*L*aboratoire de *M*écanique des *F*luides et d'*A*coustique
LMFA UMR 5509

CENTRALE LYON

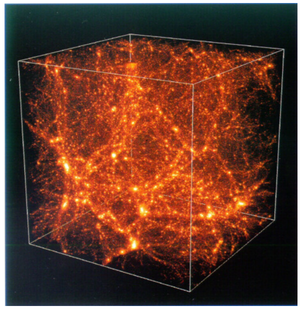
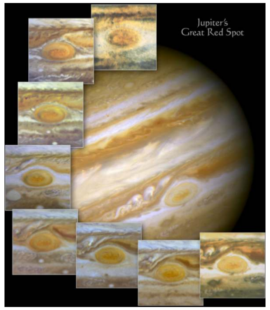
# Physics of turbulent flow

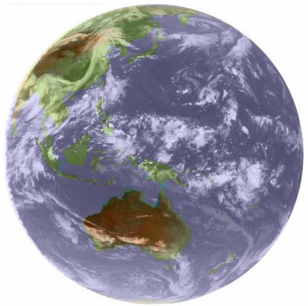
**Christophe Bailly**

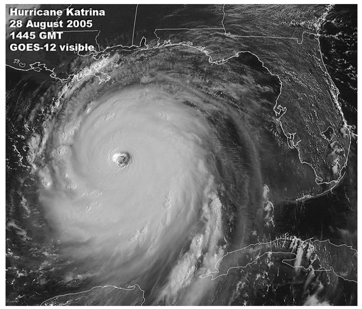
Centrale Lyon – MOD 3rdYear & MSc. – version 25-11-2024

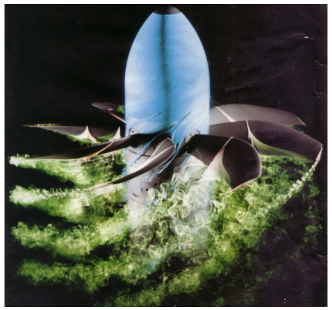
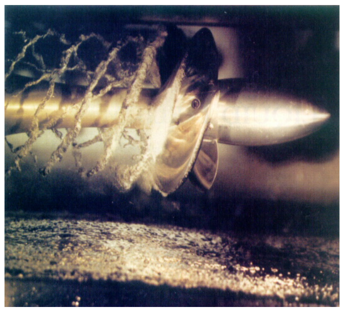
# Outline

# Homogeneous and isotropic turbulence

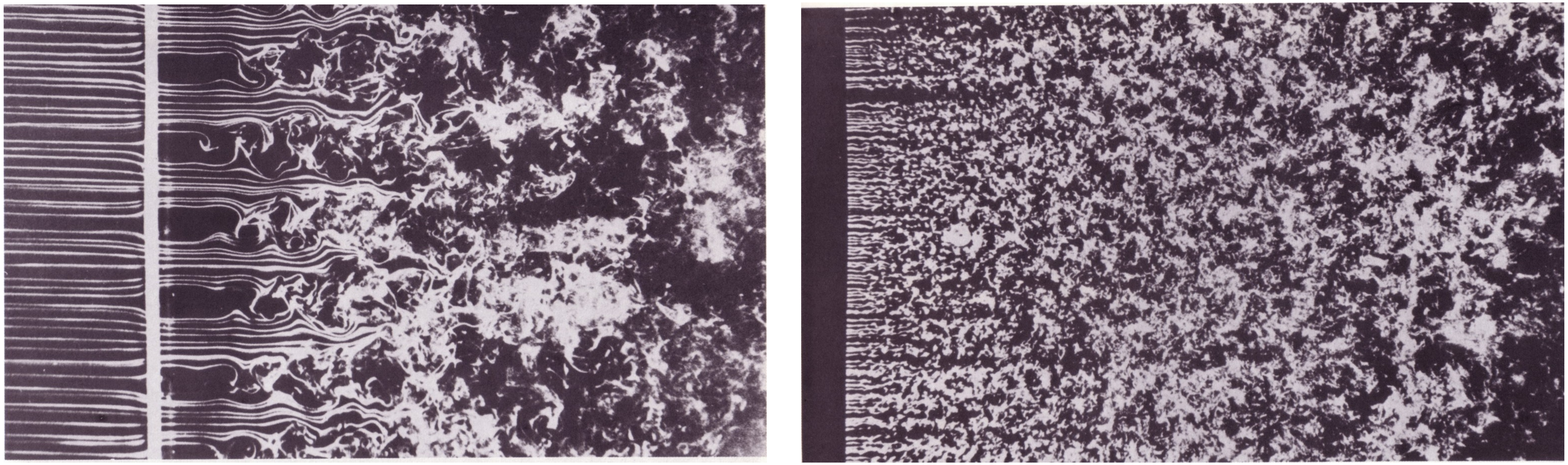








## Homogeneous turbulence

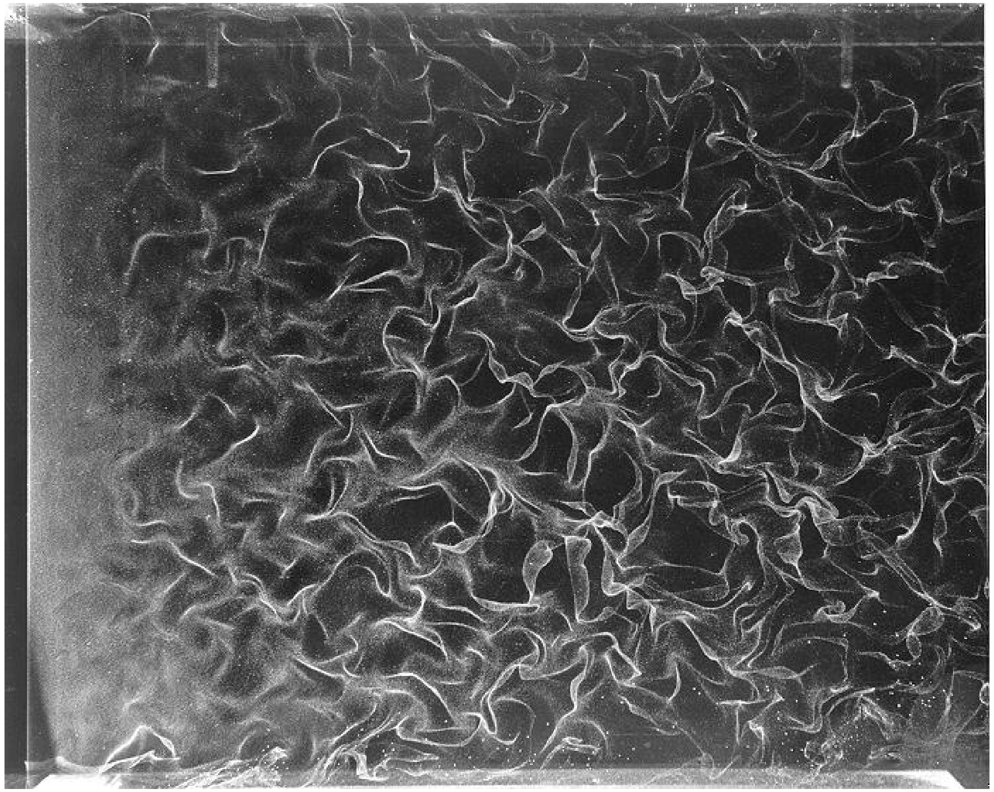


Generation of turbulence behind a grid, $\mathrm{Re}_M = 1500$ & $M = 2.54$ cm
Corke & Nagib, in *Van Dyke*, figs. 152 & 152 (1982)

Statistics are independants of space coordinates in homogeneous directions. In the present case, the turbulent flow is homogeneous in the $x_2$ and $x_3$ directions (transverse plane), *e.g.* the Reynolds tensor $-\overline{u_i'u_j'}$ is only a function of $x_1$ (and $t$).

The objective is to obtain simple configurations, without transport term

## Homogeneous turbulence



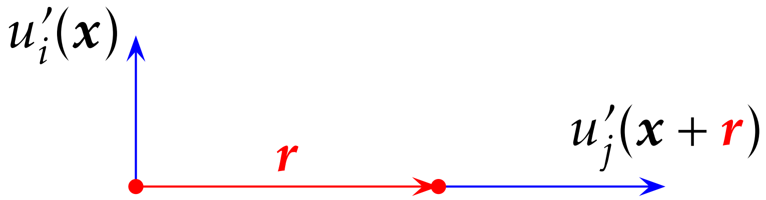
Wrinkling of a fluid surface in isotropic turbulence
Karweit in *Van Dyke*, fig. 155 (1982)

A platinum wire generates a continuous sheet of hydrogen bubbles, which is then deformed by the nearly isotropic turbulence behind the grid.

## Velocity correlation tensor

Definition :

$$R_{ij}(\boldsymbol{x}, \boldsymbol{r}, t) \equiv \overline{u'_i(\boldsymbol{x}, t)\, u'_j(\boldsymbol{x} + \boldsymbol{r}, t)} = R_{ij}(\boldsymbol{r}, t)$$




The function $R_{ij}$ is only a function of the separation vector $\boldsymbol{r}$, between the two measurement points $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$ : invariance by translation of the observer location $\boldsymbol{x}$.

Correlation coefficient $\mathcal{R}_{ij}$ (normalized correlation function $R_{ij}$)

$$-1 \leq \mathcal{R}_{ij}(\boldsymbol{r}) \equiv \frac{\overline{u'_i(\boldsymbol{x})\, u'_j(\boldsymbol{x}')}}{\sqrt{\overline{u'^2_i}(\boldsymbol{x})}\sqrt{\overline{u'^2_j}(\boldsymbol{x}')}} \leq +1$$

## Velocity correlation tensor (cont.)

A few remarks

- Autocorrelation
  $R_{11}(r,0,0) = \overline{u_1'^2}\,\mathcal{R}_{11}(r,0,0)$ with $\boldsymbol{r} = (r,0,0)$
  $R_{11}(\boldsymbol{r}) = R_{11}(-\boldsymbol{r})$, the autocorrelation function is an even function

- $R_{ij}(\boldsymbol{r}) = R_{ji}(-\boldsymbol{r})$

- Incompressibility of the turbulent field

$$\frac{\partial u_j'}{\partial x_j} = 0 \quad \Longrightarrow \quad \frac{\partial}{\partial r_j} R_{ij}(\boldsymbol{r}) = 0 \qquad \frac{\partial}{\partial r_i} R_{ij}(\boldsymbol{r}) = 0$$

- $R_{ii}(0) = \overline{u_1'^2} + \overline{u_2'^2} + \overline{u_2'^2} = 2k_t$

- **Turbulent kinetic energy budget $k_t$** (refer to this slide)

General case of homogeneous turbulence

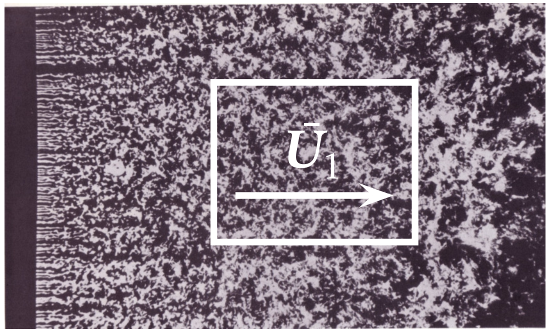
$$\frac{\partial(\rho k_t)}{\partial t} = -\rho\overline{u_i'u_j'}\frac{\partial \bar{U}_i}{\partial x_j} - \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}} \quad (= \mathcal{P} - \rho\epsilon)$$

with $\partial\bar{U}_i/\partial x_j = \text{cst}$ to preserve homogeneous turbulence (Craya, 1958)

Decaying turbulence generated behind a grid,

Stationary turbulence, homogeneous in the plane $(x_2, x_3)$ only

$$\bar{U}_1\frac{\partial k_t}{\partial x_1} = -\epsilon$$

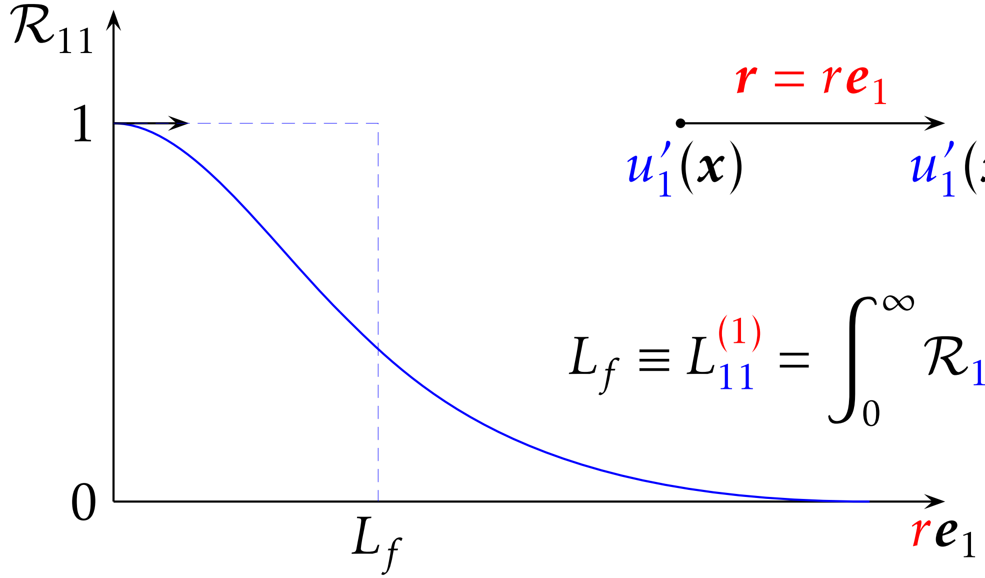



In a frame moving with the mean velocity $\bar{U}_1$,

$$\frac{\partial k_t}{\partial t} = -\epsilon$$

## Integral length scales

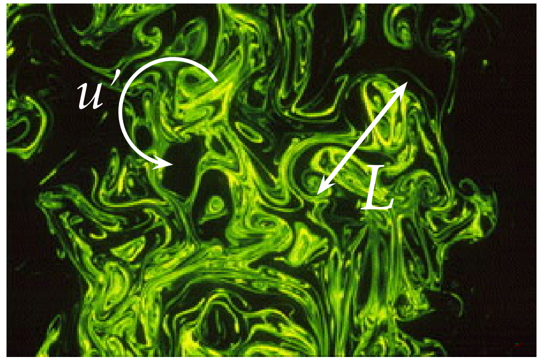
Longitudinal integral length scale : an estimate of the size of the most energetic turbulent structures, given by the integration of the correlation coefficient of the velocity component $u_1'$ between two points in the $x_1$ direction




