Physics of turbulent flow

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http://acoustique.ec-lyon.fr
• A short introduction to turbulent flow
• Statistical description of turbulent flow
• Wall-bounded turbulent flow
• Anatomy of a RANS model: the k-epsilon model
• Dynamics of vorticity
• **Homogeneous and isotropic turbulence - Kolmogorov’s theory**
• An overview of numerical simulation and experimental techniques
Homogeneous and isotropic turbulence

Homogeneous turbulence

Generation of turbulence behind a grid, $\text{Re}_M = 1500$ & $M = 2.54 \text{ cm}$

_Corke & Nagib_ in _Van Dyke_ (1982)

- statistics are independants of space coordinates, e.g. the Reynolds tensor $-\bar{u_i'u_j'}$. The established flow in a channel is homogeneous in the $x_2$ and $x_3$ directions.

$\leadsto$ (almost) the simplest configuration!
Homogeneous and isotropic turbulence

- Homogeneous turbulence

Wrinkling of a fluid surface in isotropic turbulence, Karweit (1968)

A platinum wire generates a continuous sheet of hydrogen bubbles, which is then deformed by the nearly isotropic turbulence behind the grid.
Homogeneous turbulence: velocity correlation tensor

Definition:

\[ R_{ij}(x, r, t) \equiv u_i'(x, t) u_j'(x + r, t) = R_{ij}(r, t) \]

The function \( R_{ij} \) is only a function of the separation vector \( r \), between the two measurement points \( x \) and \( x' = x + r \): invariance by translation of the observer location \( x \).

Correlation coefficient \( R_{ij} \) (normalized correlation function \( R_{ij} \))

\[ -1 \leq R_{ij}(r) \equiv \frac{u_i'(x) u_j'(x')}{\sqrt{u_i'^2(x) u_j'^2(x')}} \leq +1 \]

As an illustration, \( R_{11}(r, 0, 0) = u_1'^2 R_{11}(r, 0, 0) \) with \( r = (r, 0, 0) \).
Homogeneous and isotropic turbulence

**Homogeneous turbulence**

Turbulent kinetic energy budget $k_t$ (general case)

\[
\frac{\partial (\rho k_t)}{\partial t} = -\rho u'_i u'_j \frac{\partial \bar{U}_i}{\partial x_j} - \tau'_{ij} \frac{\partial u'_i}{\partial x_j} = P - \rho \epsilon = \text{production} - \text{dissipation}
\]

Decaying turbulence generated behind a grid,
Stationary turbulence, homogeneous in the plane $(x_2, x_3)$ only

\[
\bar{U}_1 \frac{\partial k_t}{\partial x_1} = -\epsilon
\]

In a frame moving with the mean velocity $\bar{U}_1$,

\[
\frac{\partial k_t}{\partial t} = -\epsilon
\]
Homogeneous and isotropic turbulence

**Homogeneous turbulence : integral length scales**

Longitudinal integral length scale: an estimate of the size of the most energetic turbulent structures, given by the integration of the correlation coefficient of the velocity component $u_1'$ between two points in the $x_1$ direction

$$L_1 \equiv L_{11}^{(1)} = \int_{0}^{\infty} \mathcal{R}_{11}(r, 0, 0) dr$$

A transverse integral length scale $L_2 \equiv L_{11}^{(2)}$ can also been introduced

$$L_2 \equiv L_{11}^{(2)} = \int_{0}^{\infty} \mathcal{R}_{11}(0, r, 0) dr$$

Tavoularis (2003), passive scalar mixing, $Sc \approx 2000$
Homogeneous and isotropic turbulence

- **Homogeneous turbulence: Taylor microscales**

Microscale or Taylor length scale, defined from the Taylor series of the velocity correlation function at the origin. The transverse Taylor microscale $\lambda_2$ is associated with the peak of the dissipation spectrum (will be illustrated later).

$$R_{11}(r, 0, 0) = 1 - \frac{r^2}{\lambda_1^2} + ...$$

![Graph showing $R_{11}$ vs $r$ with Taylor microscales $\lambda_1$ and $L_1$](image-url)
Homogeneous turbulence: Taylor microscales

