Small class #1

Guidelines

(i) You can reuse without demonstration all the results introduced in the course (by only citing the considered slide number for instance);

(ii) An essential part of the work is to provide valuable comments of your results.

Temperature fluctuations in buoyancy-driven flows

The turbulent kinetic energy budget, which has been established during the lectures, is here revisited to include temperature effects in order to illustrate the possible competition between kinetic and thermal turbulence. The fluid is a perfect gas and the flow is assumed to be incompressible, $\nabla\cdot\boldsymbol{u} = 0$. The Navier-Stokes equation and the energy conservation written for the temperature T, are recalled below in a conservative form

$$\begin{cases} \dfrac{\partial(\rho\boldsymbol{u})}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla\cdot\overline{\overline{\boldsymbol{\tau}}} + \rho\boldsymbol{g} \\ \dfrac{\partial(\rho c_p T)}{\partial t} + \nabla\cdot(\rho c_p T\boldsymbol{u}) = -\nabla\cdot\boldsymbol{q} + \overline{\overline{\boldsymbol{\tau}}} : \nabla\boldsymbol{u} + \dfrac{Dp}{Dt} \end{cases} \tag{1}$$

where c_p is the specific heat at constant pressure ($dh = c_p dT$ for a perfect gas), $\boldsymbol{q} = -\lambda\nabla T$ (Fourier's law), $\overline{\overline{\boldsymbol{\tau}}} : \nabla\boldsymbol{u} = \tau_{ij}\,\partial u_i/\partial x_j$ is the viscous dissipation, and $g_i = -g\delta_{3i}$ is the gravity with $g \simeq 9.81$ m.s^{-2}. All the fluid properties are assumed to be constant (ν, c_p, λ) to simplify algebra.

A – Simplification of the equation of state. In the framework of a low Mach number approximation, density $\rho = \rho(T,p)$ can be considered as a function of temperature only, that is $\rho \simeq \rho(T)$, as shown below.

1. Derive the expressions of the thermal expansion coefficient β and of the isothermal compressibility coefficient χ for a perfect gas. Calculate their numerical values for the reference state $P_0 = 101325$ Pa, $T_0 = 288.15$ K and $\rho_0 = 1.225$ kg.m^{-3}. As a reminder,

$$\beta = -\frac{1}{\rho}\left.\frac{\partial\rho}{\partial T}\right|_p \qquad \chi = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial p}\right|_T \tag{2}$$

2. By considering the Taylor series of density around its mean value ρ_0, show that

$$\frac{\Delta\rho}{\rho_0} = -\frac{\Delta T}{T_0} + \frac{\Delta p}{P_0} \tag{3}$$

where $\Delta\rho = \rho - \rho_0$, $\Delta T = T - T_0$ and $\Delta p = p - P_0$ denote the small perturbation in density, temperature and pressure. Retrieve this result by considering the logarithmic differentiation of the perfect gas law.

3. Assuming that the total pressure $p + (\rho u^2)/2$ of a fluid particle remains constant during its motion and using Eq. (3), demonstrate that

$$\left(1 + \frac{\gamma}{2}M^2\right)\frac{\Delta\rho}{\rho} \simeq -\frac{\Delta T}{T} - \gamma M^2\frac{\Delta u}{U} \qquad \text{with} \quad M = \frac{U}{c} \tag{4}$$

4. Plot the variation of $\Delta\rho/\rho$ as a function of M for several values of $\Delta T/T$ and $\Delta u/U$. Conclude this first part.

B – Generalised Boussinesq approximation. We now consider a reference steady state $(\rho_0, P_0, \boldsymbol{u}_0 = 0, T_0)$ at equilibrium which satisfies the hydrostatic balance $-\nabla P_0 + \rho_0 \boldsymbol{g} = 0$ and $\nabla^2 T_0 = 0$, and the presence of a turbulent atmospheric boundary layer. The flow variables are then decomposed as follows

$$\rho = \rho_0 + \rho' \qquad p = P_0 + p^\star = P_0 + \bar{P} + p' \qquad \boldsymbol{u} = \bar{\boldsymbol{U}} + \boldsymbol{u}' \qquad T = T_0 + T^\star = T_0 + \bar{T} + \theta \tag{5}$$

to remove the effects of the static stratification. From the previous part A, the equation of state takes the form $\rho \simeq \rho_0(1 - \beta T^\star)$ with $\beta = 1/T_0$. The temperature T_0 is almost constant for an adiabatic atmosphere at the scale of the turbulent field.

