Physics of turbulent flow

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Numerical simulation of turbulent flows

Turbulence models (statistical approach)
Based on Reynolds Averaged Navier-Stokes equations (RANS)

Direct Numerical Simulation (DNS)
Time-dependent Navier-Stokes equations are solved for all the scales (i.e. turbulent structures)

Large Eddy Simulation (LES)
Time-dependent filtered Navier-Stokes equations are solved, effects of smaller scales are modeled
Reynolds Averaged Navier-Stokes (RANS) equations

\[
\begin{align*}
\frac{\partial \overline{U}_i}{\partial x_i} &= 0 \quad \text{(incompressible flow)} \\
\frac{\partial (\rho \overline{U}_i)}{\partial t} + \frac{\partial (\rho \overline{U}_i \overline{U}_j)}{\partial x_j} &= -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left( \overline{\tau}_{ij} - \rho \overline{u_i' u_j'} \right)
\end{align*}
\]

Closure of the Reynolds stress tensor through a turbulent viscosity \( \mu_t \), Boussinesq’s hypothesis

\[-\rho \overline{u_i' u_j'} = 2 \mu_t \overline{S}_{ij} - \frac{2}{3} \rho k_t \delta_{ij} \quad k_t \equiv \frac{u_i' u_i'}{2} \quad \overline{S}_{ij} = \frac{1}{2} \left( \frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i} \right)\]

\[
\frac{\partial (\rho \overline{U}_i)}{\partial t} + \frac{\partial (\rho \overline{U}_i \overline{U}_j)}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \overline{P} + \frac{2}{3} \rho k_t \right) + \frac{\partial}{\partial x_j} \left[ 2(\mu + \mu_t) \overline{S}_{ij} \right]
\]
Reynolds Averaged Navier-Stokes (RANS) equations

Almost all the physics is now contained in $\nu_t$

Dimensional argument, $\nu_t \sim \text{length} \times \text{velocity}$ \[\text{[m}^2\text{.s}^{-1}\]\n
Two scales are required to evaluate $\nu_t$ (or ad-hoc transport equation on $\nu_t$)

$k_t - \epsilon$ turbulence model

\[
\begin{align*}
\text{length} & \sim k_t^{3/2}/\epsilon \\
\text{velocity} & \sim k_t^{1/2} \\
\nu_t & \equiv C_\mu \frac{k_t^2}{\epsilon} \quad (C_\mu = \text{cst})
\end{align*}
\]

Need of the transport equations of $k_t$ (okay !) and $\epsilon$ (tricky problem)

Usually, high-Reynolds number assumption

\[
\rho \epsilon = \tau_{ij}' \frac{\partial u_i'}{\partial x_j} \rightarrow \rho \epsilon^h = \mu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \quad \text{as Re} \rightarrow \infty
\]

(that will be demonstrated later in the course, see slide 168)

$\epsilon^h$ is the dissipation rate for homogeneous turbulence
Standard $k_t - \nu$ turbulence model (high-Reynolds number form)

Transport equation for the turbulent kinetic energy $k_t$ (refer to Chap. 2)

$$\frac{\partial (\rho k_t)}{\partial t} + \frac{\partial (\rho k_t \overline{U_j})}{\partial x_j} = -\rho u'_i u'_j \frac{\partial U_i}{\partial x_j} - \tau'_{ij} \frac{\partial u'_i}{\partial x_j}$$

$$- \frac{1}{2} \frac{\partial}{\partial x_j} \rho u'_i u'_i u'_j - u'_i \frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(u'_i \tau'_{ij})$$

By noting that

$$- \tau'_{ij} \frac{\partial u'_i}{\partial x_j} + \frac{\partial}{\partial x_j}(u'_i \tau'_{ij}) = \mu \frac{\partial^2 k_t}{\partial x_i^2} - \mu \left( \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right) \rho \nu^h$$

$$\frac{\bar{D}}{\bar{D}_t} (\rho k_t) = \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k_t}{\partial x_j} \right] + P - \rho \nu^h \quad (k_t \text{ transport equation})$$
**Standard \( k_t - \epsilon \) turbulence model (high-Reynolds number form)**

Transport equation for \( \epsilon^h \) \( (\epsilon^h \rightarrow \epsilon \text{ as Re } \rightarrow \infty) \)

\[
\frac{\partial (\rho \epsilon^h)}{\partial t} + \frac{\partial (\rho \epsilon^h \overline{U}_k)}{\partial x_k} = -2\mu \frac{\partial \overline{U}_k}{\partial x_j} \left( \frac{\partial u_i' \partial u_i'}{\partial x_k \partial x_j} + \frac{\partial u_k' \partial u_j'}{\partial x_i \partial x_j} \right) - 2\mu u_k' \frac{\partial \epsilon^h}{\partial x_j} \frac{\partial^2 \overline{U}_i}{\partial x_j \partial x_k}
\]

