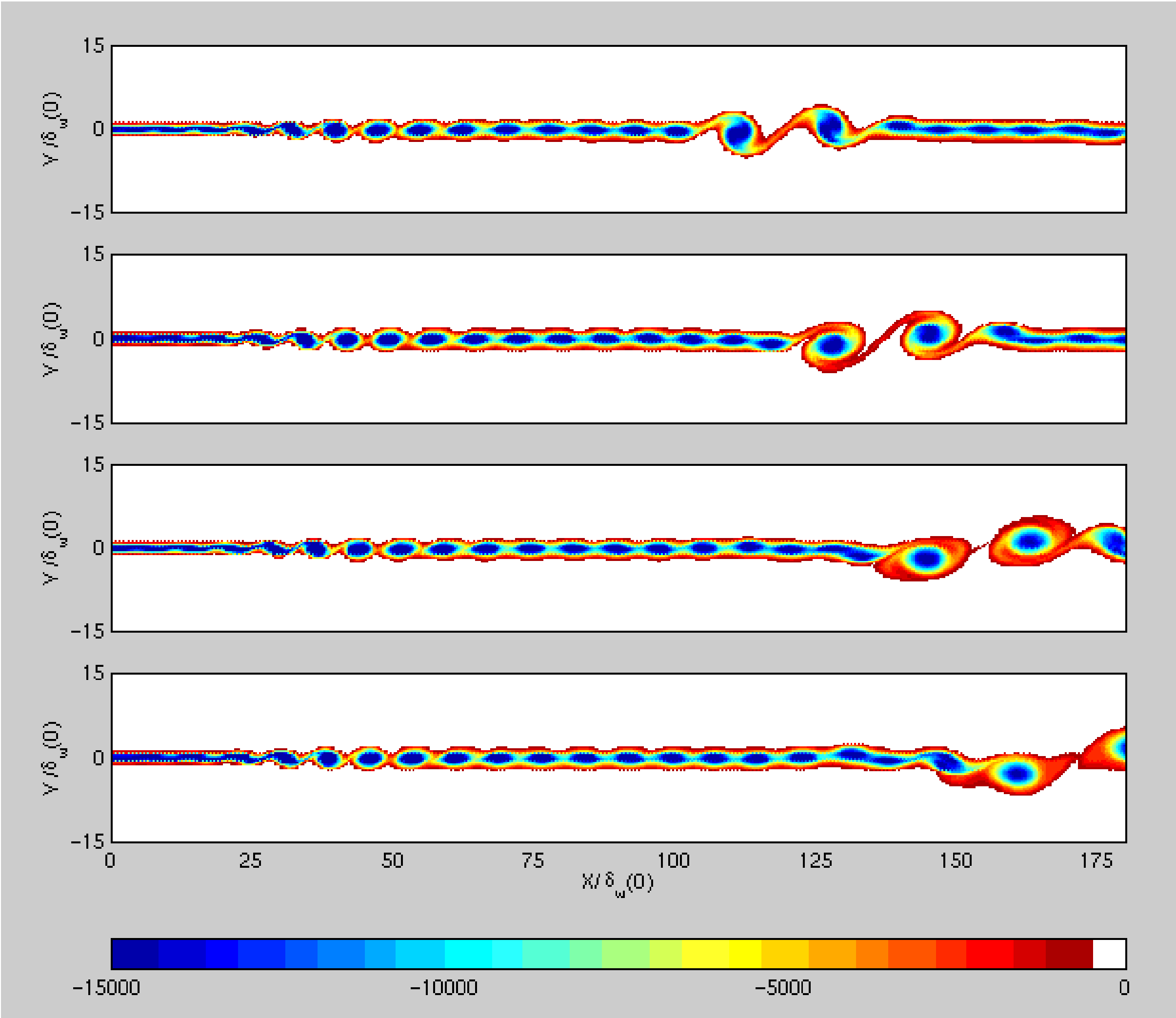
return to isotropy $\quad \overbrace{x_1\, x_2}\; \overbrace{x_2\, x_3}\; \overbrace{x_2\, x_3}\; \overbrace{x_3\, x_1}\; \overbrace{x_3\, x_1}\; \overbrace{x_1\, x_2}\; \overbrace{x_2\, x_3}\; \overbrace{x_3\, x_1}$

- **Plane mixing layer – an example of inverse energy cascade**
  Identification of vortex pairing