$\boldsymbol{r} = r\boldsymbol{e}_1$

$u_1'(\boldsymbol{x}) \qquad u_1'(\boldsymbol{x} + r\boldsymbol{e}_1)$

$$L_f \equiv L_{11}^{(1)} = \int_0^\infty \mathcal{R}_{11}(r, 0, 0)\,dr$$




Tavoularis (2003), passive scalar mixing, Sc $\simeq 2000$

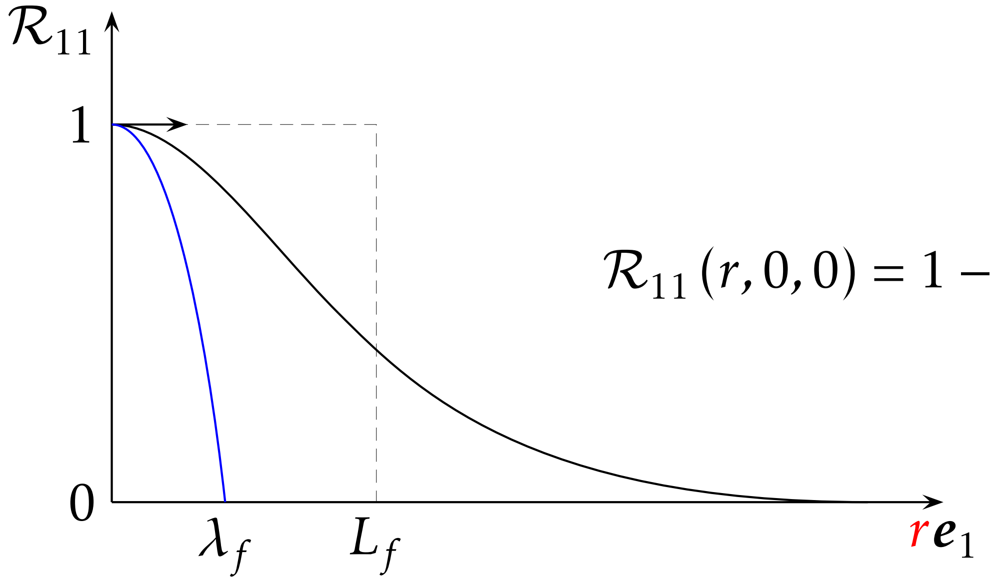
A transverse integral length scale $L_g \equiv L_{11}^{(2)}$ is also introduced

$$L_g \equiv L_{11}^{(2)} = \int_0^\infty \mathcal{R}_{11}(0, r, 0)\,dr$$

## Turbulence scales

- Large scales $(u', L)$ associated with production of larger scales by the mean shear flow ; energy containing eddies : the peak of the turbulent kinetic energy spectrum is located arround $kL \sim 1$

- We need also to introduce Taylor microscales $\lambda$ associated with large scales of the dissipation spectrum, and formally defined from the Taylor series of the velocity correlation coefficient at the origin,




$$\mathcal{R}_{11}(r,0,0) = 1 - \frac{r^2}{{\lambda_f}^2} + \ldots \qquad (\lambda_f \ll L_f)$$

## Taylor microscales

Taylor series of $u'_1(r,0,0)$ as $r \to 0$,

$$u'_1(r,0,0) = u'_1(0,0,0) + r\left.\frac{\partial u'_1}{\partial x_1}\right|_{x=0} + \frac{r^2}{2}\left.\frac{\partial^2 u'_1}{\partial x_1^2}\right|_{x=0} + \dots$$

Hence,

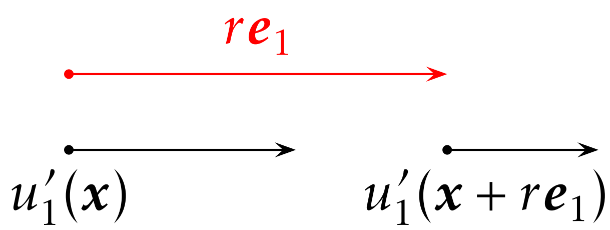
$$\begin{aligned} R_{11}(r,0,0) &= \overline{u'_1(0,0,0)\,u'_1(r,0,0)} \\ &= \overline{u'^2_1} + r\,\overline{u'_1\frac{\partial u'_1}{\partial x_1}} + \frac{r^2}{2}\,\overline{u'_1\frac{\partial^2 u'_1}{\partial x_1^2}} + \dots \\ &= \overline{u'^2_1} + r\,\frac{\partial}{\partial x_1}\left(\frac{\overline{u'^2_1}}{2}\right) + \frac{r^2}{2}\,\frac{\partial}{\partial x_1}\left(\overline{u'_1\frac{\partial u'_1}{\partial x_1}}\right) - \frac{r^2}{2}\,\overline{\left(\frac{\partial u'_1}{\partial x_1}\right)^2} + \dots \end{aligned}$$

$$\mathcal{R}_{11}(r,0,0) = 1 - \frac{r^2}{2\overline{u'^2_1}}\,\overline{\left(\frac{\partial u'_1}{\partial x_1}\right)^2} \equiv 1 - \frac{r^2}{\lambda_f^2} + \dots$$

## Taylor microscales (cont.)

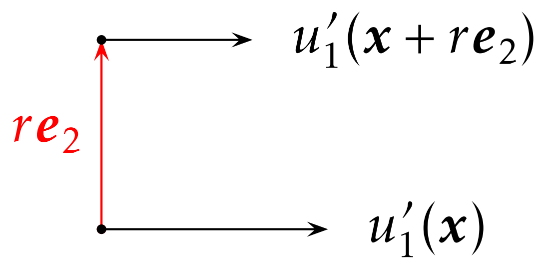
### Longitudinal Taylor microscale $\lambda_f$

$$\frac{1}{\lambda_f^2} \equiv -\frac{1}{2}\frac{d^2\mathcal{R}_{11}}{dr_1^2}\bigg|_{r=0} = \frac{1}{2\overline{u_1'^2}}\overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}$$




### Transverse Taylor microscale $\lambda_g$

$$\frac{1}{\lambda_g^2} \equiv -\frac{1}{2}\frac{d^2\mathcal{R}_{11}}{dr_2^2}\bigg|_{r=0} = \frac{1}{2\overline{u_1'^2}}\overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2}$$



## Dissipation rate $\epsilon$ of the turbulent kinetic energy

$$\rho\epsilon = \overline{\tau'_{ik}\frac{\partial u'_i}{\partial x_k}} = 2\mu\overline{s'_{ij}s'_{ij}} = 2\mu\overline{\frac{1}{4}\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)^2} = \mu\underbrace{\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}}_{(a)} + \mu\underbrace{\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}}}_{(b)}$$

$$(a) \equiv \epsilon^h = \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}} \sim \nu\frac{u'^2}{\lambda^2}$$

correlation of the turbulent velocity gradients, dominant term for the dissipation since $\lambda \ll L$

$$(b) = \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i}} = \frac{\partial^2\overline{u'_i u'_j}}{\partial x_i \partial x_j} \sim \nu\frac{u'^2}{L^2}$$

derivative of the turbulent velocity correlation (using the incompressibility condition)

for homogeneous turbulence, $(b)$ is identically zero and $\epsilon = \epsilon^h$

$\epsilon^h$ is an approximation of the dissipation $\epsilon$ when $\lambda \ll L$, that is for high Reynolds number turbulent flow (the $\epsilon^h$ equation is solved in the standard $k_t - \epsilon$ model)

## Spectral tensor

The spectral tensor $\phi_{ij}(\boldsymbol{k})$ is defined as the Fourier transform of the velocity correlation tensor $R_{ij}(\boldsymbol{r})$

$$\begin{cases} \phi_{ij}(\boldsymbol{k}) = \dfrac{1}{(2\pi)^3} \displaystyle\int_{\mathrm{R}^3} R_{ij}(\boldsymbol{r})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r} \\ R_{ij}(\boldsymbol{r}) = \displaystyle\int_{\mathrm{R}^3} \phi_{ij}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \end{cases}$$

The incompressibility condition formulated in Fourier space reads,
$k_i\phi_{ij}(\boldsymbol{k}) = k_j\phi_{ij}(\boldsymbol{k}) = 0$

It is essential in practice to introduce one-dimensional spectra, which can be measured or computed numerically,

$$E_{ij}^{(1)}(k_1) = \iint_{\mathrm{R}^2} \phi_{ij}(\boldsymbol{k})\ dk_2 dk_3$$

## One-dimensional spectrum

Let us consider the case $i = j = 1$ with a zero separation vector $\boldsymbol{r} = 0$,

$$\overline{u_1'^2} = R_{11}(\boldsymbol{r} = 0) = \int_{\mathrm{R}^3} \phi_{11}(\boldsymbol{k}) d\boldsymbol{k} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1)\ dk_1$$

The relation between the autocorrelation $R_{11}(\boldsymbol{r})$ with $\boldsymbol{r} = (r_1, 0, 0)$, and the one-dimensional spectrum $E_{11}^{(1)}(k_1)$ is found to be

$$R_{11}(r_1, 0, 0) = \int_{\mathrm{R}^3} \phi_{11}(\boldsymbol{k})\ e^{ik_1 r_1} d\boldsymbol{k} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1)\ e^{ik_1 r_1} dk_1$$

and conversely by Fourier transform, one has

$$E_{11}^{(1)}(k_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{11}(r_1, 0, 0)\ e^{-ik_1 r_1} dr_1$$

For $k_1 = 0$, $\quad E_{11}^{(1)}(0) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{+\infty} R_{11}(r_1, 0, 0)\ dr_1 = \dfrac{1}{2\pi} 2\overline{u_1'^2} L_f$

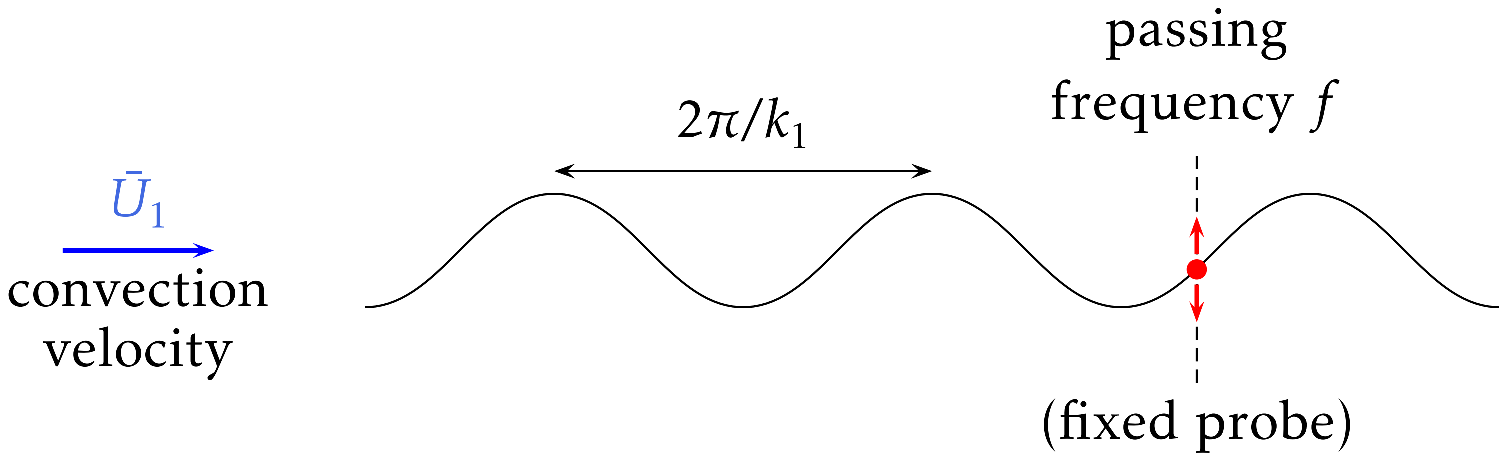
$$\boxed{L_f = \pi \frac{E_{11}^{(1)}(0)}{\overline{u_1'^2}}}$$

## Frozen turbulence approximation or Taylor's hypothesis (1938)

The velocity spectral tensor and the corresponding one-dimensional spectra cannot be directly measured from the Fourier transform of velocity correlation functions in general. Only the time evolution of the velocity in one given point is known, that is $u_1'(t)$.