Taylor series of $u'_1(r,0,0)$ as $r \to 0$,

\[
u'_1(r,0,0) = u'_1(0,0,0) + r \frac{\partial u'_1}{\partial x_1}\bigg|_{x=0} + \frac{r^2}{2} \frac{\partial^2 u'_1}{\partial x_1^2}\bigg|_{x=0} + \ldots
\]

Hence,

\[
R_{11}(r,0,0) = \frac{u'_1(0,0,0) u'_1(r,0,0)}{
= \overline{u'_1^2} + r \overline{u'_1} \frac{\partial u'_1}{\partial x_1} + \frac{r^2}{2} \overline{u'_1} \frac{\partial^2 u'_1}{\partial x_1^2} + \ldots
= \overline{u'_1^2} + r \frac{\partial}{\partial x_1} \left( \frac{\overline{u'_1^2}}{2} \right) + \frac{r^2}{2} \frac{\partial}{\partial x_1} \left( \overline{u'_1} \frac{\partial u'_1}{\partial x_1} \right) - \frac{r^2}{2} \left( \frac{\partial u'_1}{\partial x_1} \right)^2 + \ldots
= R_{11}(r,0,0) = 1 - \frac{r^2}{2 \overline{u'_1^2}} \left( \frac{\partial u'_1}{\partial x_1} \right)^2 \equiv 1 - \frac{r^2}{\lambda_1^2} + \ldots
\]
Homogeneous and isotropic turbulence

**Homogeneous turbulence : Taylor microscales**

**Longitudinal Taylor microscale**

\[
\frac{1}{\lambda_1^2} \equiv -\frac{1}{2} \frac{d^2 \mathcal{R}_{11}}{dr_1^2} = \frac{1}{2u_1'^2} \left( \frac{\partial u_1'}{\partial x_1} \right)^2
\]

**Transverse Taylor microscale**

\[
\frac{1}{\lambda_2^2} \equiv -\frac{1}{2} \frac{d^2 \mathcal{R}_{11}}{dr_2^2} = \frac{1}{2u_1'^2} \left( \frac{\partial u_1'}{\partial x_2} \right)^2
\]
An application example: approximation of the dissipation rate $\epsilon$ of the turbulent kinetic energy

$$\rho \epsilon = \tau'_{ik} \frac{\partial u'_i}{\partial x_k} = 2 \mu s'_{ij} = 2 \mu \frac{1}{4} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right)^2 = \mu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} + \mu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_j}$$

(a) $\equiv \epsilon^h = \nu \left( \frac{\partial u'_i}{\partial x_j} \right)^2 \sim \nu \frac{u''^2}{\lambda^2}$

(b) $= \nu \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_j}{\partial x_i} = \frac{\partial^2 u'_i u'_j}{\partial x_i \partial x_j} \sim \nu \frac{u''^2}{L^2}$

In addition, for homogeneous turbulence, the term (b) is zero and

$$\epsilon = \epsilon^h = \nu \left( \frac{\partial u'_i}{\partial x_j} \right)^2$$

$\epsilon^h$ is an approximation of the dissipation $\epsilon$ when $\lambda \ll L$, that is for high Reynolds number turbulent flow (see the $k_t - \epsilon^h$ model)
Homogeneous and isotropic turbulence

**Velocity spectrum**

The Fourier transform of the velocity field \( u'(x, t) \) in space for homogeneous turbulence is defined by

\[
\hat{u}(k) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} u'(x) e^{-ik\cdot x} \, dx \quad \text{and} \quad u'(x) = \int_{\mathbb{R}^3} \hat{u}(k) e^{ik\cdot x} \, dk
\]

Furthermore, for an incompressible velocity field, the condition \( \nabla \cdot u' = 0 \) provides

\[
\nabla \cdot u' = 0 = \int i k \cdot \hat{u}(k) e^{ik\cdot x} \, dk \quad \forall \, x
\]

Consequently, \( k \cdot \hat{u}(k) \equiv 0 \quad \forall \, k \)

decomposition of the velocity field in elementary transverse plane waves
Illustration of a Fourier mode

Fourier mode corresponding to the wavenumber $k = (2, 1, 0)$ and its opposite $-k = (-2, -1, 0)$. Velocity vector $u(x)$ is real and perpendicular to the wavevector $k$, its module varies as $\cos(k \cdot x + \varphi)$ with a phase shift $\varphi$ at the origin, linked to the imaginary part of $\hat{u}(k)$.
Spectral tensor

