Small class #3

Guidelines

(i) You can reuse without demonstration all the results mentioned in the slides of the course (by only citing the considered slide number for instance)

(ii) An essential part of your assignments is to provide valuable comments of your results

Work to do – At the end of this session, you will be asked to write a personal report with your answers and comments, due on Monday, January 8, 2024.

Turbulent boundary layer model

The mean velocity of a zero-pressure-gradient turbulent boundary layer is governed by the following equation introduced in classroom,

$$\int_0^{x_2} \rho\left(\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2}\right) dx_2 = \bar{\tau}_t - \tau_w \tag{1}$$

where $\bar{\tau}_t$ is the total shear stress, defined and modeled as follows

$$\bar{\tau}_t = -\rho\overline{u_1'u_2'} + \mu\frac{\partial \bar{U}_1}{\partial x_2} = (\mu + \mu_t)\frac{\partial \bar{U}_1}{\partial x_2} \tag{2}$$

Eq. (2) can be recast as follows using wall unit variables,

$$\frac{d\bar{U}_1^+}{dx_2^+} = \frac{\bar{\tau}_t^+}{1+\nu_t^+} \tag{3}$$

where a turbulent model must be used to compute the turbulent viscosity ν_t^+, and where τ_t^+ must be approximated from Eq. (1). A simple approximation developed in the course is to choose $\bar{\tau}_t^+ = 1 - x_2/\delta \simeq 1$ for $x_2/\delta \ll 1$, that is for the inner part of the profile. The edge of the boundary layer corresponding to the wake region is not investigated in this first part.

1. Solve numerically Eq. (3) for the case of a mixing-length model

$$l_m^+ = \kappa x_2^+ \tag{4}$$

to compute the turbulent viscosity ν_t^+. The value of the von Kármán constant is chosen to be equal to $\kappa \simeq 0.41$. In order to integrate Eq. (3), we can obtain an analytical expression for the derivative $d\bar{U}_1^+/dx_2^+$ by solving a quadratic equation, and then numerically integrate this equation from $x_2^+ = 0$ using a Runge-Kutta algorithm (with Matlab using *ode45.m* for instance, see Appendix).

2. Estimate the constant B of the log-law by computing the expression

$$B = \bar{U}_1^+ - \frac{1}{\kappa}\ln x_2^+$$

from your numerical solution for $x_2^+ = 200, 300, 400$ and 500. Comment on this result.

3. Compare your numerical solution to the exact analytical solution of Eq. (3), provided by Hinze[3]

$$U_1^+ = \frac{1}{\kappa}\frac{1-\sqrt{1+4\,(\kappa x_2^+)^2}}{2\,\kappa x_2^+} + \frac{1}{\kappa}\ln\left[2\kappa x_2^+ + \sqrt{1+4\,(\kappa x_2^+)^2}\right]$$

4. Repeat the numerical integration of Eq. (3) with the following mixing-length model including a dumping function, proposed by Van Driest[6]

$$l_m^+ = \kappa x_2^+ \left(1 - e^{-x_2^+/A_0^+}\right) \qquad \text{with} \qquad A_0^+ = 26 \tag{5}$$

5. Plot on the same graph both numerical solutions, as well as the viscous sublayer and the inner log laws. Comment.

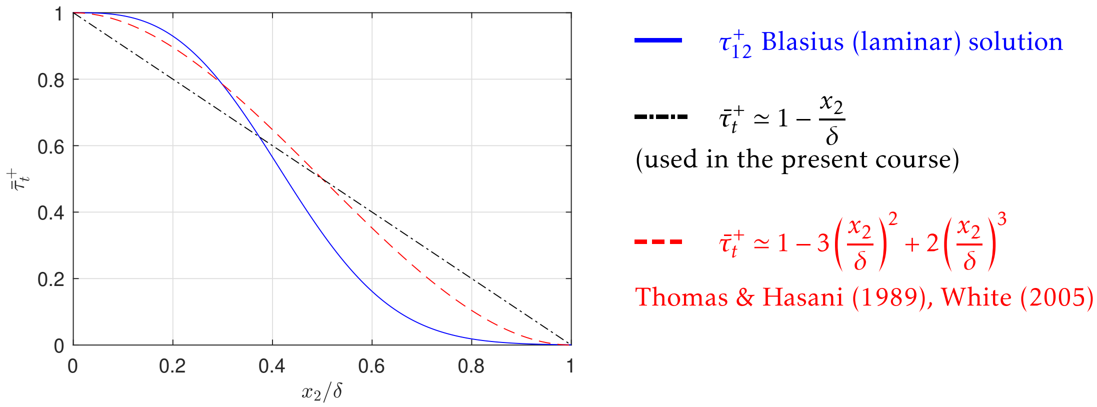
6. Perform a Taylor series of the functions l_m^+ and $-\overline{u_1' u_2'}^+$ for both mixing-length models (4) and (5), and provide the behaviour of these two functions as $x_2^+ \to 0$.