In order to estimate these spectral functions, it is usually assumed that the turbulent flow is frozen during the measurement, meaning that the observed quantity is simply convected by the local mean flow $\bar{U}_1$, which leads to

$$\frac{\partial}{\partial t} = -\bar{U}_1 \frac{\partial}{\partial x_1} \qquad k_1 = 2\pi f / \bar{U}_1 \qquad \text{Taylor's hypothesis}$$



## Frozen turbulence approximation or Taylor's hypothesis (1938)

**Geoffrey Ingram Taylor** (right) at age 69 (in 1956), in his laboratory with his assistant Walter Thompson *(Physics Today, May 2000)*

At Stanford (1968)

Application to the estimation of $L_1$, $\quad u_1'(t) \rightarrow \Phi_{11}(f) \quad \overline{u_1'^2} \equiv \int_0^{\infty} \Phi_{11}(f)df$

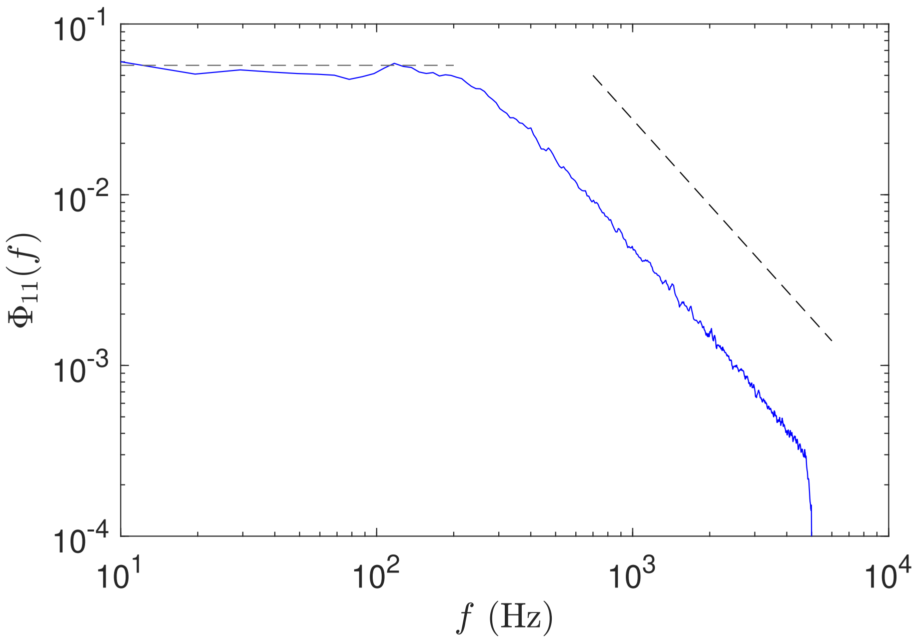
$$\overline{u_1'^2} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1)\; dk_1 \equiv \int_0^{\infty} \frac{\bar{U}_1}{2\pi} \Phi_{11}\left(f = k_1\bar{U}_1/2\pi\right) dk_1$$

$$L_{11}^{(1)} = \pi \frac{E_{11}^{(1)}(k_1 = 0)}{\overline{u_1'^2}} = \frac{1}{4}\, \bar{U}_1 \frac{\Phi_{11}(f = 0)}{\overline{u_1'^2}}$$

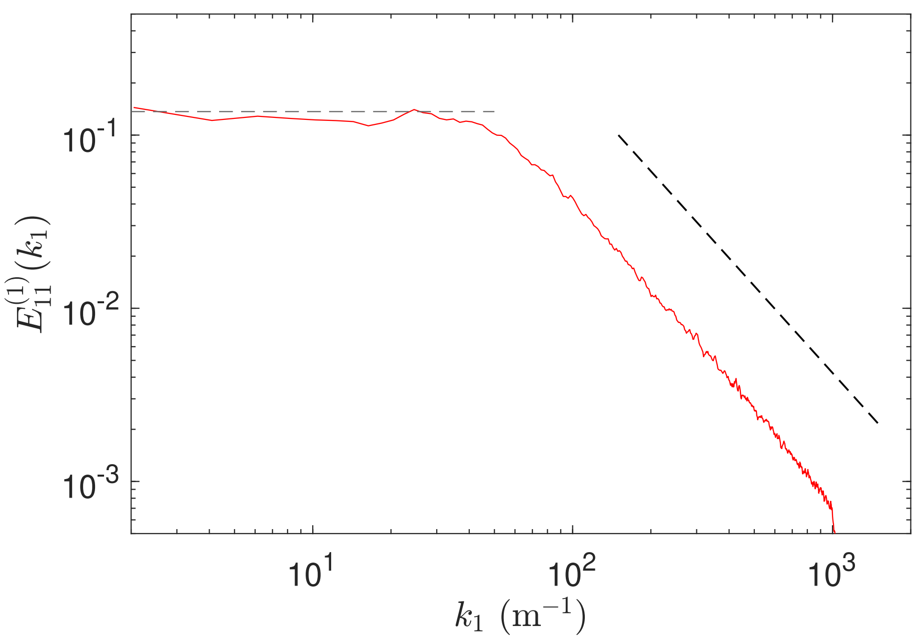
## Frozen turbulence approximation or Taylor's hypothesis (1938)

Spectrum of longitudinal velocity fluctuations
free round jet, $\mathrm{Re}_D \simeq 10^5$, hot-wire located at $x_1 = 2D$ and $x_2 = D/2$
(see also the time correlation function, $\Theta \simeq L_f/\bar{U}_1$)

$\Phi_{11}(f)$ for $f \geq 0$
– – – to estimate $L_f$ and $f^{-5/3}$ law




$E_{11}^{(1)}(k_1)$ for $-\infty \leq k_1 \leq \infty$
– – – to estimate $L_f$ and $k_1^{-5/3}$ law



## Turbulent kinetic energy and dissipation spectra

### Turbulent kinetic energy spectrum

$$k_t = \frac{\overline{u_i' u_i'}}{2} = \frac{1}{2} R_{ii}(\boldsymbol{r} = 0) = \frac{1}{2} \int_{\mathrm{R}^3} \phi_{ii}(\boldsymbol{k})\, d\boldsymbol{k}$$

### Dissipation spectrum

Usually, it is more convenient to first calculate the enstrophy spectrum from the Fourier transform of the vorticity vector, $\hat{\omega}(\boldsymbol{k}) = i\boldsymbol{k} \times \hat{\boldsymbol{u}}(\boldsymbol{k})$. It can be shown that,

$$\frac{\overline{\omega_i' \omega_i'}}{2} = \frac{1}{2} \int_{\mathrm{R}^3} k^2 \phi_{ii}(\boldsymbol{k})\, d\boldsymbol{k}$$

Then, by noting that $\epsilon = \nu \overline{\omega_i' \omega_i'}$, the following expression is obtained from the dissipation spectrum

$$\epsilon = \nu \overline{\omega_i' \omega_i'} = \nu \int_0^\infty k^2 \phi_{ii}(\boldsymbol{k}) d\boldsymbol{k}$$

## Isotropic turbulence

An isotropic turbulent flow is a class of homogeneous turbulent flow whose statistics are invariant under rotation of the coordinate axes and under reflection in a plane.

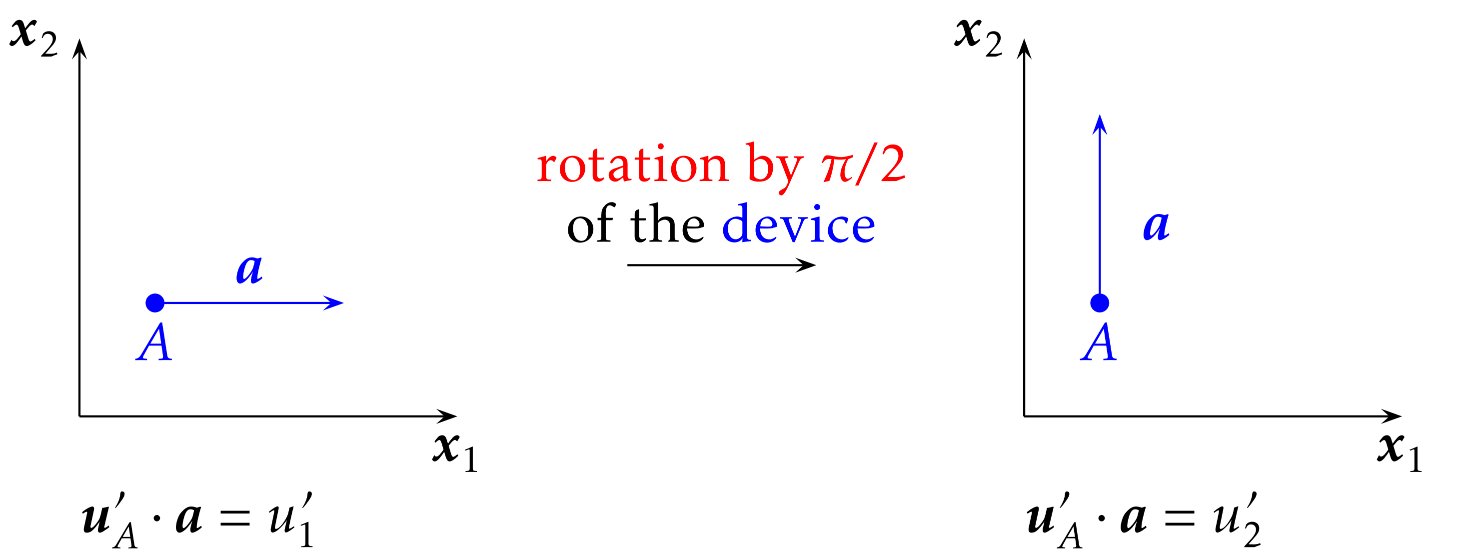
Impossible to distinguish any privileged direction
*a priori*, the most simple configuration ! (ideal theoretical framework)

In order to characterize properties induced by homogeneous and isotropic turbulence, a virtual device is introduced to measure

– a fluctuating scalar quantity : temperature, pressure, ...

– a fluctuating vector quantity : projection on a given unit vector of the turbulent velocity, ...

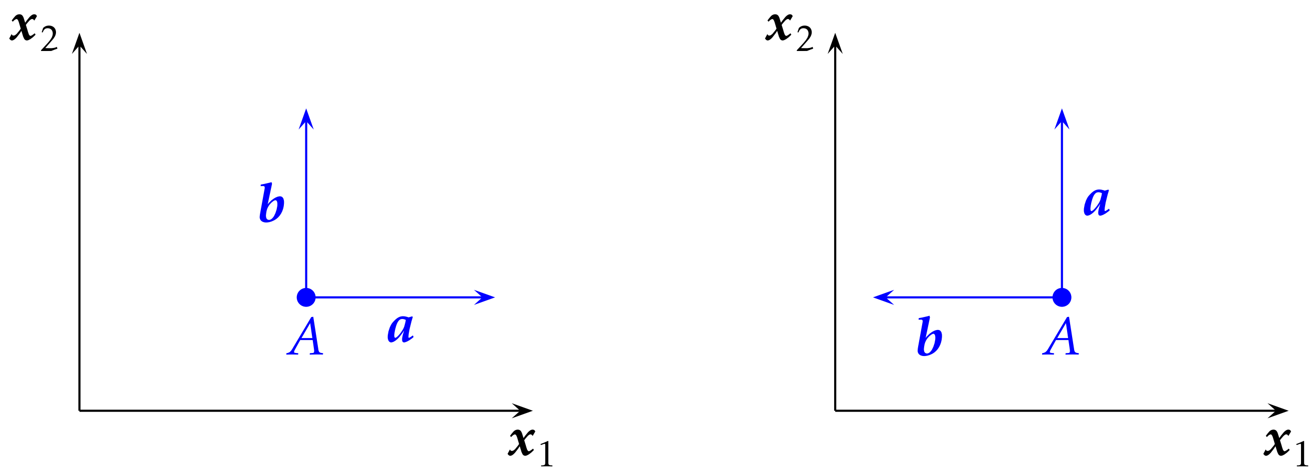
- **Second-order correlation in one point : Reynolds tensor**




The two measurements must be equal for isotropic turbulence, and therefore $\overline{u_1'^2} = \overline{u_2'^2}$. More generally,

$$\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2} = u'^2 \qquad \text{by noting } u' \equiv \left(\overline{u'^2}\right)^{1/2}$$

- **Second-order correlation in one point : Reynolds tensor**




$(\boldsymbol{u}'_A \cdot \boldsymbol{a})(\boldsymbol{u}'_A \cdot \boldsymbol{b}) = u'_1 u'_2$ $\qquad$ $(\boldsymbol{u}'_A \cdot \boldsymbol{a})(\boldsymbol{u}'_A \cdot \boldsymbol{b}) = -u'_1 u'_2$

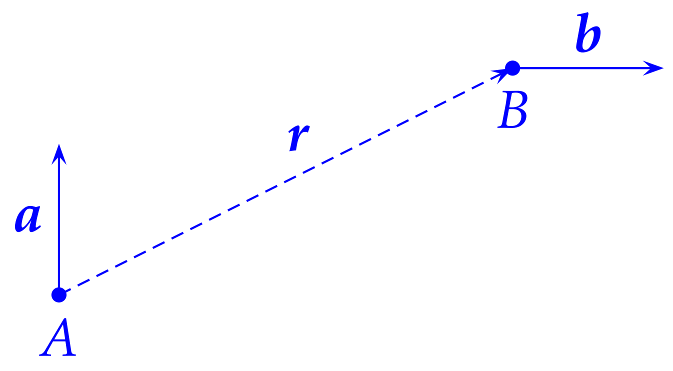
Consequently, $\overline{u'_1 u'_2} = -\,\overline{u'_1 u'_2}$ and $\overline{u'_1 u'_2} = 0$

$$\boxed{\overline{u'_i u'_j} = u'^2\,\delta_{ij} = \frac{2}{3} k_t\,\delta_{ij}}$$

- **Second-order velocity correlation in two points**

$A$ at $\boldsymbol{x}$ and $B$ at $\boldsymbol{x}+\boldsymbol{r}$ :

$$\mathcal{F} \equiv \frac{\overline{(\boldsymbol{u}'_A \cdot \boldsymbol{a})(\boldsymbol{u}'_B \cdot \boldsymbol{b})}}{\sqrt{\overline{\boldsymbol{u}'^2_A}}\sqrt{\overline{\boldsymbol{u}'^2_B}}} = \frac{\overline{u'_{iA}u'_{jB}}(\boldsymbol{r})}{u'^2}\, a_i b_j = \mathcal{R}_{ij}\, a_i b_j$$