The spectral tensor $\phi_{ij}$ is defined as the Fourier transform of the velocity correlation tensor $R_{ij}$

\[
\begin{align*}
\phi_{ij}(k) &= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} R_{ij}(r) e^{-ik \cdot r} dr \\
R_{ij}(r) &= \int_{\mathbb{R}^3} \phi_{ij}(k) e^{ik \cdot r} dk
\end{align*}
\]

Two important results from this representation in Fourier space

$\hat{u}_i^*(k)\hat{u}_j(k') = \phi_{ij}(k')\delta(k-k')$, that is two Fourier components are only correlated for the same wavenumber $k$.

Incompressibility condition:

\[k_i \phi_{ij}(k) = k_j \phi_{ij}(k) = 0\] (see exercises)
One-dimensional spectrum

It is common practice to introduce one-dimensional spectra, which can be measured or computed numerically,

$$E_{ij}^{(1)}(k_1) = \int \int_{\mathbb{R}^2} \phi_{ij}(\mathbf{k}) \, dk_2 dk_3$$

Let us consider the case $i = j = 1$ with a zero separation vector

$$u_1'^2 = R_{11}(r = 0) = \int_{\mathbb{R}^3} \phi_{11}(\mathbf{k}) \, e^{i \mathbf{k} \cdot \mathbf{r}} \, d\mathbf{k} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1) \, dk_1$$
One-dimensional spectrum (cont.)

The relation between \( R_{11}(r) \) with \( r = (r_1, 0, 0) \), and the one-dimensional spectrum \( E_{11}^{(1)}(k_1) \) is found to be

\[
R_{11}(r_1, 0, 0) = \int_{\mathbb{R}^3} \phi_{11}(k) \ e^{i k_1 r_1} \, dk = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1) \ e^{i k_1 r_1} \, dk_1
\]

and conversely by Fourier transform, one has

\[
E_{11}^{(1)}(k_1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{11}(r_1, 0, 0) \ e^{-i k_1 r_1} \, dr_1
\]

For \( k_1 = 0 \),

\[
E_{11}^{(1)}(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{11}(r_1, 0, 0) \, dr_1 = \frac{1}{2\pi} 2u_1'^2 L_1
\]

and the following expression is obtained

\[
L_1 \equiv L_{11}^{(1)} = \pi \frac{E_{11}^{(1)}(0)}{u_1'^2}
\]
Homogeneous and isotropic turbulence

**Frozen turbulence approximation or Taylor’s hypothesis (1938)**

The velocity spectral tensor and the corresponding one-dimensional spectra cannot be directly measured from the Fourier transform of velocity correlation functions in general. Only the time evolution of the velocity in one given point is known, that is \( u'_1(t) \).

In order to estimate these spectral functions, it is usually assumed that the turbulent flow is frozen during the measurement, meaning that the observed quantity is simply convected by the local mean flow \( \overline{U}_1 \), which leads to

\[
\frac{\partial}{\partial t} = -\overline{U}_1 \frac{\partial}{\partial x_1} \quad k_1 = \frac{2\pi f}{\overline{U}_1} \quad \text{Taylor’s hypothesis}
\]
Homogeneous and isotropic turbulence

**Frozen turbulence approximation or Taylor’s hypothesis (1938)**

Geoffrey Ingram Taylor (right) at age 69 (in 1956), in his laboratory with his assistant Walter Thompson (Physics Today, May 2000)

At Stanford (1968)

Application to the estimation of $L_1$, $u'_1(t) \rightarrow \Phi_{11}(f)$

$$\overline{u'^2_1} \equiv \int_0^{\infty} \Phi_{11}(f) \, df$$

$$\overline{u'^2_1} = \int_{-\infty}^{+\infty} E_{11}^{(1)}(k_1) \, dk_1 \equiv \int_0^{\infty} \frac{\bar{U}_1}{2\pi} \Phi_{11}\left(f = k_1 \frac{\bar{U}_1}{2\pi}\right) \, dk_1$$

$$L_{11}^{(1)} = \pi \frac{E_{11}^{(1)}(k_1 = 0)}{u'^2_1} = \frac{1}{4} \bar{U}_1 \frac{\Phi_{11}(f = 0)}{u'^2_1}$$
Homogeneous and isotropic turbulence