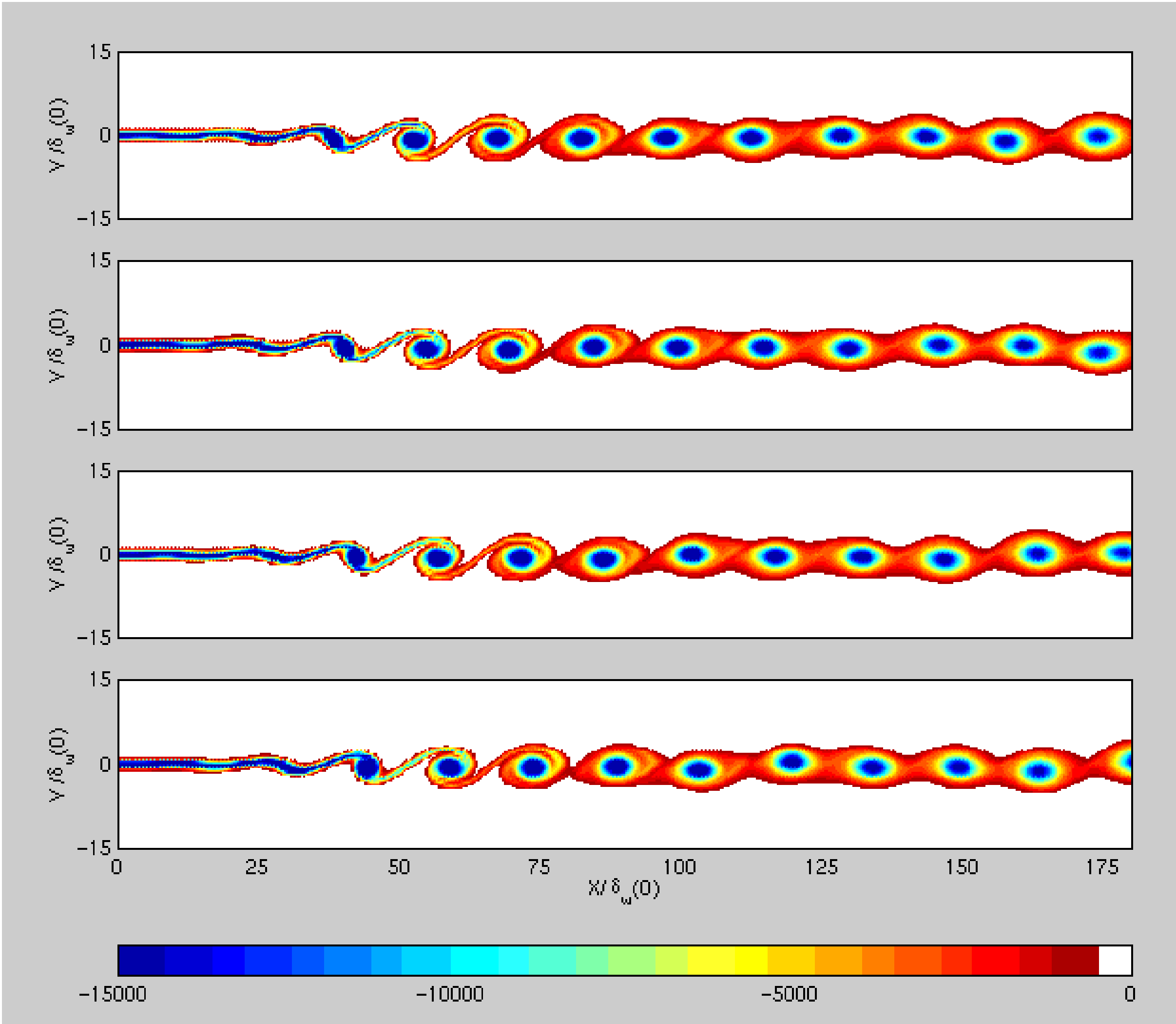


Simulation of a plane mixing layer ($M_1 = 0.12$, $M_2 = 0.48$, $\mathrm{Re}_{\delta_\omega} = 1.28 \times 10^4$), snaphots of the vorticity field at 4 consecutive times separated by $17\delta_\omega/U_c$, where U_c is the convection velocity. (Bogey, Bailly & Juvé, *AIAA Journal*, 2000)