The bilinear function $\mathcal{F}$ can only be a function of the invariants associated with the measurement device, that is distances and angles :
$\boldsymbol{r}^2 = r_i r_i$, $\boldsymbol{a}\cdot\boldsymbol{r} = a_i r_i$, $\boldsymbol{b}\cdot\boldsymbol{r} = b_j r_j$, $\boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i = a_i b_j \delta_{ij}$
and also the volume defined by $(\boldsymbol{r},\boldsymbol{a},\boldsymbol{b})$, given by the mixed product
$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{r} = \epsilon_{ijk} a_i b_j r_k$

General expression of an isotropic second-order two-point tensor (Robertson, 1940)

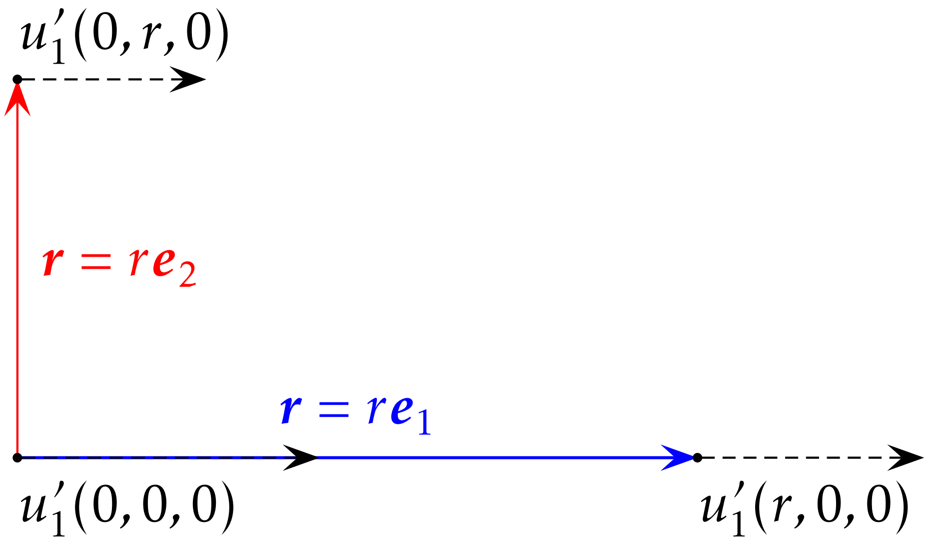
$$\mathcal{R}_{ij}(\boldsymbol{r}) = \alpha(r)\, r_i r_j + \beta(r)\,\delta_{ij}$$

where $\alpha$ and $\beta$ are two scalar functions of $r$.

## Second-order two-point velocity correlation (cont.)

It is generally found more convenient to introduce two functions $f(r)$ and $g(r)$ that can be measured in practice, rather than the two arbitrary functions $\alpha(r)$ and $\beta(r)$. Hence,

$$f(r) \equiv \mathcal{R}_{11}(r,0,0) \quad \text{longitudinal correlation function}$$
$$g(r) \equiv \mathcal{R}_{11}(0,r,0) \quad \text{transverse correlation function}$$




Kármán & Howarth (1938)

$$\mathcal{R}_{ij}(\boldsymbol{r}) = (f-g)\frac{r_i r_j}{r^2} + g\delta_{ij}$$

Take care of $R_{ij}(\boldsymbol{r}) = u'^2\mathcal{R}_{ij}(\boldsymbol{r})$

- **Compressibility condition** applied to the second-order two-point velocity correlation recast by Kármán & Howarth

$$\frac{\partial \mathcal{R}_{ij}(\boldsymbol{r})}{\partial r_i} = 0 \qquad \Longrightarrow \qquad \frac{\partial}{\partial r_i}\left[\frac{f-g}{r^2} r_i r_j + g\delta_{ij}\right] = 0$$

which leads for a 3-D turbulence to the following expression (the details are left as an exercise),

$$g = f + \frac{r}{2} f' = \frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{2} f\right)$$

The correlation coefficient $\mathcal{R}_{ij}$ is determined by a single scalar function, the longitudinal autocorrelation in space $f(r)$, for incompressible isotropic turbulence.

- **Turbulent kinetic energy and dissipation spectra**

Using a similar approach applied now to the spectral tensor $\phi_{ij}(\boldsymbol{k})$, and taking account for the incompressibility condition, it can be shown that only one scalar function $E(k)$ is required to specify $\phi_{ij}(\boldsymbol{k})$, that is

$$\phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \qquad \text{with} \qquad k_t \equiv \int_0^\infty E(k)\, dk$$

The expression of the dissipation spectrum is then deduced from the relationship established for homogeneous turbulence, see here,

$$\epsilon = 2\nu \int_0^\infty k^2 E(k)\, dk$$

## Isotropic turbulence

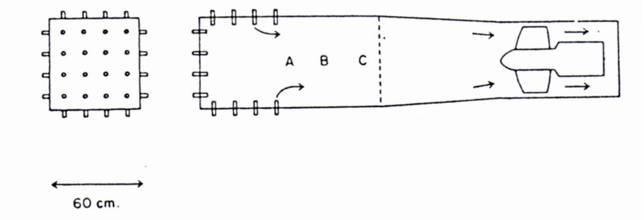
Many other remarkable results can be established for homogeneous and isotropic turbulence : refer to textbooks mentioned in the introduction of this course.

Three points must be however still considered to provide a first full overview of isotropic turbulence

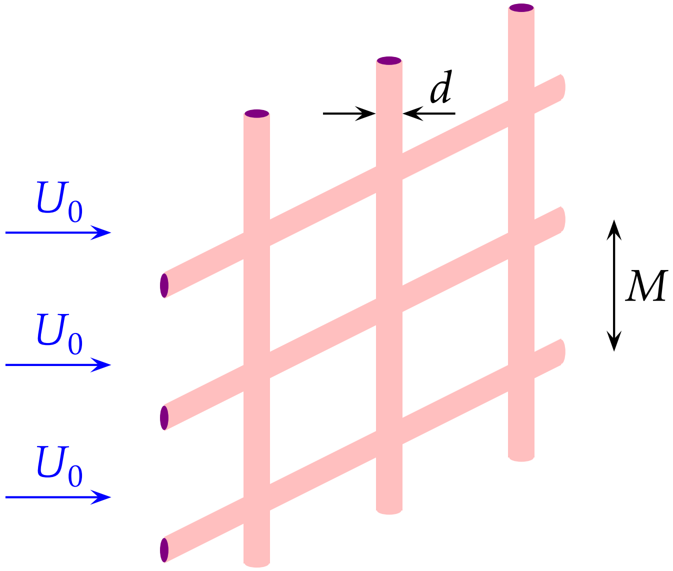
- How to generate isotropic turbulence in laboratory ?
- What is the time evolution of isotropic turbulence ?
- Can we measure or derive analytically the expression of $E(k)$ ?

## Isotropic turbulence in laboratory

Various configurations have been investigated to generate isotropic turbulence. One of the most famous is the so-called "Porcupine" by Betchov (1957)



The ' Porcupine '. The mixing of 80 small jets produces a strong turbulence in the region marked $A$, $B$, $C$.




Turbulence behind a grid, homogeneous but not fully isotropic turbulent flow

$$\overline{u_1'^2} = 1.2\, \overline{u_2'^2} = 1.2\, \overline{u_3'^2}$$

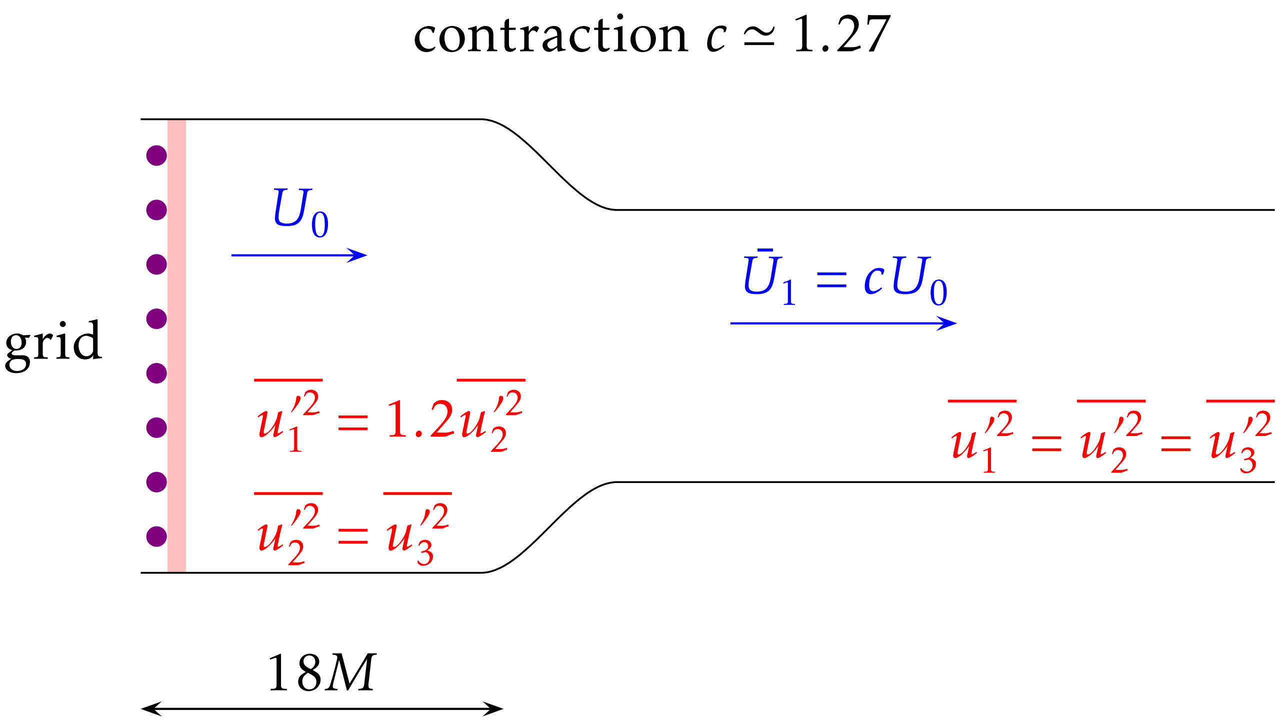
and one typically gets for turbulence intensity

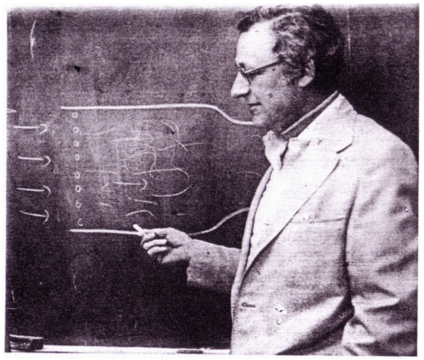
$$\frac{u'}{U_0} \simeq 2\% \qquad \mathrm{Re}_M = \frac{U_0 M}{\nu} \simeq 10^4 \text{ to } 10^5$$

## Isotropic turbulence in laboratory (cont.)

Experiences by Comte-Bellot & Corrsin at Johns Hopkins University

*J. Fluid Mech.*, 1966, **25**(4) & 1971 **48**(2)



## Stanley Corrsin

# Hopkins researcher finds fascination in turbulence

By Albert Sehlstedt, Jr.

Stanley Corrsin is a specialist in turbulence, a very complex scientific problem subject that deals with airplanes flying through the clouds, curling cigarette smoke rising under a lampshade and blood flowing through human bodies.

Explaining these seemingly commonplace occurrences poses a problem that has puzzled scientists for decades.

"It is sufficiently difficult [a subject] that the problem is not likely to be solved in my lifetime," Dr. Corrsin observed in his Maryland Hall office on the Homewood campus of the Johns Hopkins University.

"That means I'm not in danger of being unemployed," the 62-year-old scientist added with a smile. "Also, I think it is aesthetically interesting. Turbulent flows make beautiful pictures."

Turbulent flows are movements of matter in which the velocity at a

**"Stan is not only a person who himself has contributed [through research], but his discourses have been stimulating to other people."**

**— Lawrence Talbot Berkeley professor**

Stanley Corrsin, winner of the American Physical Society's 1983 Fluid Dynamics Prize, is respected as both researcher and teacher.

critical review of fluid dynamics," which "have touched a legion of students and associates."

"Stan is not only a person who himself has contributed, but his discourses have been stimulating to other people," said Lawrence Talbot,

decision-making that he would rather not undertake.

Better to talk of airplanes, soaring albatrosses, flowing water — and swallowing.

There is a "swallowing center," a complex assembly of muscles and

called non-uniform surface tension may be the answer, he said.

There is also the question of why contact lenses stay attached to the surface of the eye. Dr. Corrsin and his colleagues examined this mystery, too, but he said, "We never did

who was helping to edit a monograph on jet propulsion at a place that has since become famous for guiding spacecraft to the planets — the Jet Propulsion Laboratory in Pasadena, Calif.