Frozen turbulence approximation or Taylor’s hypothesis (1938)

Spectrum of longitudinal velocity fluctuations
free round jet, $\text{Re}_D \simeq 10^5$, $x_1 = 2D$, $x_2 = D/2$

![Graphs showing the spectrum of longitudinal velocity fluctuations](image)
Homogeneous and isotropic turbulence

Homogeneous turbulence

Turbulent kinetic energy spectrum

\[ k_t = \frac{\bar{u}_i^2}{2} = \frac{1}{2} R_{ii}(r = 0) = \frac{1}{2} \int_{\mathbb{R}^3} \phi_{ii}(k) dk \]

Dissipation spectrum

Usually, it is more convenient to first calculate the enstrophy spectrum from the Fourier transform of the vorticity vector, \( \hat{\omega}(k) = i k \times \hat{U}(k) \). It can be shown that,

\[ \frac{\bar{\omega}_i^2}{2} = \frac{1}{2} \int_{\mathbb{R}^3} k^2 \phi_{ii}(k) dk \]

Then, by noting that \( \epsilon = \nu \bar{\omega}_i^2 \), the following expression is obtained from the dissipation spectrum

\[ \epsilon = \nu \bar{\omega}_i^2 = \nu \int_0^\infty k^2 \phi_{ii}(k) dk \]
Isotropic turbulence

An isotropic turbulent flow is a class of homogeneous turbulent flow whose statistics are invariant under rotation of the coordinate axes and under reflection in a plane.

Impossible to distinguish any privileged direction \textit{a priori}, the most simple configuration! (ideal theoretical framework)

In order to characterize homogeneous and isotropic turbulence, a virtual device is introduced to measure

- fluctuating scalar quantity: temperature, pressure, ...
- fluctuating vector quantity: projection on a given unit vector of velocity, ...
Isotropic turbulence

Second-order correlation in one point: Reynolds tensor

\[ u'_A \cdot a = u'_1 \]

\[ u'_A \cdot a = u'_2 \]

The two measurements must be equal for isotropic turbulence, and therefore \( u'^2_1 = u'^2_2 \). More generally,

\[ u'^2_1 = u'^2_2 = u'^2_3 = u'^2 \]

by noting \( u'^2 \equiv u'^2 \)
Isotropic turbulence

Second-order correlation in one point: Reynolds tensor

\[(u'_A \cdot a)(u'_A \cdot b) = u'_1 u'_2\]

\[(u'_A \cdot a)(u'_A \cdot b) = -u'_1 u'_2\]

Consequently, \(u'_1 u'_2 = -u'_1 u'_2\) and \(u'_1 u'_2 = 0\)

\[u'_i u'_j = u'^2 \delta_{ij} = \frac{2}{3} k_t \delta_{ij}\]
Isotropic turbulence

Second-order velocity correlation in two points A at $x$ and B at $x + r$:

$$
F \equiv \frac{(u'_A \cdot a)(u'_B \cdot b)}{\sqrt{u'^2_A} \sqrt{u'^2_B}} = \frac{u'_{iA} u'_{jB}(r)}{u'^2} a_i b_j = \mathcal{R}_{ij} a_i b_j
$$

The bilinear function $F$ can only be a function of the invariants associated with the measurement device, that is distances and angles, $r^2 = r_i r_i$, $a \cdot r = a_i r_i$, $b \cdot r = b_j r_j$, $a \cdot b = a_i b_i = a_i b_j \delta_{ij}$, and also the volume defined by $(r, a, b)$, that is $(a \times b) \cdot r = \epsilon_{ijk} a_i b_j r_k$.

General expression of an isotropic second-order two-point tensor (Robertson, 1940)

$$
\mathcal{R}_{ij}(r) = \alpha(r) r_i r_j + \beta(r) \delta_{ij}
$$
Isotropic turbulence

Second-order two-point velocity correlation (cont.)