\( (i) \)

\[
-2\mu \frac{\partial u_i' \partial u_k' \partial u_i'}{\partial x_j \partial x_j \partial x_k}
\]

\( (ii) \)

\[
-2\mu \frac{\partial u_i' \partial u_k' \partial u_i'}{\partial x_j \partial x_j \partial x_k} - \mu \frac{\partial}{\partial x_k} \left( u_k' \frac{\partial u_i' \partial u_i'}{\partial x_j \partial x_j} \right) - 2\nu \frac{\partial}{\partial x_i} \left( \frac{\partial u_i' \partial p'}{\partial x_j \partial x_j} \right)
\]

\( (iii) \) \( (iv) \) \( (v) \)

\[
+ \mu \frac{\partial^2 \epsilon^h}{\partial x_k \partial x_k} - 2\rho \nu^2 \frac{\partial^2 u_i'}{\partial x_k \partial x_j} \frac{\partial^2 u_i'}{\partial x_k \partial x_j}
\]

\( (vi) \) \( (vii) \)

The \( \epsilon^h \) transport equation is finally derived by mimicking the structure of the \( k_t \) equation.
Standard $k_t - \epsilon$ turbulence model (high-Reynolds number form)

Jones & Launder (1972), Launder & Spalding (1974)

\[
\begin{align*}
\frac{\partial}{\partial t}(\rho k_t) &= \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_k} \right) \frac{\partial k_t}{\partial x_j} \right] + P - \rho \varepsilon^h \\
\frac{\partial}{\partial t}(\rho \varepsilon^h) &= \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon^h}{\partial x_j} \right] + \varepsilon^h \left( C_{\varepsilon 1} P - C_{\varepsilon 2} \rho \varepsilon^h \right)
\end{align*}
\]

\[
P = -\rho u_i^'u_j^' \frac{\partial U_i}{\partial x_j} \quad \mu_t = C_\mu \rho \frac{k_t^2}{\varepsilon^h}
\]

Constants are determined from canonical experiments

$C_\mu = 0.09$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$

Refer to lecture notes: low-Reynolds number form, Favre average & compressible form, near-wall treatment
Turbulent mean flow through a diaphragm (standard $k_t - \epsilon$ model)

Lafon (LaMSID - UMR EDF/CNRS 2832, 1999)

See also Gloerfelt & Lafon, *Comput. & Fluids* (2008) for LES results
**Turbulent mean flow in a diaphragm (standard $k_t - \epsilon$ model)**

Duct height $2.29h$ & width $w = 2.86h$, $h = 35$ mm obstruction height, $Re_h = \overline{U}_m h/\nu = 4.8 \times 10^4$. Streamlines in $xy$-planes computed with the $\overline{U}_1$ and $\overline{U}_2$ components of the 3-D mean velocity field at three spanwise locations $z/w = 0.1$, 0.5 and 0.9.

Realizability conditions

Galilean invariance & thermodynamics

Realizability conditions for the Reynolds stress tensor
(no summation for Greek indices)

\[
\begin{align*}
    \overline{u'_\alpha u'_\alpha} & \geq 0 \\
    \overline{u'_\alpha u'_\beta}^2 & \leq \overline{u'_\alpha u'_\alpha \, u'_\beta u'_\beta} \\
    \det(\overline{u'_\alpha u'_\beta}) & \geq 0
\end{align*}
\]


A turbulent closure should guarantee realizable conditions for the computed Reynolds stress tensor

Renormalization group approach: RNG \( k_t - \epsilon \) model
(rather weak impact on turbulence closure)

Yakhot, Orszag, Thangam, Gatski & Speziale (1992)
The $k_t - \epsilon$ turbulence model

Anomalous growth of $k_t$ near a stagnation point (spurious production)

Contours of $k_t/U_\infty$

(a) with realizability
(b) without constraint


\[ \nu_t = C_\mu k_t \times \frac{k_t}{\epsilon} \]

\[ T \equiv \frac{k_t}{\epsilon} = \min \left( \frac{k_t}{\epsilon}, \frac{1}{C_\mu} \sqrt{\frac{3}{8S^2}} \right) \]

\[ \bar{S} \equiv \sqrt{2S_{ij} \bar{S}_{ij}} \]
The $k_t - \varepsilon$ turbulence model

How interprete Unsteady RANS (URANS) simulation?

obtained through time-marching algorithms, \( \partial_t \overline{U} + \mathcal{F}(\overline{U}) = 0 \)

Let’s consider a mean shear flow \( \overline{U}_1 = \overline{U}_1(x_2) \overline{U}_2 = \overline{U}_3 = 0 \)

The Schwarz’s inequality \( u'_1 u'_2 \leq u'^2_1 u'^2_2 \)

provides for the \( k_t - \epsilon \) model,

\[
9C^2_{\mu}S^2_{12} \leq \frac{\epsilon^2}{k^2_t} \sim \text{frequency}^2
\]

Implicit low-pass filter imposed by the mean shear: no development of energy cascade with grid refinement.