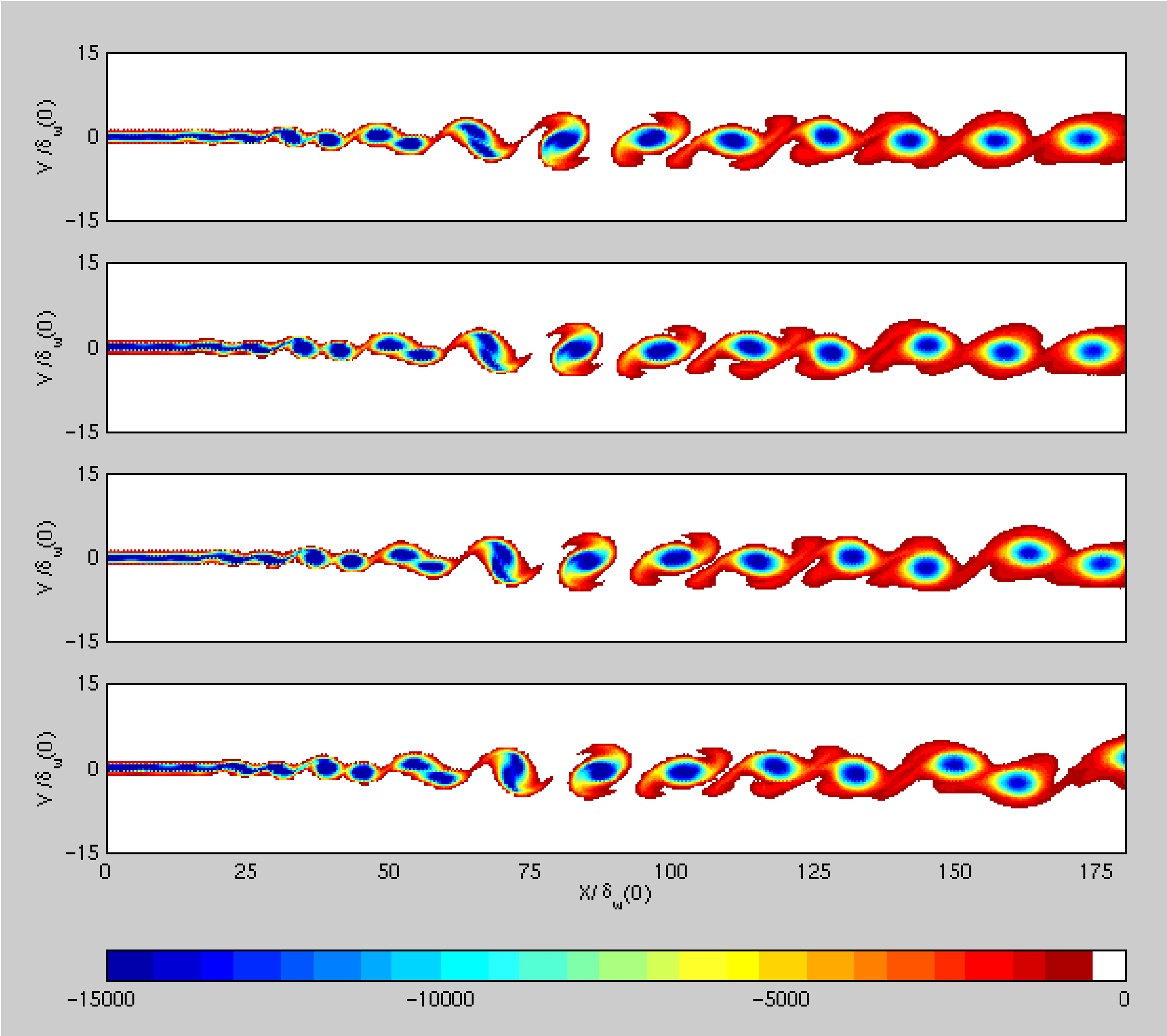
- **Plane mixing layer forced at f_0**
  (f_0 fundamental frequency corresponding to most amplified perturbations)