It is generally found more convenient to introduce two functions $f(r)$ and $g(r)$ that can be measured in practice, rather than the two arbitrary functions $\alpha(r)$ and $\beta(r)$. Hence,

\[ f(r) \equiv R_{11}(r, 0, 0) \]  
longitudinal correlation function

\[ g(r) \equiv R_{11}(0, r, 0) \]  
transverse correlation function

\[
R_{ij}(r) = (f - g) \frac{r_i r_j}{r^2} + g \delta_{ij}
\]

Kármán & Howarth (1938)
Isotropic turbulence: turbulent kinetic energy spectrum

Using a similar approach applied now to the spectral tensor $\phi_{ij}(k)$, and taking account for the incompressibility condition, it can be shown that only one scalar function $E(k)$ is required to specify $\phi_{ij}(k)$, that is

$$\phi_{ij}(k) = \frac{E(k)}{4\pi k^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \quad \text{with} \quad k_t \equiv \int_0^\infty E(k) \, dk$$

(and only one scalar function $f(r)$ is needed to specify the second-order two-point velocity tensor in physical space, using the incompressibility condition)

The expression of the dissipation spectrum is then deduced from the relationship established for homogeneous turbulence,

$$\epsilon = 2\nu \int_0^\infty k^2 E(k) \, dk$$
Isotropic turbulence

Many other remarkable results can be established for homogeneous and isotropic turbulence: refer to textbooks mentioned in the introduction of this course.

Three points must be however still developed to provide a first full overview of isotropic turbulence:

− isotropic turbulence in laboratory?
− time evolution of isotropic turbulence
− expression of $E(k)$?

Kolmogorov’s theory
Homogeneous and isotropic turbulence in laboratory
Betchov (1957) - The “Porcupine”

The ‘Porcupine’. The mixing of 80 small jets produces a strong turbulence in the region marked $A$, $B$, $C$. 
A homogeneous - but not fully isotropic - turbulent flow is obtained, \( \overline{u'_1^2} = 1.2 \overline{u'_2^2} = 1.2 \overline{u'_3^2} \)

and one typically gets \( \frac{u'}{U_0} \approx 2\% \quad \text{Re}_M = \frac{U_0 M}{\nu} \approx 10^4 \text{ to } 10^5 \)
Homogeneous and isotropic turbulence

Homogenous and isotropic turbulence in laboratory (cont.)

Experiences by Comte-Bellot & Corrsin (1966) at Johns Hopkins University

A contraction $c \approx 1.27$

\[
\begin{align*}
\overline{u_1'^2} &= 1.2 \overline{u_2'^2} \\
\overline{u_2'^2} &= \overline{u_3'^2} \\
\overline{u_1'^2} &= \overline{u_2'^2} = \overline{u_3'^2}
\end{align*}
\]
Homogeneous and isotropic turbulence

Stanley Corrsin

Hopkins researcher finds fascination in turbulence

By Albert Sehstedt, Jr.

Stanley Corrsin is a specialist in turbulence, a very complex scientific problem subject that deals with airplanes flying through the clouds, curling cigarette smoke rising under a lampshade and blood flowing through human bodies. Explaining these seemingly commonplace occurrences poses a problem that has puzzled scientists for decades.

“It is sufficiently difficult [a subject] that the problem is not likely to be solved in my lifetime,” Dr. Corrsin observed in his Maryland Hall office on the Homewood campus of the Johns Hopkins University.

“That means I’m not in danger of being unemployed,” the 62-year-old scientist added with a smile. “Also, I think it is aesthetically interesting. Turbulent flows make beautiful pictures. Turbulent flows are movements of matter in which the velocity at a critical review of fluid dynamics,” which “have touched a legion of students and associates.”

“Stan is not only a person who himself has contributed [through research], but his discourses have been stimulating to other people.”

— Lawrence Talbot
Berkeley professor

Stanley Corrsin, winner of the American Physical Society’s 1983 Fluid Dynamics Prize, is respected as both researcher and teacher.

Better to talk of airplanes, soaring albatrosses, flowing water — and swallowing. There is a “swallowing center,” a complex assembly of muscles and called non-uniform surface tension may be the answer, he said.