The constant \( C_\mu \) (and thus \( \nu_t \)) is reduced in practice, 3-D is required.

Semi-deterministic modelling (Ha Minh, 1991)
The $k_t – \omega –$ SST (shear-stress transport) model

Formal change of variable $\epsilon = C_\mu \omega k_t$, and thus $\nu_t = k_t/\omega$, $k_t – \epsilon$ model transformed into a $k_t – \omega$ model

$$\frac{\bar{d}\omega}{dt} = \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_\epsilon} \right) \frac{\partial \omega}{\partial x_j} \right] + \frac{\alpha}{\nu_t} \mathcal{P} - \beta \rho \omega^2$$

$$+ \frac{\omega}{k_t} \frac{\partial}{\partial x_j} \left[ \left( \frac{\mu_t}{\sigma_\epsilon} - \frac{\mu_t}{\sigma_k} \right) \frac{\partial k_t}{\partial x_j} \right] + \frac{2}{k_t} (\mu + \sigma_\omega \mu_t) \frac{\partial k_t}{\partial x_j} \frac{\partial \omega}{\partial x_j}$$

From standard $k_t – \epsilon$ model, $\alpha = C_{\epsilon_1} – 1$, $\beta = C_\mu (C_{\epsilon_2} – 1)$, $\sigma_\omega = 1/\sigma_\epsilon$, the transport term in gray is neglected.

Without the cross-diffusion term in blue and specific values of $\alpha$ and $\beta$, Wilcox’s model (1988, 2008, AIAA Journal)
- better calibrated, no damping function near the wall
- too sensitive to freestream values
(Menter, 1991; Spalart & Rumsey, 2007; AIAA Journal)
The $k_t - \omega - $SST (shear-stress transport) model

One of the most widely used model

Should be automatically used as default model in CFD codes

*Ad hoc* modifications (constant calibration, $F_1$ blending function, ...)

Menter (1994) *AIAA Journal*

$$\frac{d\omega}{dt} = \frac{\partial}{\partial x_j} \left[ \left( \mu + \frac{\mu_t}{\sigma_\varepsilon} \right) \frac{\partial \omega}{\partial x_j} \right] + \frac{\alpha}{\nu_t} Pr - \beta \rho \omega^2 + 2(1 - F_1) \frac{\rho}{\sigma_\omega^2 \omega} \frac{\partial k_t}{\partial x_j} \frac{\partial \omega}{\partial x_j}$$

Compressibility corrections

Dezitter *et al.* AIAA Paper 2009-3370

Rumsey *J. Spacecraft & Rockets* (2010)
The $k_t - \omega - \text{SST (shear-stress transport) model}$

Supersonic underexpanded jet with flight effects $M_j = 1.35$ (NPR = 2.97, convergent nozzle), $M_f = 0.4$
eLSA solver, $k_t - \omega - \text{SST (ONERA)}$

\[ \overline{U_1} \quad k_t \]

SNECMA / AIRBUS
The $k_t - \omega - \text{SST}$ (shear-stress transport) model

Dezitter et al., AIAA Paper 2009-3370 (VITAL project)
The $k_t - \omega - SST$ (shear-stress transport) model

Dezitter et al., AIAA Paper 2009-3370 (VITAL project)
Other strategies for solving RANS equations

Reynolds stress models: RSM or \( R_{ij} - \epsilon \) or \( R_{ij} - \epsilon_{ij} \)
(not so often used in industrial context)


Spalart-Allmaras (SA) model (1992, 1994)
Ad-hoc transport equation on the turbulent viscosity

\[
\frac{d\nabla}{dt} = c_{b_1} \tilde{S} \nabla + \frac{1}{\sigma} \left\{ \nabla \cdot \left[ (\nu + \nabla) \nabla \right] + c_{b_2} (\nabla \nabla)^2 \right\} - c_{w_1} f_w \left( \frac{\nabla}{d_w} \right)^2
\]

\( \nabla \) related to the turbulent viscosity \( \nu_t = \nabla f_{\nu_1} \)

\( \tilde{S} \) vorticity magnitude sensor

\( d_w \) distance to the wall

\( f_w \) and \( f_{\nu_1} \) are damping functions (near-wall corrections)
Advantages

Simple and easy to use and implement

May handle complex geometries, many developments to model complex flows (reactive flows, combustion, two-phase flows, transport & pollution)

Various \((ad-hoc)\) models!