5. Show that the momentum conservation equation can be recast as follows

$$\rho_0(1 - \beta T^\star)\frac{D\boldsymbol{u}}{Dt} = -\nabla p^\star + \nabla \cdot \overline{\overline{\boldsymbol{\tau}}} - \rho_0 \beta T^\star \boldsymbol{g} \tag{6}$$

6. Deduce that for small temperature perturbations, the Boussinesq approximation is obtained : density fluctuations are only taken into account in the buoyancy force $\rho \boldsymbol{g}$ in governing equations.

C – Temperature fluctuations in the lowest layer of Earth's atmosphere – In this third part, we use the previous Boussinesq approximation to derive the turbulent kinetic energy budget including buoyancy effects. We assume a mean velocity field $\bar{U}_1(x_3)$ consistent with geophysical practice.

7. Derive the equation for the mean temperature $\bar{T}$ by introducing the Reynolds decomposition (5). How can we simplify this equation for low Mach number flows ?

8. Show that the total mean heat flux $\bar{\boldsymbol{q}}^t$ can be written as

$$\bar{q}_j^t = -\rho_0 c_p \left(a \frac{\partial \bar{T}}{\partial x_j} - \overline{\theta u_j'} \right)$$

What does the coefficient a represent ? Provide its dimension. How could you simplify the expression of the total heat flux for a high value of the Peclet number ?

9. Compute the additional term associated with gravity in the equation for the fluctuating velocity component u_i', and in the equation for the turbulent kinetic energy k_t.

10. In order to quantify the importance of the buoyancy force effects, we can compare the additional term in the transport equation for k_t with the production term $\mathcal{P}$, by introducing the Richardson number Ri defined as

$$\mathrm{Ri} \equiv -\frac{\rho_0 g \overline{\theta u_3'}/T_0}{\mathcal{P}} \qquad \text{where} \qquad \mathcal{P} = -\rho_0\, \overline{u_1' u_3'}\, \frac{\partial \bar{U}_1}{\partial x_3}$$

Show that if $d\bar{T}/dx_3 < 0$, the additional term acts as a production term for the turbulent kinetic energy. What is the sign of the Richardson number ? Comment the dispersion of a pollutant for an unstable thermal stratification, that is when the temperature decreases with the altitude.

11. Repeat your analysis for the case of a stable thermal stratification.

12. A temperature inversion over the first few hundred meters is regularly observed in the atmospheric layer, leading to the trap of air pollution. Find a recent example of this phenomenon (internet, scientific paper written for a large audience), where the temperature inversion is documented and comment your finding in a few sentences.

Temperature fluctuations in buoyancy-driven flows

A – Simplification of the equation of state.

1. For a perfect gas, the equation of state reads $p = \rho r T$, leading to

$$\beta = -\frac{1}{\rho}\left.\frac{\partial \rho}{\partial T}\right|_{p,0} = \frac{1}{T_0} \simeq 3.5\times 10^{-3}\ \mathrm{K}^{-1} \qquad \chi = \frac{1}{\rho}\left.\frac{\partial \rho}{\partial p}\right|_{T,0} = \frac{1}{P_0} \simeq 9.9\times 10^{-6}\ \mathrm{Pa}^{-1}$$

2. The Taylor series of density at the first order is given by

$$\rho = \rho(T,p) \simeq \rho_0\Big[1 - \beta(T - T_0) + \chi(p - P_0)\Big] \tag{7}$$

by noting $\Delta\rho = \rho - \rho_0$, $\Delta T = T - T_0$ and $\Delta p = p - P_0$ a small perturbation with respect to the reference value. Using the perfect gas law, the previous result reads also as

$$\frac{\Delta\rho}{\rho_0} = -\frac{\Delta T}{T_0} + \frac{\Delta p}{P_0} \tag{8}$$

By considering the differentiation of the logarithm of $p = \rho r T$, it is straigthforward to obtain,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

in agreement with Eq. (8).