Dr. Corrsin said he chose Hopkins

## Decaying isotropic turbulence

In a frame moving with the mean velocity,

- Decay of the normal stresses

$$\frac{\overline{u'^2}}{U_0^2} = \frac{1}{A}\left(\frac{tU_0}{M} - \frac{t_0U_0}{M}\right)^{-n} \qquad \text{with} \qquad n \simeq 1.3$$

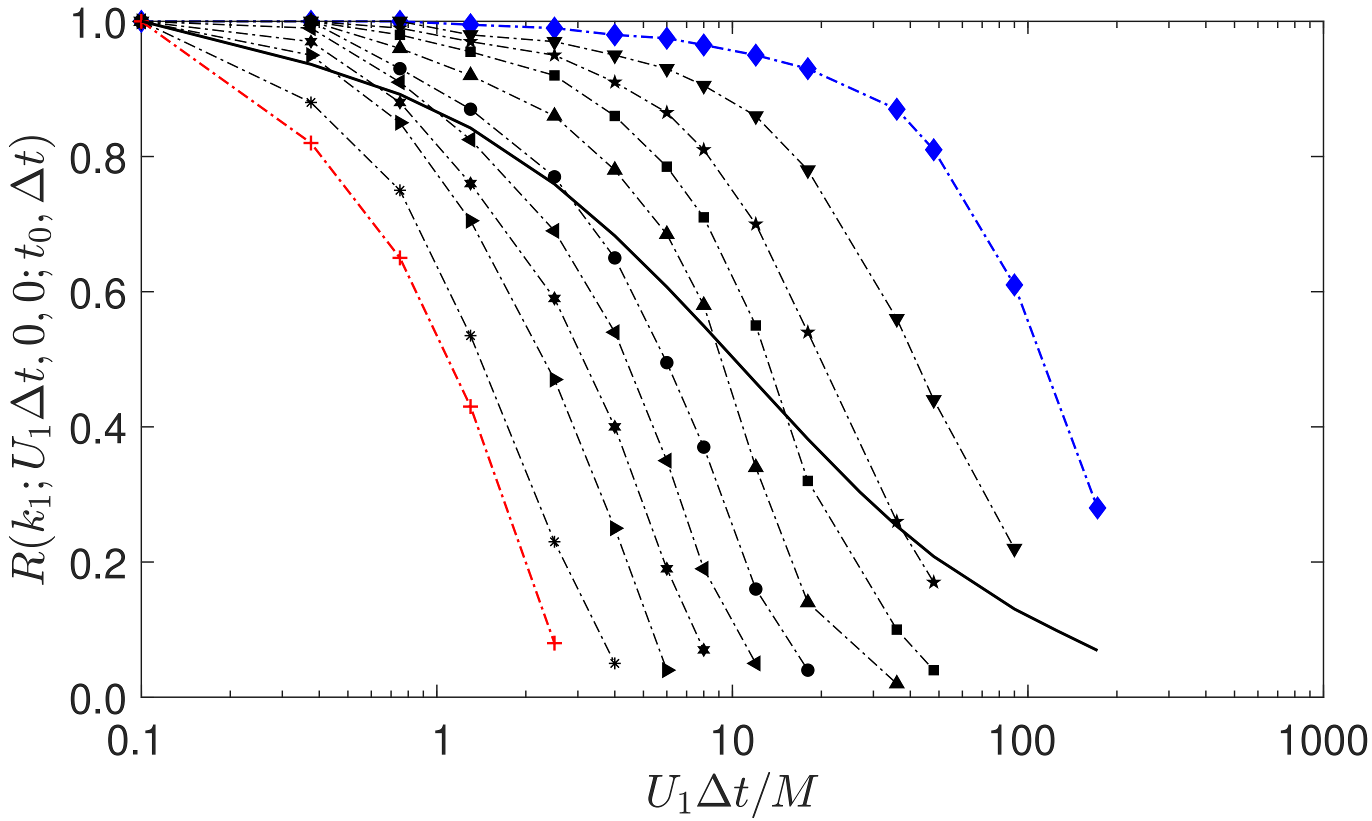
Comte-Bellot & Corrsin (1966), Mohamed & Larue (1990)

- The dissipation rate of the turbulent kinetic energy is imposed by larger turbulent structures,

$$\boxed{\epsilon \simeq \frac{u'^3}{L_f}} \qquad\qquad \frac{\partial k_t}{\partial t} = -\epsilon \sim \frac{u'^2}{L_f/u'}$$

where $L_f$ is the longitudinal integral length scale, $L_f = \int_0^\infty f(r)dr$

## Decaying isotropic turbulence

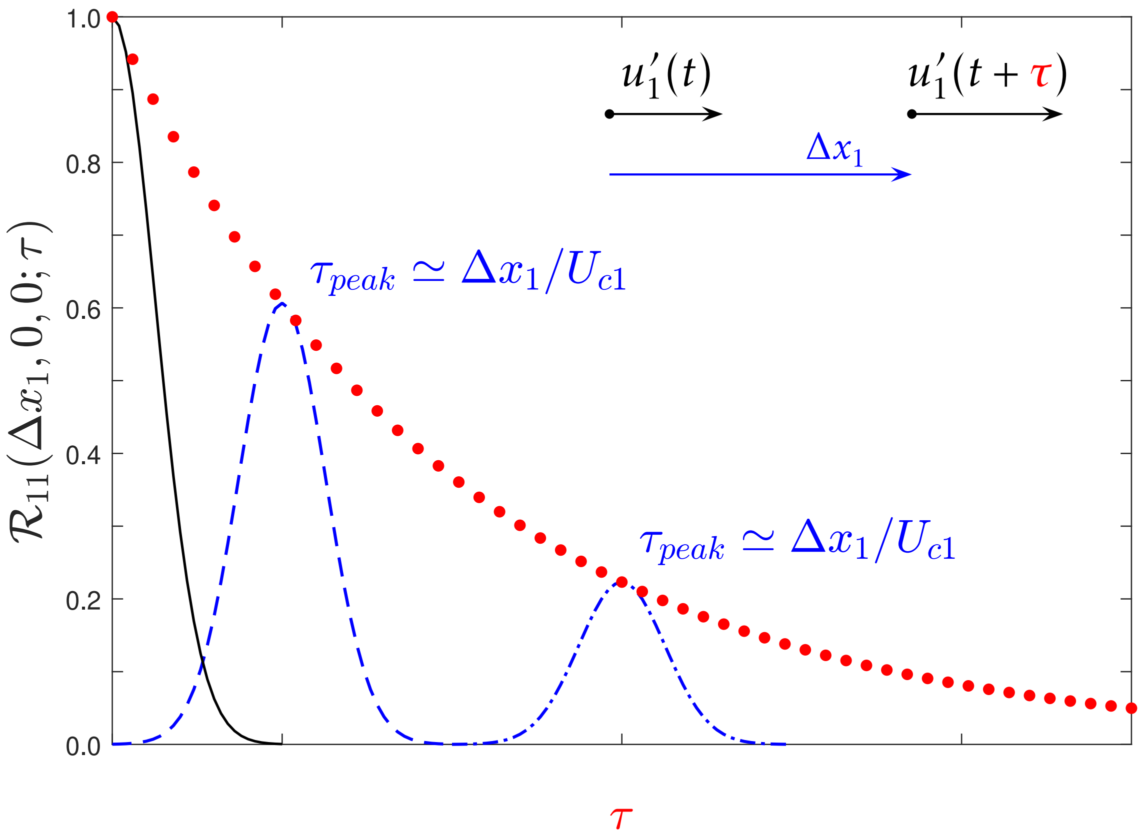



Time correlation in a frame travelling with the mean velocity $\bar{U}_1$ for different values of the wavenumber, from $k_1 = 0.25$ cm$^{-1}$ (♦) to $k = 10.10$ cm$^{-1}$ (+)

—— total signal (full-band case)

## Space-time correlations

$$\mathcal{R}_{11}(\Delta x_1, 0, 0; \tau) = \overline{u_1'(\boldsymbol{x}, t)\, u_1'(\boldsymbol{x} + \Delta x_1 \boldsymbol{e}_1, t + \tau)} / \overline{u_1'^2}$$




— time autocorrelation function ($\Delta x_1 = 0$), which provides the integral time scale in the fixed frame $\Theta_1 \sim L_f / U_{c1}$ (Taylor)

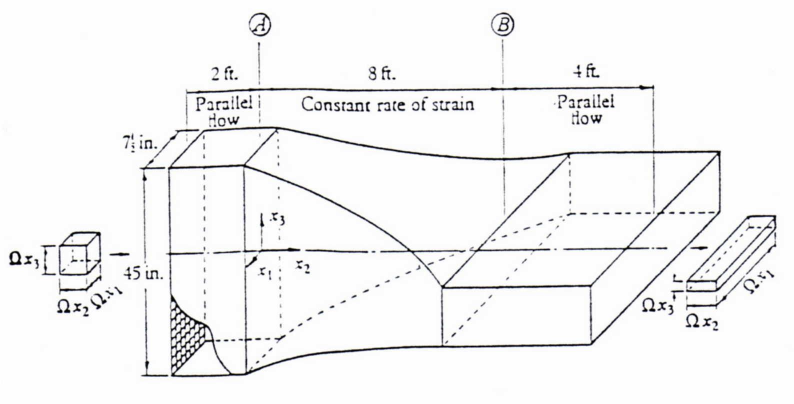
- - - time correlation for a given separation $\Delta x_1$ of the two probes

— time autocorrelation function in the convected frame

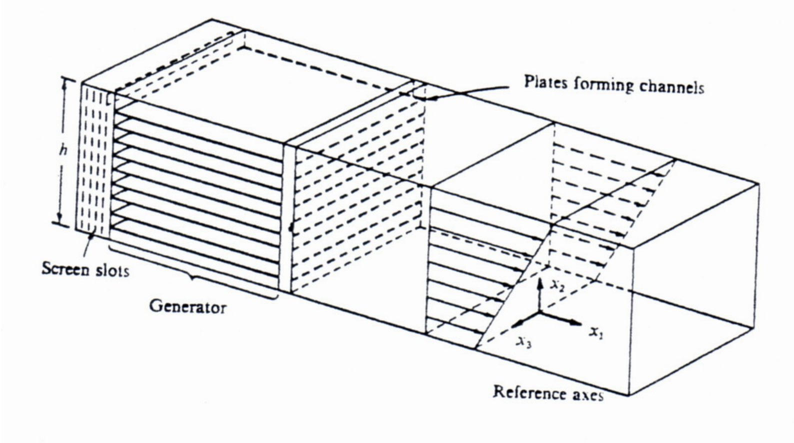
$$\Theta_{c1} = \int_0^\infty \mathcal{R}_{c11}(\tau)\, d\tau$$

$\Theta_{c1} \sim L_f / u_1'$ represents the time characterizing the loss of coherence or the memory time of turbulence

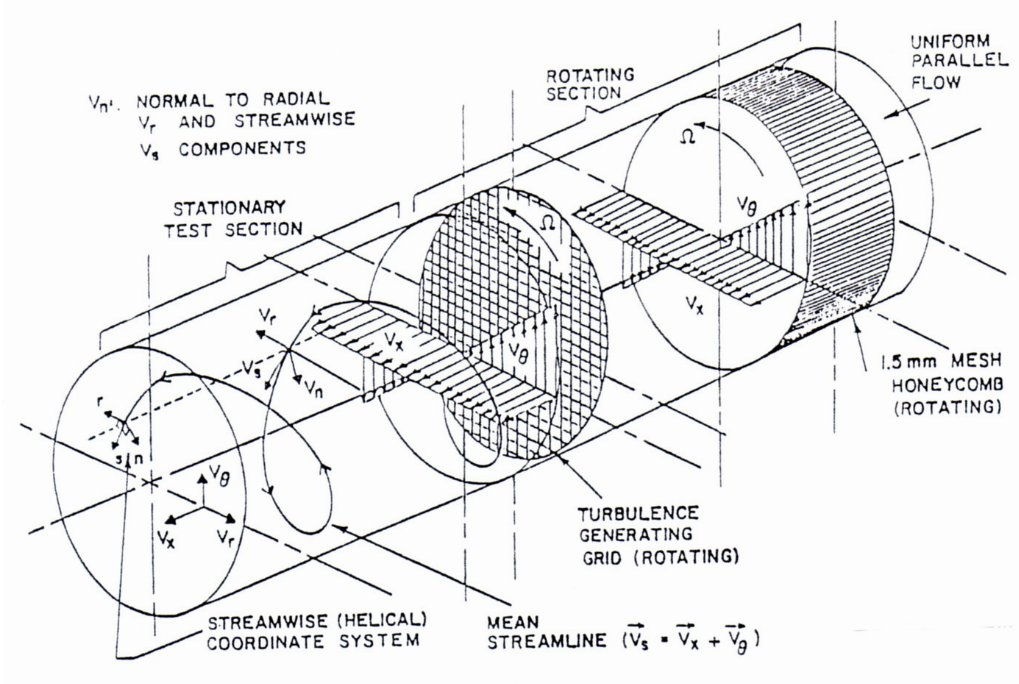
## Isotropic turbulence submitted to ...




Tucker & Reynolds - Plane strain

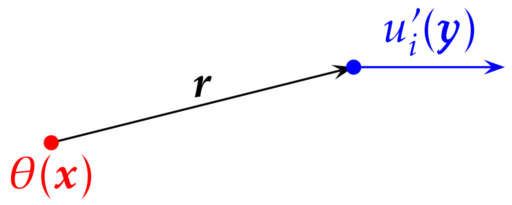



Champagne et al. - Sheared mean flow




Wigeland & Nagib - Solid body rotation

## Exercise #1




Correlation between temperature and a velocity component in two points $\boldsymbol{x}$ and $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{r}$ where $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$ is the separation vector

1. Expression of the two-point correlation $\overline{\theta(\boldsymbol{x})u_i'(\boldsymbol{y})}$ for isotropic turbulence ?
2. Can we generalize the previous result for any scalar quantity ? (temperature, pressure, concentration, ...)