There is also the question of why contact lenses stay attached to the surface of the eye. Dr. Corrsin and his colleagues examined this mystery, too, but he said. “We never did who was helping to edit a monograph on jet propulsion at a place that has since become famous for guiding spacecraft to the planets — the Jet Propulsion Laboratory in Pasadena, Calif.

Dr. Corrsin said he chose Hookins...
Decaying isotropic turbulence

In a frame moving with the mean velocity,

Decay of the normal stresses

\[
\frac{\overline{u'^2}}{U_0^2} = \frac{1}{A} \left( \frac{tU_0}{M} - \frac{t_0 U_0}{M} \right)^{-n} \quad \text{with} \quad n \approx 1.3
\]

Comte-Bellot & Corrsin (1966), Mohamed & Larue (1990)

The dissipation rate of the turbulent kinetic energy is imposed by larger turbulent structures,

\[
\epsilon \approx \frac{u'^3}{L_f}
\]
Decaying isotropic turbulence

Time correlation in a frame travelling with the mean velocity $\overline{U}_1$ for different values of the wavenumber, from $k_1 = 0.25 \text{ cm}^{-1}$ (♦) to $k = 10.10 \text{ cm}^{-1}$ (+)

— total signal (full-band case)
Homogeneous and isotropic turbulence

- Isotropic turbulence submitted to ...

- **Tucker & Reynolds** - plane strain

- **Wigeland & Nagib** - solid body rotation

- **Champagne et al.** - sheared mean flow
Homogeneous and isotropic turbulence

**Space-time correlations (homogeneous in the \(x_1\)-direction)**

\[
R_{11}(\Delta x_1, 0, 0; \tau) = \frac{u_1'(x, t)u_1'(x + \Delta x_1 e_1, t + \tau)}{\bar{u}_1'^2}
\]

- Time autocorrelation function \((\Delta x_1 = 0)\)
- \(\sim\) integral time scale in the fixed frame \(\Theta_1 \sim L_f/U_{c1}\) (Taylor)
- Time correlation for a given separation \(\Delta x_1\) of the two probes \((\Delta x_1 > \Delta x_1)\)
- Time autocorrelation function in the convected frame

\[
\Theta_{c1} = \int_0^\infty R_{c11}(\tau) d\tau
\]

\(\Theta_{c1} \sim L_f/u_1'\) represents the time characterizing the loss of coherence or the memory time of turbulence.
Theory of Kolmogorov: energy cascade

The spectrum of turbulent kinetic energy is the key function for an isotropic turbulence. Can we determine the form of $E(k)$ and its time evolution?

Two families of scales have been introduced to characterize turbulence,

<table>
<thead>
<tr>
<th>energy-containing eddies</th>
<th>smallest eddies</th>
</tr>
</thead>
<tbody>
<tr>
<td>large scales</td>
<td>dissipative scales ($\text{Re}_\eta = 1$)</td>
</tr>
<tr>
<td>velocity</td>
<td>$u_\eta = \nu^{1/4} \epsilon^{1/4}$</td>
</tr>
<tr>
<td>$u' \sim k_t^{1/2}$</td>
<td>$l_\eta = \nu^{3/4} \epsilon^{-1/4}$</td>
</tr>
<tr>
<td>length</td>
<td>$\tau_\eta = \nu^{1/2} \epsilon^{-1/2}$</td>
</tr>
<tr>
<td>$L_f$</td>
<td></td>
</tr>
<tr>
<td>time</td>
<td></td>
</tr>
<tr>
<td>$L_f/u'$</td>
<td></td>
</tr>
</tbody>
</table>

Moreover, the rate of dissipation is imposed by larger structures of the turbulent flow, i.e. $\epsilon \simeq u'^3/L_f$, and we know the spectrum shapes of the turbulent kinetic energy $k_t$ and of the dissipation $\epsilon$ ...
Theory of Kolmogorov: energy cascade for high Reynolds numbers

\[ D_y(k) = 2\nu k^2 E(k) \]

\[ \epsilon \approx u'^3/L \]

\[ \frac{L_f}{l_\eta} = \frac{L_f}{\nu^{3/4} \epsilon^{-1/4}} = \left(\frac{u'L_f}{\nu}\right)^{3/4} = \text{Re}_{L_f}^{3/4} \]

\[ \text{Re}_{L_f} \equiv \frac{u'L_f}{\nu} \quad \text{Reynolds number of large structures} \]
Dynamics of isotropic turbulence

- Theory of Kolmogorov: energy cascade

small $k$  \hspace{1cm} intermediate $k$  \hspace{1cm} high $k$

Fourier space  \hspace{1cm} physical space

Physics of turbulent flow – Christophe Bailly – ECL'19
Dynamics of isotropic turbulence

Theory of Kolmogorov: energy cascade

The higher the Reynolds number is, the more spectra of the kinetic energy and dissipation will be separated: fully developed turbulence.