Shortcomings

Strong assumptions regarding the behaviour of the mean flow (concept of turbulent viscosity)

Modeling of the laminar-turbulent transition prediction

Boundary layer with a mean pressure gradient, detached flows, ...

No information (or poor/bad information) about the time-dependence
Compressible Reynolds Averaged Navier-Stokes (RANS) equations

- Formulation dramatically simplified by introducing the Favre average (1965) or density-weighted average for compressible flows

\[ \rho = \bar{\rho} + \rho' \]  
\( \bar{\rho} \) conventional Reynolds-averaged density

Favre decomposition of \( u_i = \tilde{U}_i + u''_i \)

\[ \tilde{U}_i = \frac{\rho u_i}{\bar{\rho}} \quad \rho u''_i = \bar{\rho} u''_i = 0 \quad u''_i \neq 0 \]

- Equation of state \( \bar{P} = \bar{\rho} \bar{r} \tilde{T} \) or \( c_p \bar{\rho} \tilde{T} = c_v \bar{\rho} \tilde{T} + \bar{P} \)

- Introducing \( \rho = \bar{\rho} + \rho' \) \( u_i = \tilde{U}_i + u''_i \) \( p = \bar{P} + p' \) \( T = \tilde{T} + T'' \)

\( \sim \) Favre-averaged equations for compressible flows
Favre-Averaged Navier-Stokes equations

\[
\begin{align*}
\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \bar{U}_j)}{\partial x_j} &= 0 \\
\frac{\partial (\bar{\rho} \bar{U}_i)}{\partial t} + \frac{\partial}{\partial x_j} \left( \bar{\rho} \bar{U}_i \bar{U}_j + \bar{P} \delta_{ij} - \bar{\tau}_{ij} - T_{ij} \right) &= 0 \\
\frac{\partial \bar{\rho} \tilde{E}_t}{\partial t} + \frac{\partial}{\partial x_j} \left[ (\bar{\rho} \tilde{E}_t + \bar{P}) \bar{U}_j - \bar{\tau}_{ij} \bar{U}_i + \bar{Q}_j - \mathcal{E}_j - \mathcal{P}_j + \mathcal{D}_j \right] &= 0
\end{align*}
\]

\[
\tilde{E}_t = c_v \bar{T} + \frac{\bar{U}_i^2}{2} + k_t \quad \quad k_t \equiv \frac{\bar{u}_i^\prime \bar{u}_i^\prime \prime}{2}
\]

\[
T_{ij} = -\bar{\rho} u_i u_j + \bar{\rho} \bar{U}_i \bar{U}_j = -\bar{\rho} u_i^\prime u_j^\prime \quad \quad \bar{\tau} \simeq \bar{\tau} \quad \quad \bar{Q} \simeq \bar{Q}
\]

\[
\mathcal{E}_j = -\bar{\rho} e_t u_j + \bar{\rho} \bar{E}_t \bar{U}_j = -\bar{\rho} e^\prime u_j^\prime - \bar{U}_j T_{ij} - \frac{1}{2} \bar{\rho} u_i^\prime u_j^\prime
\]

\[
\mathcal{P}_j = -\bar{p} u_j + \bar{P} \bar{U}_j = -p^\prime u_j^\prime \quad \quad \mathcal{D}_j = -\bar{\tau}_{ij} u_i + \bar{\tau}_{ij} \bar{U}_i = -\bar{\tau}_{ij} u_i^\prime
\]
**Favre-Averaged Navier-Stokes equations**

- Closure based on the introduction of a **turbulent viscosity** $\mu_t$  
  (generalization of Boussinesq’s hypothesis)

$$T_{ij} = -\overline{\rho u'_i u'_j} = 2\mu_t \tilde{S}_{ij} - \frac{2}{3} \bar{\rho} k_t \delta_{ij}$$

- **Constant turbulent Prandtl number** $\sigma_t$ for the turbulent heat flux, 
  Reynolds analogy (1874) between momentum and heat transfer

$$-\rho e'' u_j' - p' u_j' = -c_p \rho T'' u_j' = c_p \nu_\theta \frac{\partial \bar{T}}{\partial x_j} = \frac{\mu_t c_p}{\sigma_t} \frac{\partial \bar{T}}{\partial x_j} \quad \sigma_t \equiv \frac{\nu_t}{\nu_\theta}$$
**$k_t - \epsilon$ turbulence model with compressible corrections**

Supersonic jet $M_j = 1.33$, $T_j/T_\infty = 1$ (estet - EDF)

**$k_t - \epsilon$ turbulence model with compressible corrections**

Supersonic jet $M_j = 2.0$, $T_j/T_\infty = 1$ (estet - EDF)