## Exercise #2

Scales

**1.** Show from Kármán & Howarth's relation, that for 3-D incompressible turbulence,

$$g = f + \frac{r}{2} f'$$

**2.** Deduce that $L_f = 2L_g$ and that $\lambda_f = \sqrt{2}\lambda_g$, by noting that

$$\frac{1}{\lambda_f^2} = -\frac{1}{2} f''(0) \qquad \frac{1}{\lambda_g^2} = -\frac{1}{2} g''(0)$$

**3.** Deduce the two followwing addtional expressions of dissipation,

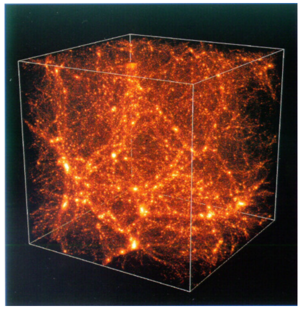
$$\epsilon = \frac{15}{2}\nu \overline{\left(\frac{\partial u_1'}{\partial x_2}\right)^2} = 15\nu \overline{\left(\frac{\partial u_1'}{\partial x_1}\right)^2}$$

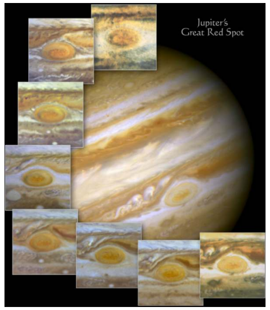
## Exercise #3

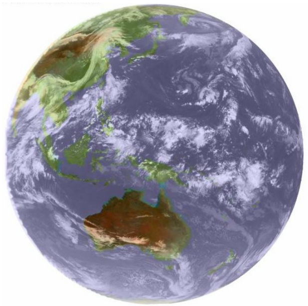
Cumulus clouds : the length scale of the large eddies is about 250 m and the fluctuating velocity is 1 m.s$^{-1}$

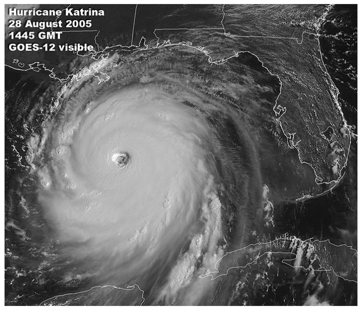
1. Estimate the energy dissipation rate in a cumulus cloud, both per unit mass and for the entire cloud (from Tennekes & Lumley, 1972). Compute the total dissipation rate in kilowatts. Also estimate the Kolmogorov scale. Compare with the power received at the surface of the Earth from the Sun.

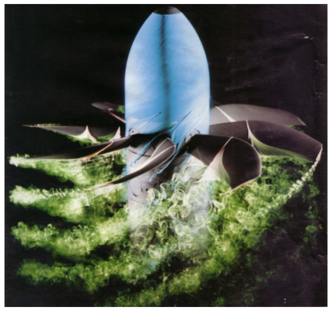
# Dynamics of isotropic turbulence – Kolmogorov’s theory

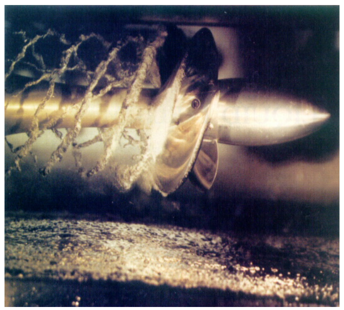






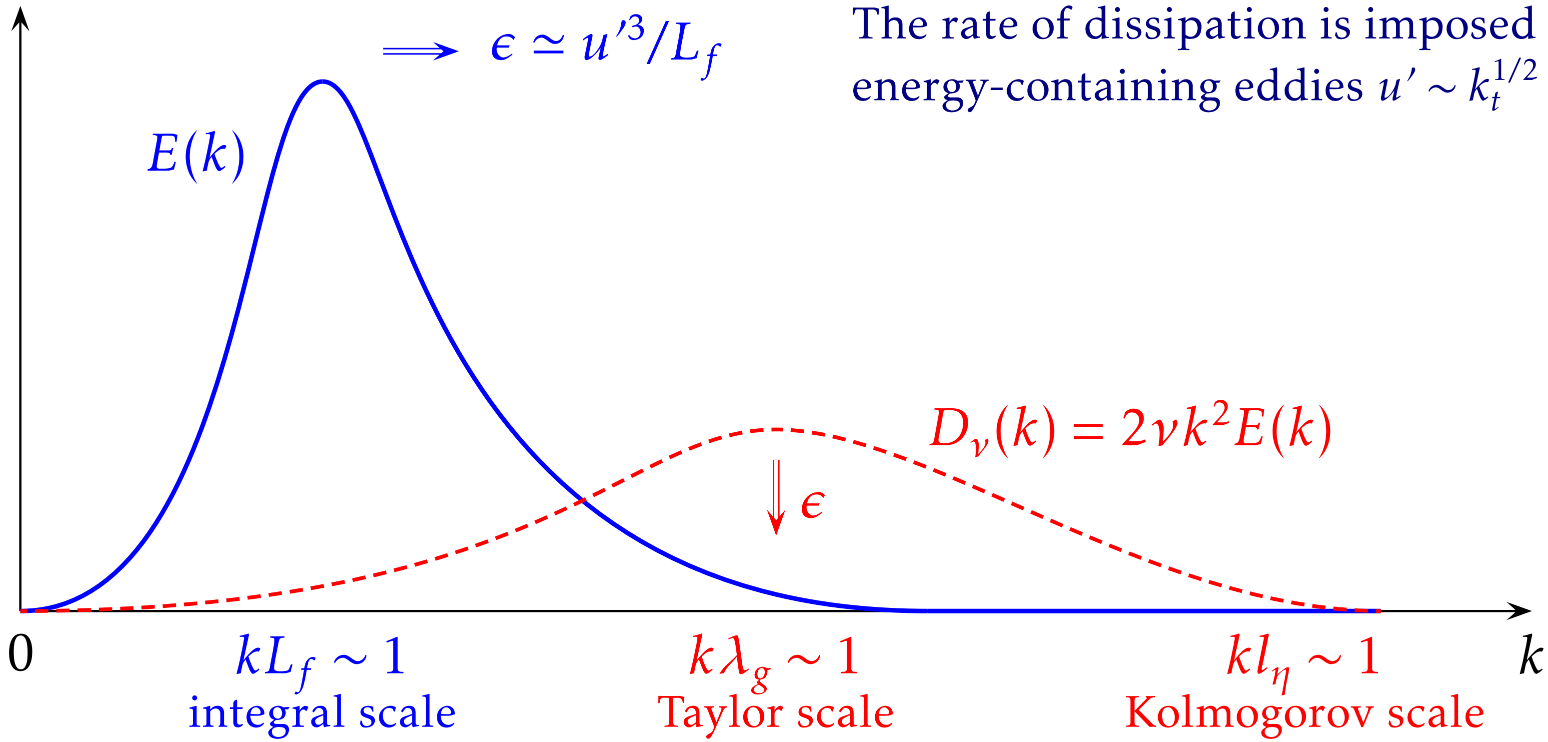






## Introduction

The spectrum of turbulent kinetic energy is the key function for isotropic turbulence. Can we determine the form of $E(k)$ and its time evolution ?

The rate of dissipation is imposed by larger structures, energy-containing eddies $u' \sim k_t^{1/2}$ time $\sim L_f/u'$




viscous scales ($\mathrm{Re}_\eta = 1$)

$l_\eta = \nu^{3/4}\epsilon^{-1/4} \qquad u_\eta = \nu^{1/4}\epsilon^{1/4}$

## Energy cascade

The higher the Reynolds number is, the more spectra of the kinetic energy and dissipation will be separated : fully developed turbulence.

$$\frac{L_f}{l_\eta} = \frac{L_f}{\nu^{3/4}\epsilon^{-1/4}} = \left(\frac{u'L_f}{\nu}\right)^{3/4} = \mathrm{Re}_{L_f}^{3/4} \qquad \mathrm{Re}_{L_f} \equiv \frac{u'L_f}{\nu}$$

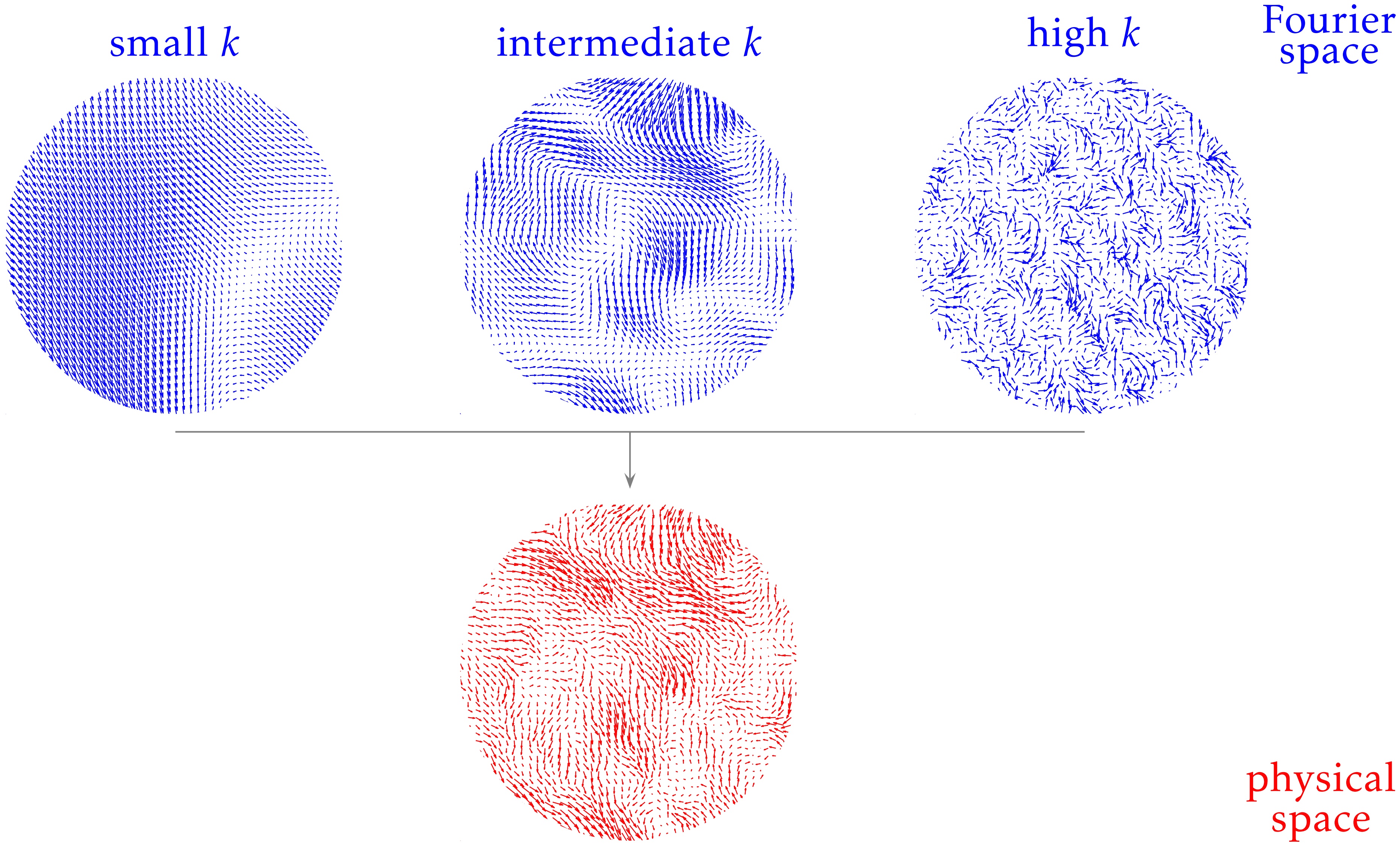
Reynolds number of large structures

Kolmogorov (1941) – energy cascade

The dissipation rate $\epsilon$ is imposed by large eddies, but carries out by the smallest ones (at Kolmogorov scales), it can be argued as assumptions that

- the dissipation rate $\epsilon$ is finite, even when $\mathrm{Re} \to \infty$,
- there is a self-similar dynamics ; velocity scale of an eddy of size $l$ varies as $u_l \sim l^p$ (that is a power law)

## Representation in spectral space



## Representation in spectral space (cont.)

For isotropic turbulence, the turbulent kinetic energy spectrum $E(k)$ is decomposed over spheres of radius $k$, with elementary turbulent structures of wavenumber $k$ as already discussed in the Introduction Chapter.

For exponential spectra, this will be the case for $E(k)$, it is interesting to introduce a linear representation in logarithmic scale. For a geometric sequence $k_n$,

$$\frac{k_n}{k_{n-1/2}} = a = \frac{k_{n+1/2}}{k_n} \qquad \Delta k_n = k_{n+1/2} - k_{n-1/2}$$

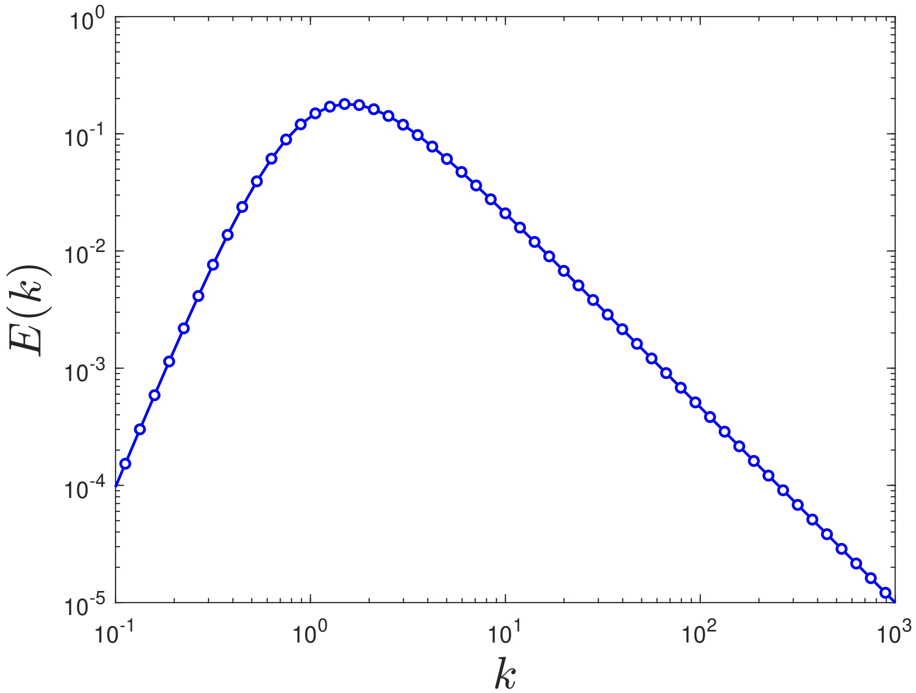
and it is always possible to choose the common ratio $a$ such as $\Delta k_n / k_n = 1$.
With a constant bandwidth for $d \ln k = dk/k$,