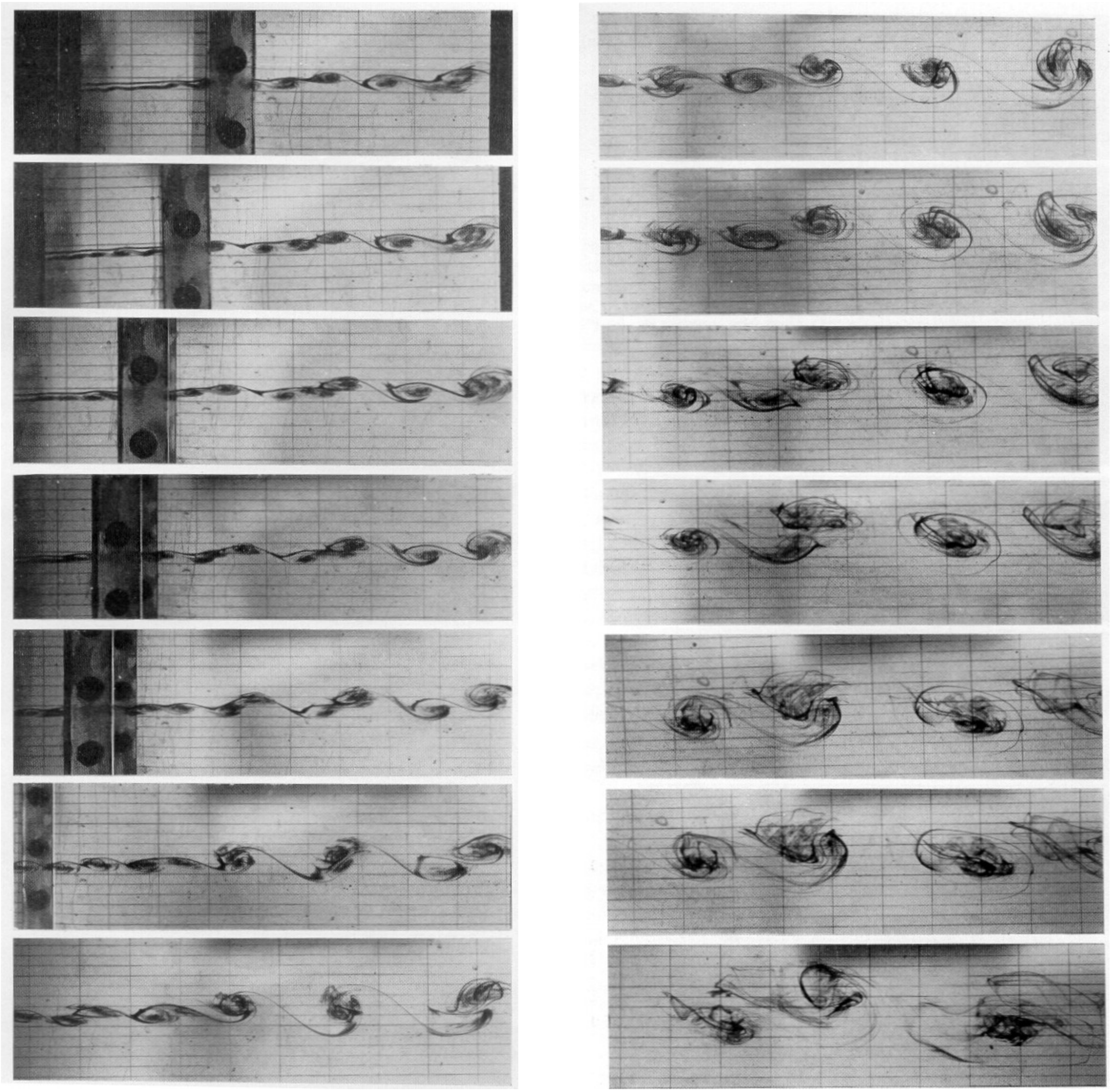
- **Plane mixing layer forced at f_1**
  ($f_1 = f_0/2$, first subharmonic frequency)



- **Plane mixing layer forced at f_0 and f_1**
  Vortex pairings occurred at fixed streamwise locations