K41 – energy cascade

The dissipation rate $\epsilon$ is imposed by large eddies, but carries out by the smallest ones (at Kolmogorov scales), it can be argued as assumptions that

- the dissipation rate is finite, even when $\text{Re} \to \infty$,
- there is a self-similar dynamics; velocity scale of an eddy of size $l$ varies as $u_l \sim l^p$
Theory of Kolmogorov: energy cascade

- Eddy of size $l$ and of velocity $u_l$, eddy-life time or turn-over time $t_l \sim l/u_l$

\[
\frac{u_l^2}{l/u_l} = \text{cst} = \epsilon \quad \implies \quad u_l \sim (\epsilon l)^{1/3}
\]

- Kinematic energy $\mathcal{E}_l$ associated with eddies of size $l \sim 1/k_l$

\[
\mathcal{E}_l \sim u_l'^2 \sim (\epsilon l)^{2/3}
\]

- Turbulent kinetic energy spectrum $\mathcal{E}_l \sim k_l E(k_l)$, and thus

\[
E(k_l) \sim \frac{\epsilon^{2/3} k_l^{-2/3}}{k_l} \sim \epsilon^{2/3} k_l^{-5/3} \quad \text{Kolmogorov’s law}
\]

Inertial subrange $1/L \ll k \ll k_\eta$ at high-Reynolds number,

\[
E(k, \epsilon, \nu) = E(k, \epsilon)
\]
Dynamics of isotropic turbulence

- Benefit of frequency weighted spectrum (compensated spectrum)

\[ I = \int_0^\infty E(k) \, dk = \int_0^\infty kE(k) \, d\ln k \]

With a constant bandwidth for \( d\ln k = \frac{dk}{k} \),
\[ I(k_0 \pm \delta k_0) \sim k_0 E(k_0) \]

\[ E(k) = \frac{k^4}{(1 + k^2)^{17/6}} \]
\[ I_{\text{exact}} \approx 1.0325 \]

\[ I \approx 0.36 \times 2.9 \approx 1.04 \]
**Benefit of frequency weighted spectra**

von Kármán spectra (arbitrary units here)

- for $k_t = 3$, — for $k_t = 1.5$

log-log scales (to observe the $-5/3$ law) versus $k \times E(k)$ on linear scales

Area of the grey rectangle, $1.25 \times \ln(10) \times 1.05 \approx 3$
Theory of Kolmogorov: energy cascade

Dimensional analysis $E = E(k, \nu, \epsilon) = E(k, \epsilon)$

$$E \sim k^\alpha \epsilon^\beta F\left(\frac{k}{\eta}\right)$$

It is straightforward to determine the two exponents $\alpha$ and $\beta$. In the inertial subrange (at sufficiently high Reynolds number), the function $F$ is simply a constant $C_K$,

$$E(k) = C_K \epsilon^{2/3} k^{-5/3} \quad C_K \approx 1.5$$

Richardson (1922)

Kolmogorov (1941) - Obukhov (1941)

Original formulation by Kolmogorov (1941, 1962) is based on structure functions, and is still debated (e.g. self-similarity at small scales)
Theory of Kolmogorov: energy cascade

\[ \overline{u_1'^2} = \int_{-\infty}^{+\infty} E_{11}(k_1) dk_1 \]

\[ 2 \times E_{11}(k_1) \left( \frac{\varepsilon^{1/2}}{v^{1/4}} \right) \sim k_1^{-5/3} \]

--- Pao (1965)
Dynamics of isotropic turbulence

- **Theory of Kolmogorov**: energy cascade
  Measurements of Grant, Stewart & Moilliet (1962)
Dynamics of isotropic turbulence