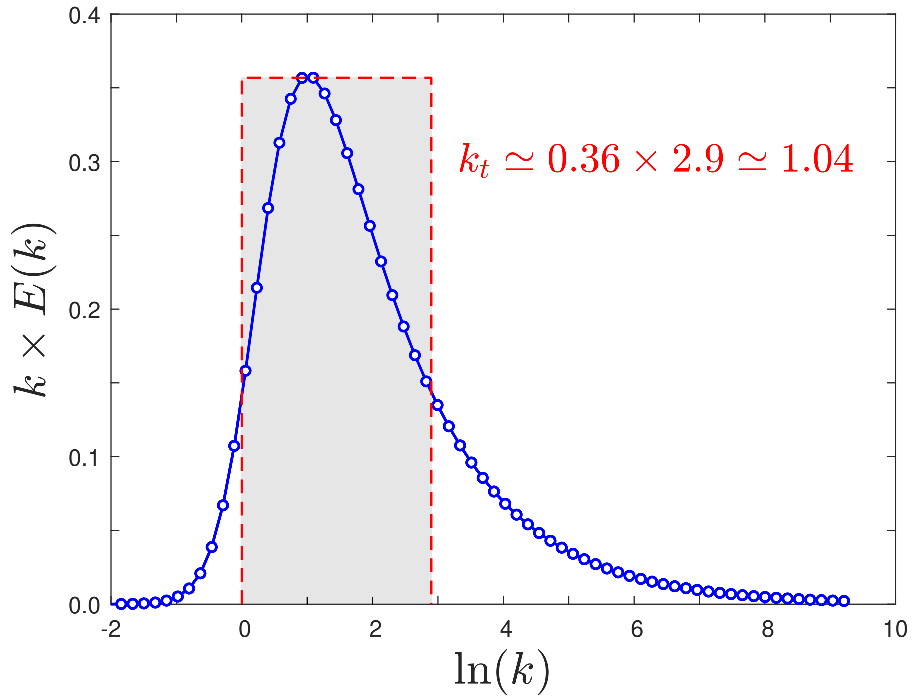
$$\int_{k_{n-1/2}}^{k_{n+1/2}} E(k)\, dk = \int_{k_{n-1/2}}^{k_{n+1/2}} kE(k)\, d\ln k \sim k_n E(k_n)$$

In the same way, the importance of frequency weighted spectra or compensated spectra is underlined for exponential form.

## Benefit of frequency weighted spectrum

equal areas = equal contributions using log-axes



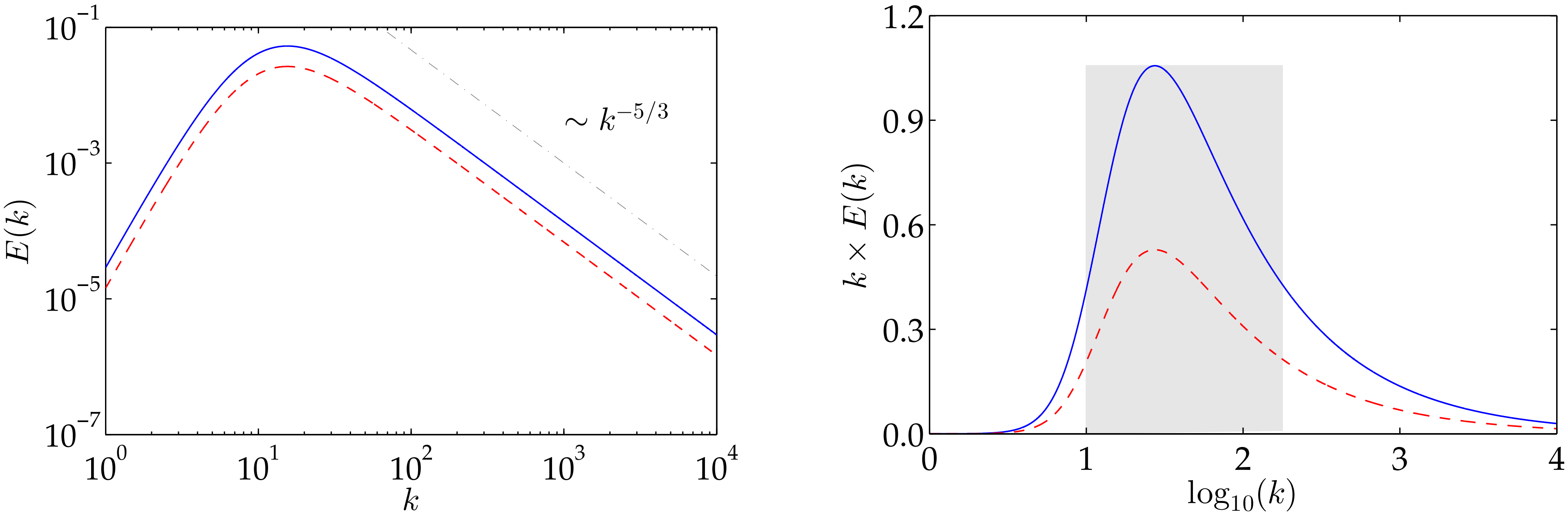




$$k_t = \int_0^\infty E(k)\, dk = \int_0^\infty kE(k)\, d\ln k$$

von Kármán spectrum model,

$$E(k) = \frac{k^4}{(1+k^2)^{17/6}} \qquad k_{t\,\mathrm{exact}} \simeq 1.0325$$

## Benefit of frequency weighted spectrum (cont.)




von Kármán spectrum (arbitrary units here)

—— for $k_t = 3$    - - - for $k_t = 1.5$

log-log scales (to observe the $-5/3$ law) versus $k \times E(k)$ on linear scales

On the right, area of the grey rectangle, $1.25 \times \ln(10) \times 1.05 \simeq 3$

(error detection is straightforward)

## Theory of Kolmogorov – K41

- Eddy of size $l$ and of velocity $u_l$,
  eddy-life time or turn-over time $t_l \sim l/u_l$

$$\frac{u_l^2}{l/u_l} = \text{cst} = \epsilon \qquad \Longrightarrow \qquad u_l \sim (\epsilon l)^{1/3}$$

- Kinematic energy $\mathcal{E}_l$ associated with eddies of size $l \sim 1/k_l$
  $\mathcal{E}_l \sim u_l'^2 \sim (\epsilon l)^{2/3}$

- Turbulent kinetic energy spectrum $\mathcal{E}_l \sim k_l E(k_l)$, and thus

$$E(k_l) \sim \frac{\epsilon^{2/3} k_l^{-2/3}}{k_l} \sim \epsilon^{2/3} k_l^{-5/3} \qquad \text{Kolmogorov's law}$$

Inertial subrange between $kL_f$ and $kl_\eta$ at high-Reynolds number,
where $E(k,\epsilon,\nu) = E(k,\epsilon)$

## Theory of Kolmogorov – K41 (cont.)

In the inertial subrange (fine turbulence structures) of the turbulent kinetic energy spectrum, there is a universal spectrum shape

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \quad C_K \simeq 1.5$$

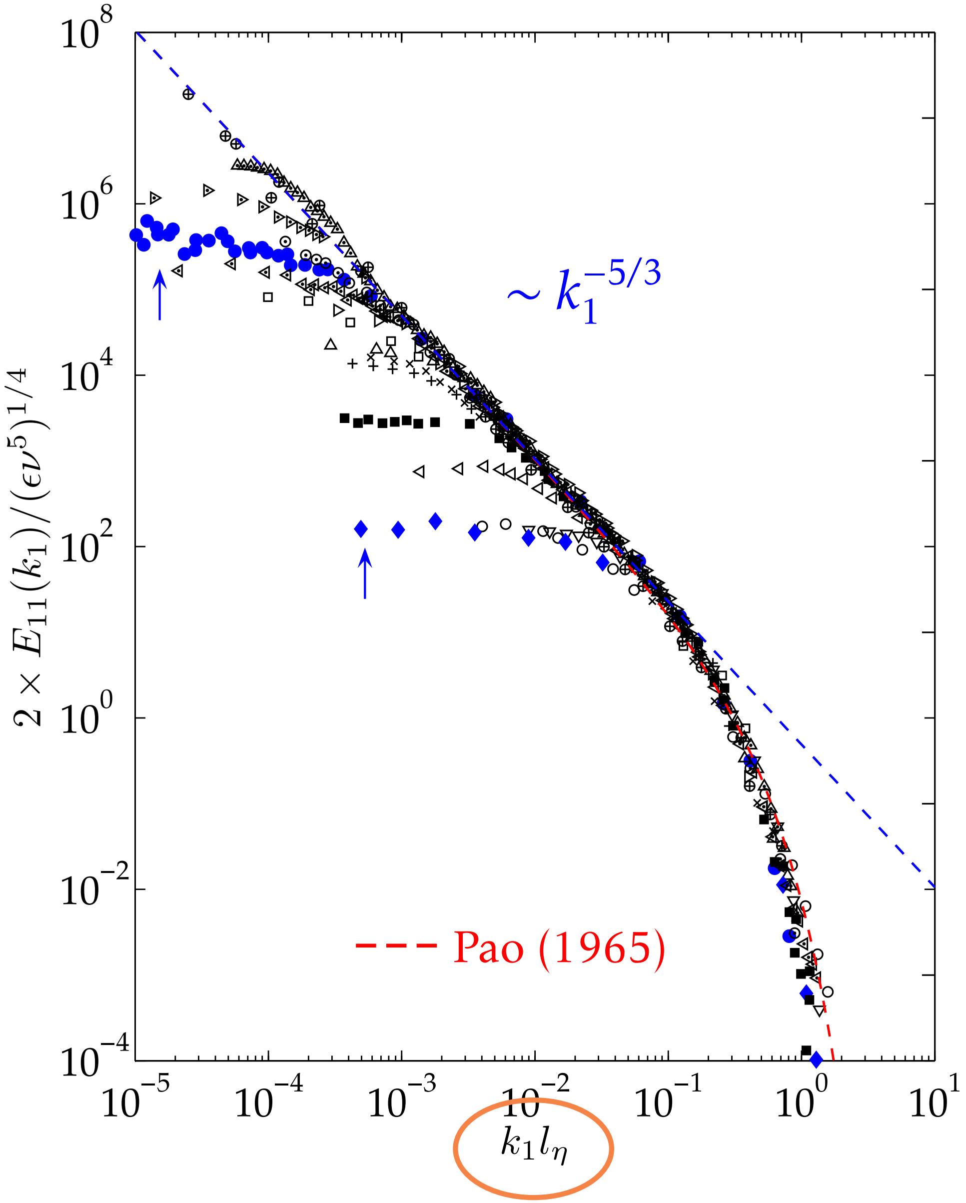
where $C_K$ is the Kolmogorov constant, and this region widens as the Reynolds number increases.

The original formulation by Kolmogorov (1941, 1962) is based on structure functions, see exercises. The theory is still debated, *e.g.* self-similarity at small scales, intermittence in the dissipation process.

The concept of energy cascade has been introduced by Richardson (1922), and first developed theoretically by Kolmogorov (1941) and also Obukhov (1941).

Arnold Kolmogorov (1903-1987)

## Theory of Kolmogorov – Experimental evidence



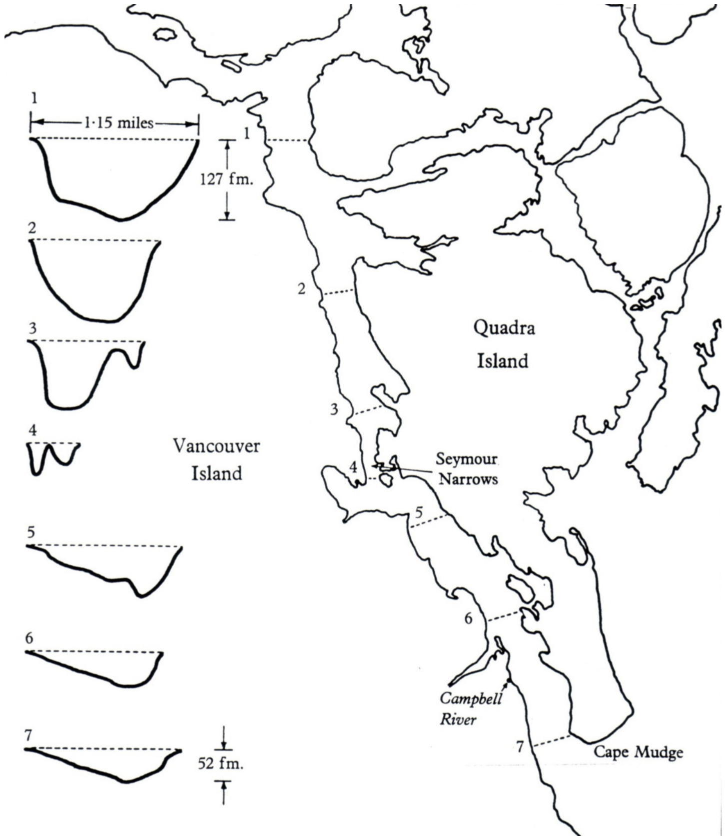

$$\overline{u_1'^2} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1)\,dk_1$$