3. If the total pressure P_t remains constant during the motion of a fluid particle, one has

$$P_t = p + \frac{1}{2}\rho u^2 = \mathrm{cst} \tag{9}$$

and by differentiation,

$$\begin{aligned} \Delta p + \frac{1}{2}U^2\,\Delta\rho + \rho U\,\Delta u &= 0 \\ \frac{\Delta p}{P} &= -\frac{1}{2}U^2\frac{\Delta\rho}{P} - \frac{\rho}{P}U\,\Delta u \end{aligned} \tag{10}$$

Combining Eqs. (8) and (10) to express the relative variation of density as a function of temperature and velocity, one obtains

$$\left(1 + \frac{1}{2}U^2\frac{\rho}{P}\right)\frac{\Delta\rho}{\rho} = -\frac{\Delta T}{T} - \frac{\rho}{P}U^2\frac{\Delta u}{U}$$

For a perfect gas, one has $c^2 = \gamma P/\rho$ where c is the speed of sound, and the previous relation can be recast as follows,

$$\left(1 + \frac{\gamma}{2}M^2\right)\frac{\Delta\rho}{\rho} = -\frac{\Delta T}{T} - \gamma M^2\frac{\Delta u}{U} \tag{11}$$

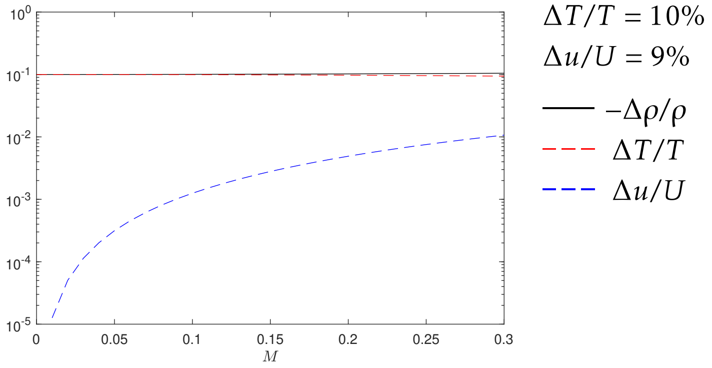
where $M = U/c$ is the Mach number. As expected, $\rho \simeq \rho(T)$ as M goes to zero.

As a remark, Eq. (9) can be recast as follows,

$$P_t = P\left(1 + \frac{\gamma}{2}M^2\right) \qquad \text{and} \qquad \frac{P_t}{P} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma}{\gamma-1}} \simeq 1 + \frac{\gamma}{2}M^2$$

when $M \to 0$.

4. As an illustration,



B – Generalised Boussinesq approximation.

5. By substracting the hydrostatic balance to the momentum conservation equation, using the simplification of the equation of state $\rho \simeq \rho_0(1-\beta T^\star)$ with $\beta = 1/T_0$, one gets

$$\rho_0(1-\beta T^\star)\frac{D\boldsymbol{u}}{Dt} = -\nabla p^\star + \nabla\cdot\overline{\overline{\tau}} - \rho_0\beta T^\star \boldsymbol{g}$$

6. For small temperature perturbations, the previous equation reduces to

$$\rho_0\frac{D\boldsymbol{u}}{Dt} = -\nabla p^\star + \nabla\cdot\overline{\overline{\tau}} - \rho_0\beta T^\star \boldsymbol{g}$$