7. By considering the incompressibility condition $\nabla \cdot \boldsymbol{u}' = 0$, provide the theoretical behaviour of $-\overline{u_1' u_2'}^+$ as $x_2^+ \to 0$.

8. Repeat the numerical integration using the Van Driest model (5) for solving Eq. (3), by using now the expression of Thomas & Hasani[5,7] for $\bar{\tau}_t^+$

$$\bar{\tau}_t^+ \simeq 1 - 3\left(\frac{x_2}{\delta}\right)^2 + 2\left(\frac{x_2}{\delta}\right)^3$$

which improves the linear approximation[1] $\bar{\tau}_t^+ \simeq 1 - (x_2/\delta)$. The following numerical value $\mathrm{Re}^+ \simeq 2675$ can be used to compare to experimental data.



9. Consider now the following improved mixing length model[4] including the wake region

$$\begin{cases} l_{m\mathrm{inner}}^+ = \kappa x_2^+ \sqrt{\bar{\tau}^+}\left(1 - e^{-x_2^+/A_0}\right) \\ l_{m\mathrm{outer}}^+ = A_w \mathrm{Re}^+ \\ l_m^+ = l_{m\mathrm{outer}}^+ \tanh\left(l_{m\mathrm{inner}}^+ / l_{m\mathrm{outer}}^+\right) \end{cases} \tag{6}$$

where the constant value is $A_w = 0.085$. Compute the mean velocity profile for the case $\mathrm{Re}^+ \simeq 2675$. Comment your results and the construction of the model (6).

10. Bonus : retrieve the laminar solution (see figure in question 8) by using Appendix B.

References

[1] Bailly, C. & Comte Bellot, G., 2015, *Turbulence* (in english), Springer, Heidelberg.

[2] Cebeci, T. & Keller, H. B., 1971, Shooting and parallel shooting methods for solving the Falkner-Skan boundara leyer equation, *Comput. Phys.*, **7**, 289-300.

[3] Hinze, J. O., 1975, *Turbulence*, McGraw-Hill International Book Company, New York.

[4] Prigent, S.L. & Bailly, C., 2022, From shear stress to wall pressure spectra : a semi-analytical approach to account for mean pressure gradients in turbulent boundary layers, *Acta Acustica*, **6**, 43, 1-12.

[5] Thomas, L. C. & Hasani, S. M. F., 1989, Supplementary boundary-layer approximations for turbulent flow, *Journal of Fluids Engineering* (Transactions of the ASME), **111**, 420-427.

[6] Van Driest, E. R., 1956, On turbulent flow near a wall, *Journal of Aeronautical Sciences*, **23**(11), 1007-1011.

[7] White, F., 2005, Viscous fluid flow, 3ed Ed., *McGraw-Hill, Inc.*, New-York (1st Ed. 1974).

Appendix A : Runge-Kutta scheme

In order to integrate ordinary differential equation, you can directly used the *ode45.m* function from Matlab. You can also write your own script from the following 4th-order Runge-Kutta algorithm. To integrate the first-order differential equation $\partial \boldsymbol{U}/\partial t = \boldsymbol{F}(\boldsymbol{U},t)$, consider

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \Delta t\left(b_1\boldsymbol{K}^1 + b_2\boldsymbol{K}^2 + b_3\boldsymbol{K}^3 + b_4\boldsymbol{K}^4\right)$$

where

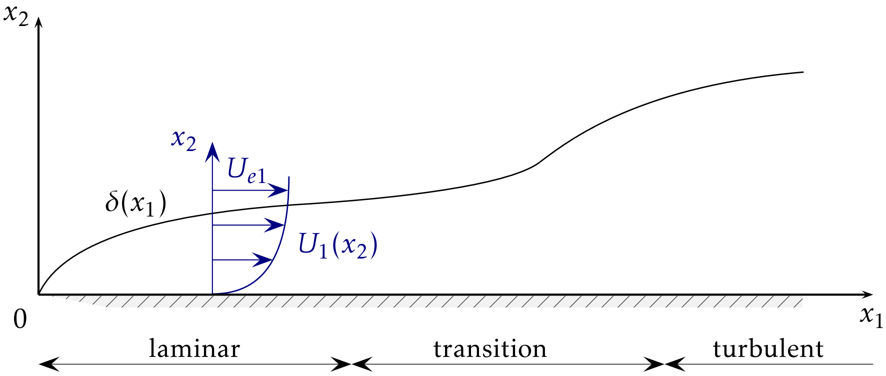
$$\begin{cases} \boldsymbol{K}^1 = \boldsymbol{F}(\boldsymbol{U}^n, t^n) \\ \boldsymbol{K}^2 = \boldsymbol{F}(\boldsymbol{U}^n + a_{21}\boldsymbol{K}^1, t^n + c_2\Delta t) \\ \boldsymbol{K}^3 = \boldsymbol{F}(\boldsymbol{U}^n + a_{32}\boldsymbol{K}^2, t^n + c_3\Delta t) \\ \boldsymbol{K}^4 = \boldsymbol{F}(\boldsymbol{U}^n + a_{43}\boldsymbol{K}^3, t^n + c_4\Delta t) \end{cases}$$

and with

$$\begin{array}{c|c} c_i & a_{ij} \\ \hline & b_i \end{array} \qquad\qquad \begin{array}{l|cccc} c_1 = 0 & 0 & & & \\ c_2 = 1/2 & 1/2 & & & \\ c_3 = 1/2 & 0 & 1/2 & & \\ c_4 = 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

Appendix B : Blasius solution

Consider a laminar two-dimensional boundary layer developing along a flat plate in the presence of a uniform and steady external flow U_{e1}.