Vortex pairing in a plane mixing layer

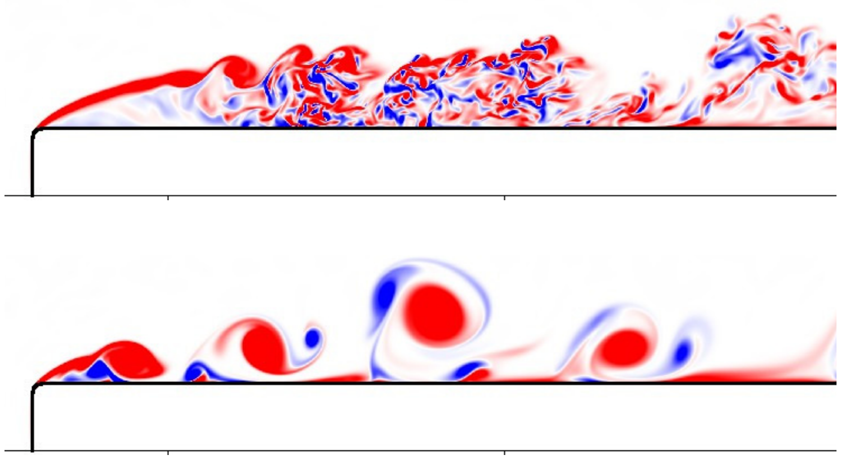


Winant & Browand

J. Fluid Mech. (1974)

- **2-D simulations must be proscribed : no energy cascade**

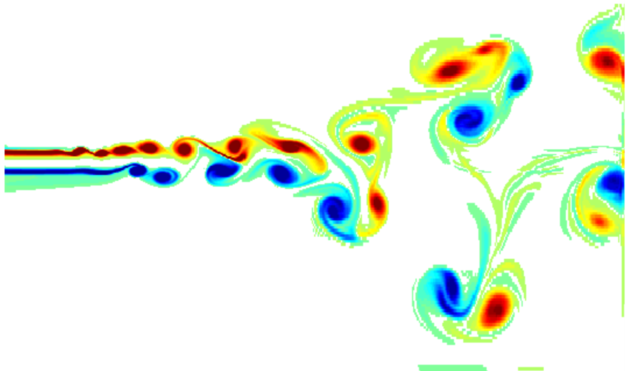
Flow separation behind a rounded leading edge (3-D versus 2-D !)



Spanwise vorticity ω_z, from red to blue with $\omega_z = \pm 5U_\infty/H$, DNS with inflow perturbations $u'_{\text{inflow}} = 0.1\% U_\infty$ ($\eta = 0.125$)

Courtesy of Lamballais, Sylvestrini & Laizet
Int. Journal Heat Fluid Flow, **31**, 2010

2-D free jet, vorticity field



Bogey (Ph.D. EC-Lyon, 1999)

- **Transport equation for the mean vorticity** $\omega_i = \bar{\Omega}_i + \omega'_i$

$$\frac{\partial \bar{\Omega}_i}{\partial t} + \bar{U}_j \frac{\partial \bar{\Omega}_i}{\partial x_j} = \bar{\Omega}_j \frac{\partial \bar{U}_i}{\partial x_j} + \underbrace{\frac{\partial}{\partial x_j}\left(\overline{\omega'_j u'_i} - \overline{\omega'_i u'_j}\right)}_{(a)} + \underbrace{\nu \frac{\partial^2 \bar{\Omega}_i}{\partial x_j \partial x_j}}_{(b)}$$

(a) $\sim$ correlation term involving turbulence fluctuations only, must be closed to solve this equation

(b) $\sim$ viscous diffusion

In practice, this equation is rarely (if ever !) solved to obtain the mean flow field : turbulence models are based on the resolution of the mean velocity field (RANS Eqs.). This equation is theoretically used to study enstrophy.

- **Enstrophy**
  Similar to the kinetic energy for velocity, that is

$$\frac{\overline{\omega_i'\omega_i'}}{2} \equiv \frac{\overline{\omega_1'^2} + \overline{\omega_2'^2} + \overline{\omega_3'^2}}{2}$$

To quickly derive its transport equation, we assume that there is no mean flow, that is $\bar{U}_i \equiv 0$ et $\bar{\Omega}_i \equiv 0$

$$\frac{\partial}{\partial t}\left(\frac{\overline{\omega_i'\omega_i'}}{2}\right) + \frac{\partial}{\partial x_j}\left(\overline{u_j'\frac{\omega_i'\omega_i'}{2}}\right) = \overline{\omega_i'\omega_j'\frac{\partial u_i'}{\partial x_j}} - \nu\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} + \nu\frac{\partial^2}{\partial x_j^2}\left(\frac{\overline{\omega_i'\omega_i'}}{2}\right)$$

As usual, this Eq. can be greatly simplified for homogeneous turbulence, in order to isolate basic physical mechanisms

$$\frac{\partial}{\partial t}\left(\frac{\overline{\omega_i'\omega_i'}}{2}\right) = \underbrace{\overline{\omega_i'\omega_j'\frac{\partial u_i'}{\partial x_j}}}_{(a)} - \underbrace{\nu\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}}}_{(b)} \tag{3}$$

Enstrophy (cont.)