C – Temperature fluctuations in the lowest layer of Earth's atmosphere

7. Equation gouverning the mean temperature $\bar{T}$

$$\rho_0 c_p\left(\frac{\partial \bar{T}}{\partial t} + \bar{\boldsymbol{U}}\cdot\nabla\bar{T}\right) = -\nabla\cdot\left(-\lambda\nabla\bar{T} + \rho_0 c_p\overline{\theta\boldsymbol{u}'}\right) + \bar{\boldsymbol{\tau}} : \nabla\bar{\boldsymbol{U}} + \overline{\boldsymbol{\tau}' : \nabla\boldsymbol{u}'} + \frac{\partial\bar{P}}{\partial t} + \bar{\boldsymbol{U}}\cdot\nabla\bar{P} + \nabla\overline{p'\boldsymbol{u}'}$$

The red terms are small terms (order two with respect to the Mach number), and can be neglected. We can identify the two dissipation terms coming from the mean and turbulent kinetic energy budget (refer to slides / Chapter 2).

For the case where there is only a single mean velocity component $\bar{U}_1 = U_(x_3)$ and for a steady mean flow,

$$\rho_0 c_p \bar{U}_1\frac{\partial\bar{T}}{\partial x_3} = -\nabla\cdot\left(-\lambda\nabla\bar{T} + \rho_0 c_p\overline{\theta\boldsymbol{u}'}\right) + 2\mu\left(\frac{\partial\bar{U}_1}{\partial x_3}\right)^2 + \rho\epsilon + \bar{U}_1\frac{\partial\bar{P}}{\partial x_1} + \frac{\partial}{\partial x_i}\overline{p'u_i'}$$

and

$$-\frac{\partial\bar{P}}{\partial x_1} + \frac{\partial}{\partial x_j}(-\rho_0\overline{u_1'u_j'}) = 0$$

8. The blue term in the previous equation is identified as the total thermal flux $\bar{\boldsymbol{q}}^t$, and $a \equiv \lambda/(\rho c_p)$ is the thermal diffusivity in $m^2.s^{-1}$ (same unit as ν the kinematic viscosity),

$$\bar{q}_j^t \equiv -\lambda\frac{\partial\bar{T}}{\partial x_j} + \rho_0 c_p\overline{\theta u_j'} = -\rho_0 c_p\left(a\frac{\partial\bar{T}}{\partial x_j} - \overline{\theta u_j'}\right)$$

The Peclet number plays the same role as the Reynolds number for the Navier-Stokes equation. For high values of the Peclet number,

$$\bar{q}_j^t \simeq \rho_0 c_p\overline{\theta u_j'}$$

9. The stating point is Eq. (6)

$$\rho_0 \frac{D\boldsymbol{u}}{Dt} = -\nabla p^\star + \nabla \cdot \overline{\overline{\boldsymbol{\tau}}} - \rho_0 \beta T^\star \boldsymbol{g}$$

that is

$$\frac{\partial}{\partial t}(\rho_0 u_i) + \frac{\partial}{\partial x_j}(\rho_0 u_i u_j) = -\frac{\partial p^\star}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho_0 \beta T^\star g \delta_{3i} \tag{12}$$

RANS equation

$$\frac{\partial}{\partial t}(\rho_0 \bar{U}_i) + \frac{\partial}{\partial x_j}(\rho_0 \bar{U}_i \bar{U}_j) = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}(\bar{\tau}_{ij} - \rho_0 \overline{u_i' u_j'}) + \rho_0 \beta \bar{T} g \delta_{3i} \tag{13}$$

Eq. for u_i' given by Eq. (12) – Eq. (13).

$$\frac{\partial}{\partial t}(\rho_0 u_i') + \frac{\partial}{\partial x_j}(\rho_0 u_i' \bar{U}_j + \rho_0 \bar{U}_i u_j' + \rho_0 u_i' u_j') = -\frac{\partial p'}{\partial x_i} + \frac{\partial}{\partial x_j}(\tau_{ij}' + \rho_0 \overline{u_i' u_j'}) + \rho_0 \beta \Theta g \delta_{3i}$$

Multiplying the resulting equation by u_i' leads to the transport equation for the turbulent kinetic energy k_t,