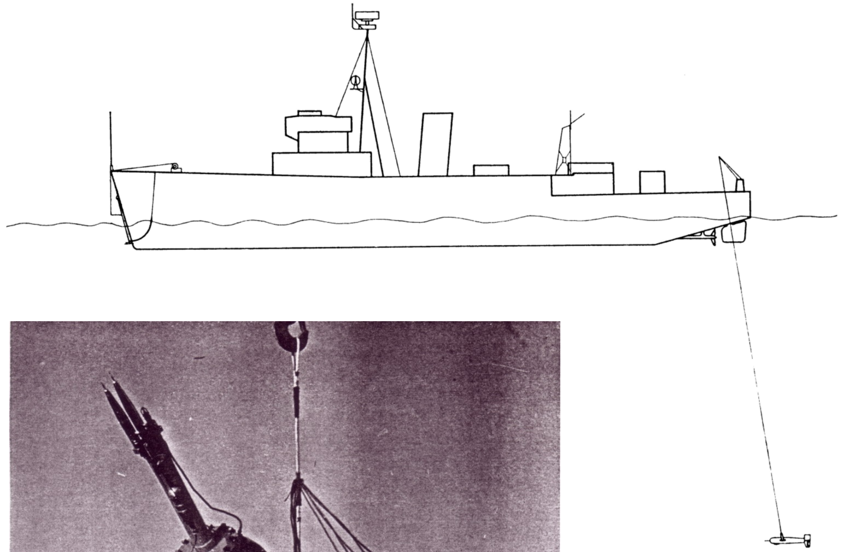
$Re_{\lambda_g}$

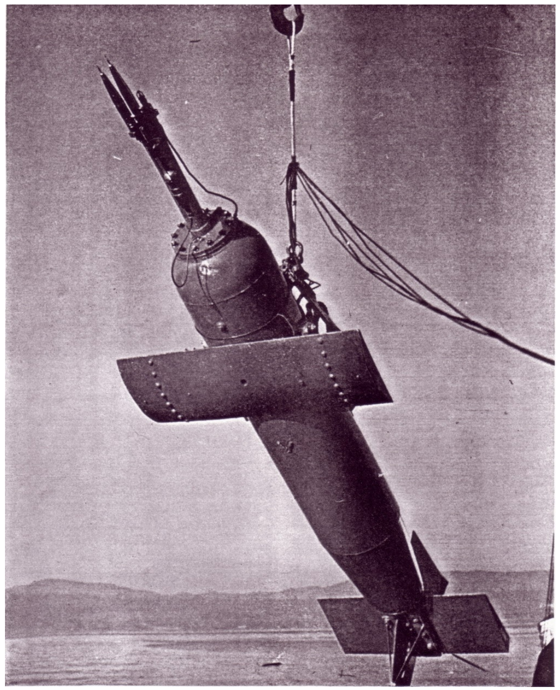
| | | |
|---|---|---|
| ○ | 23 | boundary layer (Tielman, 1967) |
| ♦ | 23 | cylinder wake (Uberoi & Freymuth, 1969) |
| ▽ | 37 | grid turbulence (Comte Bellot & Corrsin, 1971) |
| ◁ | 72 | grid turbulence (Comte Bellot & Corrsin, 1971) |
| ■ | 130 | homogeneous shear flow (Champagne *et al.*, 1970) |
| + | 170 | pipe flow (Laufer, 1952) |
| × | 282 | boundary layer (Tielman, 1969) |
| □ | 308 | cylinder wake (Uberoi & Freymuth, 1969) |
| △ | 401 | boundary layer (Sanborn & Marshall, 1965) |
| ▷ | 540 | grid turbulence (Kistler & Vrebalovich, 1966) |
| ◁ | 600 | boundary layer (Saddoughi, 1994) |
| ⊙ | 780 | round jet (Gibson, 1963) |
| ● | 850 | boundary layer (Coantic & Favre, 1974) |
| ▷ | 1500 | boundary layer (Saddoughi, 1994) |
| ⊕ | 2000 | tidal channel (Grant *et al.*, 1962) |
| △ | 3180 | return channel (CAHI Moscou, 1991) |

## Theory of Kolmogorov – Experimental evidence

Measurements of Grant, Stewart & Moilliet (1962)

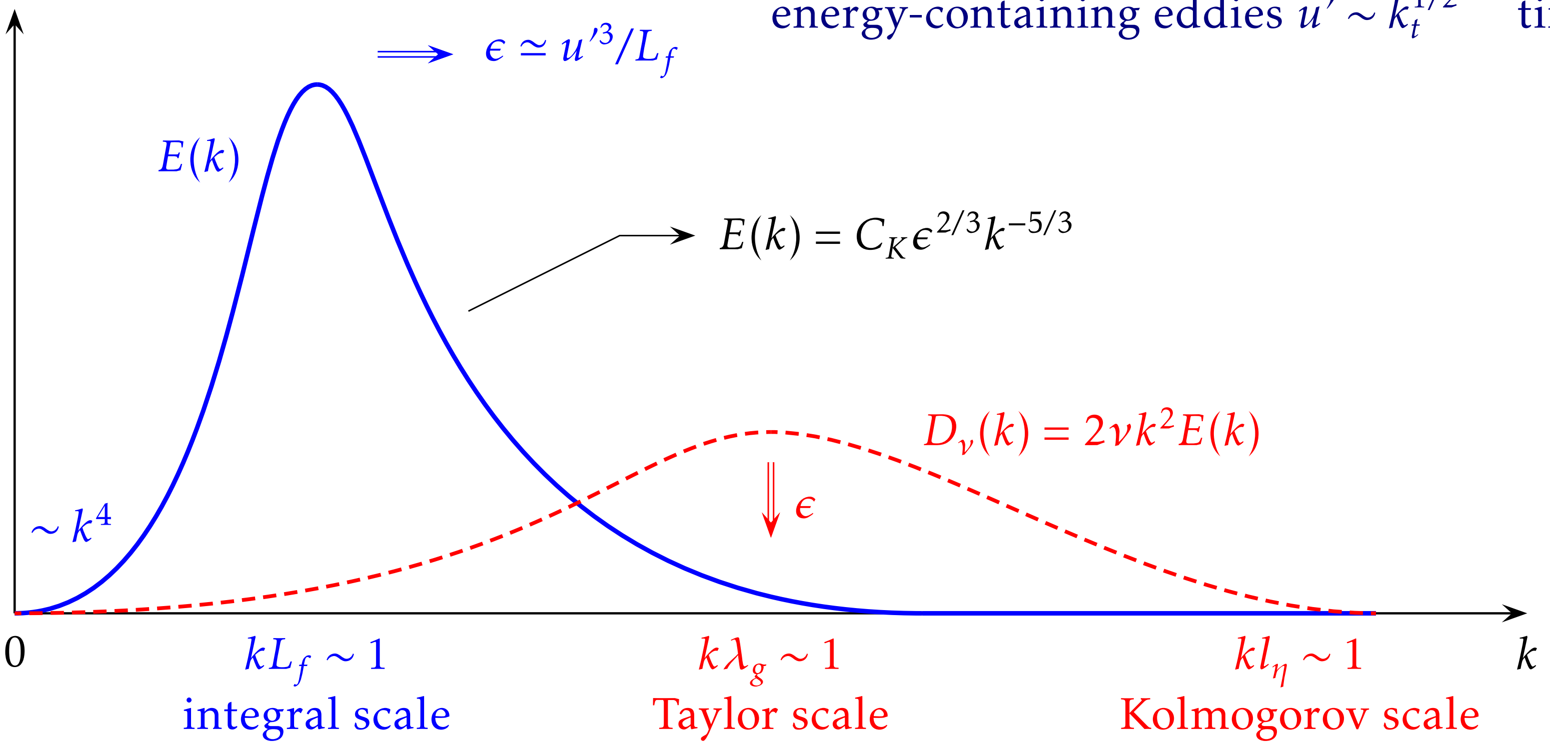




## In summary

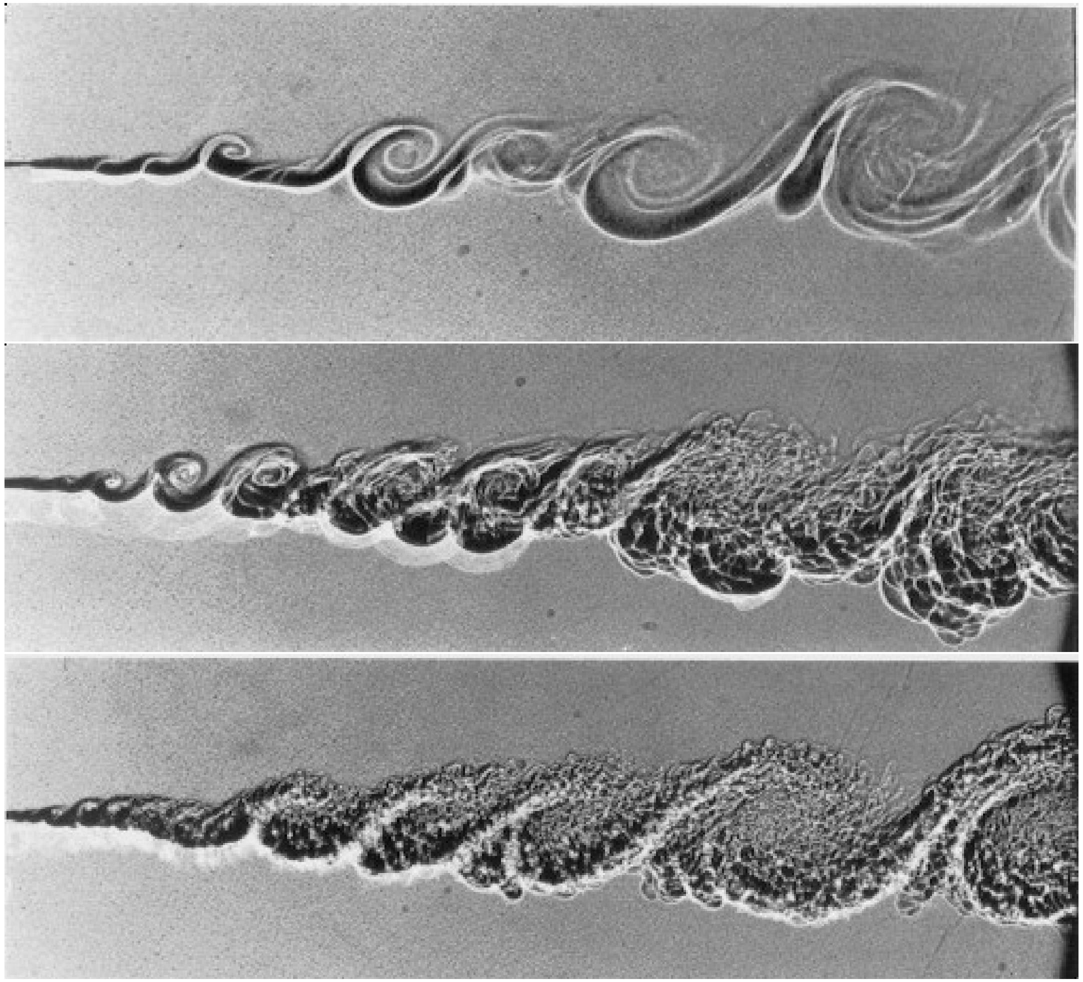
The rate of dissipation is imposed by larger structures,
energy-containing eddies $u' \sim k_t^{1/2}$ time $\sim L_f/u'$




viscous scales ($\mathrm{Re}_\eta = 1$)
$l_\eta = \nu^{3/4}\epsilon^{-1/4}$ $u_\eta = \nu^{1/4}\epsilon^{1/4}$

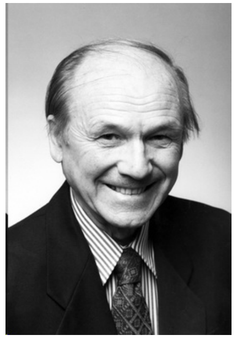
$$L/l_\eta \sim \mathrm{Re}_{L_f}^{3/4}$$

- **Energy cascade in a turbulent mixing layer (Brown & Roshko, 1974)**
  Shadowgraphs (spark source)



Energy cascade in a mixing layer by increasing the Reynolds number (through pressure and velocity, ×2 for each view)

More small-scale structures are produced without basically altering the large-scale ones



Anatol Roshko (1923-2017)

- **Lin's equation (1947)**
  Transport equation for the turbulent kinetic energy spectrum $E(k)$

  A way to derive this equation is

  - to consider the transport equation for the Reynolds tensor $R_{ij} = \overline{u_i'(\boldsymbol{x})u_j'(\boldsymbol{x}+\boldsymbol{r})}$, known as the Kármán & Howarth equation
  - to take its Fourier transform and to contract subscripts as follows $i = j$

  … which gives

$$\boxed{\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2\nu k^2 E(k,t)}$$

  where the nonlinear term $T(E)$ is linked to the third-order (triple) velocity correlation. This term can be directly associated with the energy transfer between turbulent structures of different size.

## Lin's equation (cont.)

In order to illustrate this point, Lin's equation can be integrated over all the wavenumbers $k$

$$\underbrace{\frac{\partial}{\partial t}\int_0^\infty E(k,t)\,dk}_{=\partial k_t/\partial t} = \int_0^\infty T(k,t)\,dk - \underbrace{2\nu\int_0^\infty k^2 E(k,t)\,dk}_{=\epsilon}$$

For isotropic turbulence $\partial k_t/\partial t = -\epsilon$. Consequently, the transfer term integral must be zero

$$\int_0^\infty T(k,t)\,dk = 0$$

The term $T$ corresponds to the rate of energy transferred to successively smaller and smaller scales of the turbulent field.

Note that this term $T$ is difficult to measure, and it makes sense only for high Reynolds number turbulent flows, in order to ensure the presence of an inertial region in the spectrum $E(k)$.

## Lin's equation (cont.)

Let us introduce the function $S(k,t)$ defined by

$$S(k,t) = -\int_0^k T(k',t)\,dk'$$

$S(k,t)$ represents the energy transferred from all the wavenumbers smaller than $k$ (large structures) to wavenumbers larger than $k$ (small structures)