The term (a) is linked to the stretching of vortices, and the term (b) to viscous dissipation.

Historically, the term (a) was assumed to be zero by von Kármán (1937), but Taylor (1938) demonstrated later that this term is not zero and furthermore, must be positive. It expresses that two fluid particles initially close one from the other will be later separated by turbulence in average.

Singular behaviour of two-dimensional turbulent flow again,
enstrophy can only decrease

$$\frac{\partial}{\partial t}\left(\frac{\overline{\omega'_i\omega'_i}}{2}\right) = -\nu\,\overline{\frac{\partial \omega'_i}{\partial x_j}\frac{\partial \omega'_i}{\partial x_j}}$$

Enstrophy (cont.)

In order to solve Eq. (3), the nonlinear term can be modeled with an acceptable dimensional expression. For instance,

$$\overline{\omega_i'\omega_j'\frac{\partial u_i'}{\partial x_j}} \simeq A(\overline{\omega^2})^{3/2} \qquad A = \mathrm{cst} \qquad \omega^2 \equiv \omega_i'\omega_i'$$

Neglecting viscous effects to simplify calculations, the integration leads to the following time evolution.

$$\frac{\overline{\omega^2}}{\omega_0^2} = \frac{1}{\left[1 - A\sqrt{\omega_0^2}\,(t_0 - t)\right]^2}$$

A singularity is thus obtained for a finite time ... refer to Leray (1934), Moffatt (2000) : artefact induced by the model itself and the incompressibility condition. Not so easy to derive an acceptable model for physics !

- **Helicity**
  Quantity widely studied by Moffatt (1969)

$$\mathcal{H} \equiv \int_V \boldsymbol{u} \cdot \boldsymbol{\omega} \, d\boldsymbol{x}$$

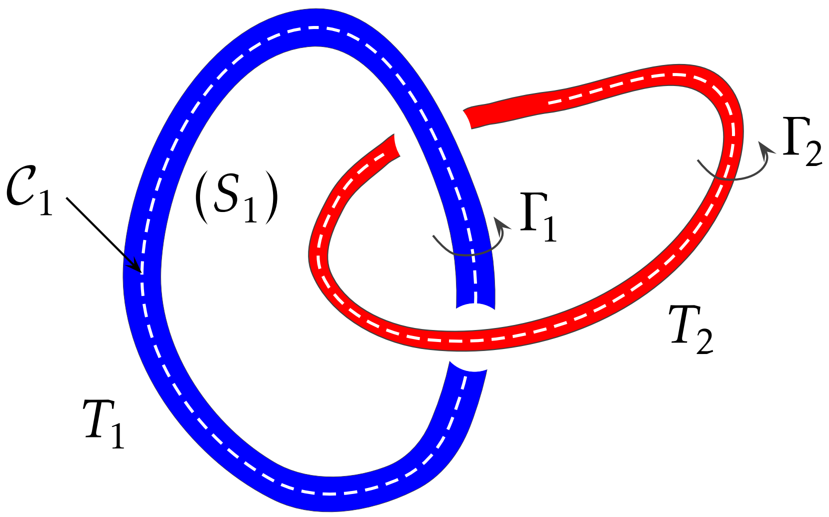
This quantity is an invariant of the flow motion, under the same assumptions introduced for Kelvin's circulation theorem.

For a two-dimensional flow, $\mathcal{H} = 0$.

Interpretation ?

Helicity (cont.)

Sketch of two linked vortex tubes T_1 and T_2



$$\mathcal{H} = \int_{\mathcal{V}} \boldsymbol{u}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} = \int_{T_1} \boldsymbol{u}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} + \int_{T_2} \boldsymbol{u}\cdot\boldsymbol{\omega}\, d\boldsymbol{x}$$

Consider the integral over the vortex T_1

$$\int_{T_1} \boldsymbol{u}\cdot\boldsymbol{\omega}\, d\boldsymbol{x} \simeq \Gamma_1 \oint_{\mathcal{C}_1} \boldsymbol{u}\cdot d\boldsymbol{l} = \Gamma_1 \int_{\mathcal{S}_1} (\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}\, d\mathcal{S}$$

$$= \begin{cases} \Gamma_1\Gamma_2 & \text{if } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are linked,} \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{H} = \pm 2n\Gamma_1\Gamma_2$ $\quad$ n linking number