$\rightsquigarrow$ additional term : $\rho_0 g \beta \overline{\Theta u_3'}$

$$\frac{\partial}{\partial t}(\rho_0 k_t) + \frac{\partial}{\partial x_i}(\rho_0 k_t \bar{U}_i) = -\rho_0 \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} + \rho_0 g \beta \overline{\Theta u_3'} - \rho_0 \epsilon - \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i' u_i' u_j'} + \overline{p' u_j'} - \overline{u_i' \tau_{ij}'}\right)$$

It is always a good recommendation to examine the case of homogeneous turbulence, to remove transport terms. The turbulent kinetic energy budget becomes

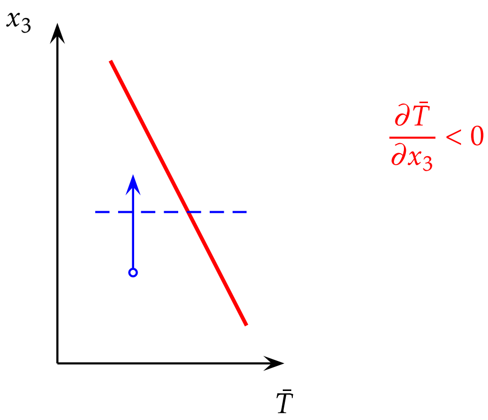
$$\begin{aligned} \rho_0 \frac{\partial k_t}{\partial t} &= -\rho_0 \overline{u_i' u_j'} \frac{\partial \bar{U}_i}{\partial x_j} + \rho_0 g \beta \overline{\Theta u_3'} - \rho_0 \epsilon \\ &= \text{kinetic production} + \text{thermal production} - \text{dissipation} \end{aligned}$$

The presence of the buoyancy term is associated with a second production term ; only through the u_3' velocity component, anisotropy is expected

10. Flux Richardson number Ri_f

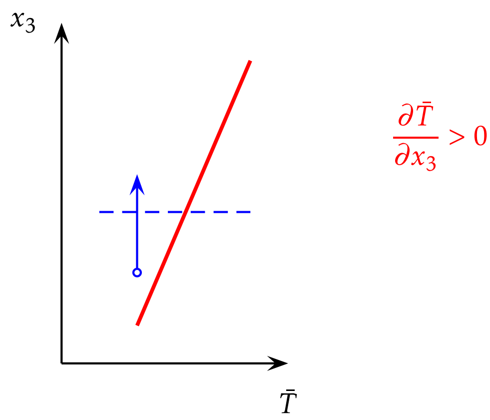
$$\mathrm{Ri}_f \equiv -\frac{\rho_0 g \beta \overline{\Theta u_3'}}{\mathcal{P}} \qquad \text{where} \qquad \mathcal{P} = -\rho_0 \overline{u_1' u_3'} \frac{d\bar{U}_1}{dx_3}$$

The production term of k_t is then given by $\mathcal{P}\,(1 - \mathrm{Ri}_f)$. Discussion about thermal vs kinematic turbulence : the sign of the flux Richardson number leads to destruction or amplification of $\overline{u_3'^2}$



$\left| \begin{array}{l} u'_3 > 0 \\ \theta > 0 \end{array} \right.$ $\overline{u'_3\theta} > 0$ $\mathrm{Ri}_f < 0$

unstable atmosphere
increase of turbulence and dispersion
$\overline{u_3'^2} > \overline{u_1'^2}, \overline{u_2'^2}$
(leakage)



$\left| \begin{array}{l} u'_3 > 0 \\ \theta < 0 \end{array} \right.$ $\overline{u'_3\theta} < 0$ $\mathrm{Ri}_f > 0$

stable atmosphere
inversion of temperature profile
destruction of $\overline{u_3'^2}$, $\sim$ 2-D turbulence
(leazy, fire in a tunnel for instance)

Critical value $\mathrm{Ri}_f^\star \geq 0.25$, $\overline{u_3'^2} \to 0$ (Chimonas, *J. Fluid Mech.*, 1970)

11. As an illustration

Fairbanks Alaska in 2012
(from The Official Blog for author Sue Ann Bowling)

Rising smoke in Lochcarron forms a ceiling over the valley due to a temperature inversion
(30 janvier 2006, ©JohanTheGhost)