Development of a zero-pression-gradient boundary layer on a flat plate. The transition occurs for $\mathrm{Re}_{x_1} = x_1 U_{e1}/\nu \simeq 3.2\times 10^5$, that is $\mathrm{Re}_\delta = \delta U_{e1}/\nu \simeq 2800$.

1. Recall without demonstration the governing equations for a laminar boundary layer, and very briefly the underlying assumptions.

2. Introduce the stream function ψ defined by $U_1 = \partial\psi/\partial x_2$ and $U_2 = -\partial\psi/\partial x_1$, and show that the problem consists in solving the equation

$$\frac{\partial\psi}{\partial x_2}\frac{\partial^2\psi}{\partial x_1\partial x_2} - \frac{\partial\psi}{\partial x_1}\frac{\partial^2\psi}{\partial x_2^2} = \nu\frac{\partial^3\psi}{\partial x_2^3}$$

associated with the following boundary conditions,

$$\left.\frac{\partial\psi}{\partial x_2}\right|_{x_2=0} = 0 \qquad \left.\frac{\partial\psi}{\partial x_1}\right|_{x_2=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial\psi}{\partial x_2}\right|_{x_2\to\infty} = U_{e1}$$

3. Recall very briefly the reasoning that leads to $\delta \sim \sqrt{\nu x_1/U_{e1}}$. The self similarity variable η is then introduced

$$\eta = x_2\sqrt{\frac{U_{e1}}{\nu x_1}} \qquad \text{and consequently,} \qquad \psi(x_1, x_2) = \sqrt{U_{e1}\nu x_1}\, f(\eta)$$

Show that the ordinary differential equation satisfied by f, known as the Blasius equation (1908), reads

$$2f''' + ff'' = 0 \tag{7}$$

and provide the associated boundary conditions.

4. In order to solve numerically Eq. (7), this equation is recast into a first-order system (which can be solved using Matlab with *ode45.m* for example)

$$\begin{cases} f' = g \\ g' = h \\ h' = -\frac{1}{2}fh \end{cases} \tag{8}$$

over the interval $0 \leq \eta \leq 10$, with the following initial conditions at $\eta = 0$: $f(0) = 0$, $g(0) = 0$ and $h(0) = \alpha$. We then seek to determine the value of α by a so-called shooting method, in order to satisfy the outer boundary condition at $\eta = 10$, that is $N(\alpha) \equiv g - 1 = 0$. A simple and efficiency method is to use a Newton algorithm,

$$\alpha^{n+1} = \alpha^n - N(\alpha^n) / \left.\frac{\partial N}{\partial\alpha}\right|_{\alpha^n}$$

and to observe that the derivative $\partial N/\partial\alpha$ can be obtained by solving, in parallel to (8), the following variational system[2]

$$\begin{cases} F' = G \\ G' = H \\ H' = -\frac{1}{2}(Fh + fH) \end{cases} \qquad \text{where} \qquad \begin{cases} F = \partial f/\partial\alpha \\ G = \partial g/\partial\alpha \\ H = \partial h/\partial\alpha \end{cases} \tag{9}$$

with the boundary conditions $F(0) = 0, G(0) = 0$ and $H(0) = 1$.
Solve both Eqs (8) and Eqs (9) to determine numerically α and thus the function f.

5. From the computation of the f function, check the following properties for a zero-pressure-gradient laminar boundary layer,

$$\frac{\delta_{0.99}}{x_1} \simeq \frac{4.91}{\mathrm{Re}_{x_1}^{1/2}} \qquad \frac{\delta_1}{x_1} \simeq \frac{1.72}{\mathrm{Re}_{x_1}^{1/2}} \qquad \frac{\delta_\theta}{x_1} \simeq \frac{0.664}{\mathrm{Re}_{x_1}^{1/2}} \qquad c_f = \frac{\tau_w}{\frac{1}{2}\rho U_{e1}^2} \simeq \frac{0.664}{\mathrm{Re}_{x_1}^{1/2}}$$

where δ_1 is the displacement thickness and δ_θ is the momentum thickness. Calculate the expression of the transverse velocity U_2. What value does U_2 take when $\eta \to \infty$? Provide an interpretation.

6. Show that

$$U_1\frac{\partial U_1}{\partial x_1} + U_2\frac{\partial U_1}{\partial x_2} = -\frac{U_{e1}^2}{2x_1}ff''$$

and then, by considering the integration of the momentum equation from the wall to a current point x_2, plot the evolution of the normalized shear stress $\tau_{12}^+ \equiv \tau_{12}/\tau_w$.