- Theory of Kolmogorov: energy cascade

Universal equilibrium of fine turbulent structures

\[ E(k) \times \frac{k}{u^2} \eta \text{ as a function of } \frac{k}{k_\eta} \]

At sufficiently high Reynolds numbers, there is a universal spectrum shape in the inertial subrange, and this region widens as the Reynolds number increases.
In summary

\[ \epsilon \approx u'^3/L \]

\[ E(k) = C_K \epsilon^{2/3} k^{-5/3} \]

\[ D_v(k) = 2\nu k^2 E(k) \]

\[ L/l_\eta \sim Re_L^{3/4} \]

\[ l_\eta = \nu^{3/4} \epsilon^{-1/4} \]
Energy cascade in a turbulent mixing layer (Brown & Roshko, 1974)
Shadowgraphs (spark source)

Energy cascade in a mixing layer by increasing the Reynolds number (through pressure and velocity, $\times 2$ for each view)

More small-scale structures are produced without basically altering the large-scale ones

Anatol Roshko (1923-2017)
Paradox of the energy cascade ...

e.g. Energy dissipated by the motion at $U_\infty$ of a sphere of diameter $D$

Consider the power developed by the drag force $F_D U_\infty$ is balanced by the energy dissipated with the flow $\rho \nu \epsilon$

$$\epsilon = \frac{1}{\rho \nu} \frac{\rho U_\infty^2}{2} C_D (\text{Re}_D) U_\infty \propto C_D (\text{Re}_D) \frac{U_\infty^3}{D}$$

with $\nu \sim D^3$, and where $\text{Re}_D \simeq \text{cst}$ as $\text{Re}_D$ goes to infinity.

For high Reynolds number flows $\text{Re}_D \gg 1$, the rate of dissipation per unit mass $\epsilon$ becomes independent of the viscosity $\nu$, whereas $\epsilon = \nu |\nabla \mathbf{u}|^2$

$\text{Re}_D \nearrow$, $\nu \searrow$ and $|\nabla \mathbf{u}| \nearrow$ leading to singularities inside the flow
Drag coefficient for a smooth sphere (adapted from Clift, Grace, & Weber, 1978)

\[ C_D = \frac{24}{Re_D} \]

\[ Re_D = 10^3 \text{ (ONERA)} \]
\[ Re_D = 1.5 \times 10^4 \text{ (Van Dyke, 1982)} \]
\[ Re_D \approx 4 \times 10^5 \text{ (Werlé, 1987)} \]
\[ Re_D^c \approx 3 \times 10^5 \]
Lin’s equation (1947)

Transport equation for the turbulent kinetic energy spectrum $E(k)$?

A way to derive this equation is

- to consider the transport equation for the Reynolds tensor $R_{ij} = u'_i(x)u'_j(x+r)$, known as the Kármán & Howarth equation
- to take the Fourier transform and to contract subscripts as follows $i = j$

which gives

$$\frac{\partial}{\partial t} E(k, t) = T(k, t) - 2\nu k^2 E(k, t)$$

where the nonlinear term $T(E)$ is linked to the third-order (triple) velocity correlation. This term can be directly associated with the energy transfer between turbulent structures of different size.
In order to illustrate this point, Lin’s equation can be integrated over all the wavenumbers $k$

$$\frac{\partial}{\partial t} \int_0^\infty E(k, t) \, dk = \int_0^\infty T(k, t) \, dk - 2\nu \int_0^\infty k^2 E(k, t) \, dk = \frac{\partial k_i}{\partial t} = \epsilon$$

For isotropic turbulence $\partial k_i/\partial t = -\epsilon$, and consequently, the transfer term integral is zero

$$\int_0^\infty T(k, t) \, dk = 0$$

The term $T$ corresponds to the rate of energy transferred to successively smaller and smaller scales of the turbulent field. The function $S(k, t)$ defined by

$$S(k, t) = -\int_0^k T(k', t) \, dk'$$
Lin’s equation (cont.)

$S(k,t)$ represents the energy transferred from all the wavenumbers smaller than $k$ to wavenumbers larger than $k$

Note that this term $T$ is difficult to measure, and it makes sense only for high Reynolds number turbulent flows, in order to have an inertial region in the spectrum $E(k)$. 