Small class #3

Work to do – At the end of this session, you will be asked to write a personal report with your answers and comments, due on Friday, January 10, 2025.

Turbulent boundary layer model

The mean velocity of a zero-pressure-gradient turbulent boundary layer is governed by the following equation introduced in the course,

$$\int_0^{x_2} \rho\left(\bar{U}_1 \frac{\partial \bar{U}_1}{\partial x_1} + \bar{U}_2 \frac{\partial \bar{U}_1}{\partial x_2}\right) dx_2 = \bar{\tau}_t - \tau_w \tag{1}$$

where $\bar{\tau}_t$ is the total shear stress. It is also defined as,

$$\bar{\tau}_t = -\rho\overline{u_1' u_2'} + \mu \frac{\partial \bar{U}_1}{\partial x_2} = (\mu + \mu_t)\frac{\partial \bar{U}_1}{\partial x_2} \tag{2}$$

and modeled by introducing a turbulent viscosity.
Eq. (2) can be recast as follows using wall unit variables,

$$\frac{d\bar{U}_1^+}{dx_2^+} = \frac{\bar{\tau}_t^+}{1 + \nu_t^+} \tag{3}$$

and by developing the expression of the turbulent viscosity ν_t^+, one gets

$$(l_m^+)^2 \left(\frac{d\bar{U}_1^+}{dx_2^+}\right)^2 + \frac{d\bar{U}_1^+}{dx_2^+} - \bar{\tau}_t^+ = 0$$

and the positive root is selected to solve $d\bar{U}_1^+/dx_2^+$, which leads to

$$\frac{d\bar{U}_1^+}{dx_2^+} = \frac{-1 + \sqrt{1 + 4\, l_m^{+2}\, \bar{\tau}_t^+}}{2\, l_m^{+2}} = \frac{2\bar{\tau}_t^+}{1 + \sqrt{1 + 4\, l_m^{+2}\, \bar{\tau}_t^+}} \tag{4}$$

The total shear stress $\bar{\tau}_t^+$ must be approximated from Eq. (1). A simple choice done in the course is to consider $\bar{\tau}_t^+ = 1 - x_2/\delta \simeq 1$ for $x_2/\delta \ll 1$, that is for the **inner part** of the velocity profile. The edge of the boundary layer corresponding to the wake region is not investigated in this first part.

Part 1 – Inner region

1. Solve numerically Eq. (3) for the case of a mixing-length model

$$l_m^+ = \kappa x_2^+ \tag{5}$$

   to compute the turbulent viscosity ν_t^+. The value of the von Kármán constant is chosen to be equal to $\kappa \simeq 0.41$. In order to integrate Eq. (4), we can obtain an analytical expression for the derivative $d\bar{U}_1^+/dx_2^+$ by solving a quadratic equation, and then numerically integrate this equation from $x_2^+ = 0$ using a Runge-Kutta algorithm (with Matlab using *ode45.m* for instance).

2. Estimate the constant B of the log-law by computing the expression

$$B = \bar{U}_1^+ - \frac{1}{\kappa}\ln x_2^+$$

   from your numerical solution for $x_2^+ = 200, 300, 400$ and 500. Comment on this result.

3. Compare your numerical solution to the exact analytical solution of Eq. (4), provided by Hinze[3]

$$U_1^+ = \frac{1}{\kappa}\frac{1-\sqrt{1+4\,(\kappa x_2^+)^2}}{2\,\kappa x_2^+} + \frac{1}{\kappa}\ln\left[2\kappa x_2^+ + \sqrt{1+4\,(\kappa x_2^+)^2}\right]$$

4. Repeat the numerical integration of Eq. (3) with the following mixing-length model including a dumping function, proposed by Van Driest[6]

$$l_m^+ = \kappa x_2^+\left(1-e^{-x_2^+/A_0^+}\right) \qquad \text{with} \qquad A_0^+ = 26 \tag{6}$$

5. Plot on the same graph both numerical solutions, as well as the viscous sublayer and the inner log laws. Comment.

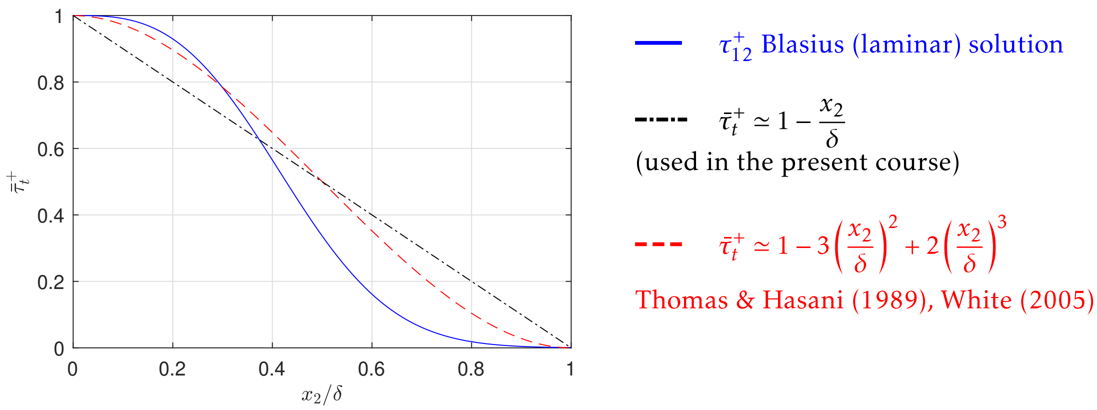
6. Perform a Taylor series of the functions l_m^+ and $-\overline{u_1' u_2'}^+$ for both mixing-length models (5) and (6), and provide the behaviour of these two functions as $x_2^+ \to 0$.

7. By considering the incompressibility condition $\nabla\cdot\boldsymbol{u}' = 0$, provide the theoretical behaviour of $-\overline{u_1' u_2'}^+$ as $x_2^+ \to 0$.

8. Repeat the numerical integration using the Van Driest model (6) for solving Eq. (4), by using now the expression of Thomas & Hasani[5,7] for $\bar{\tau}_t^+$

$$\bar{\tau}_t^+ \simeq 1 - 3\left(\frac{x_2}{\delta}\right)^2 + 2\left(\frac{x_2}{\delta}\right)^3$$

which improves the linear approximation[1] $\bar{\tau}_t^+ \simeq 1-(x_2/\delta)$. The following numerical value $\mathrm{Re}^+ \simeq 2675$ can be used to compare to experimental data.



Part 2 – Complete velocity profile

9. Consider now the following improved mixing length model[4] including the wake region

$$\begin{cases} l_{m\text{inner}}^+ = \kappa x_2^+ \sqrt{\bar{\tau}^+}(1-e^{-x_2^+/A_0}) \\ l_{m\text{outer}}^+ = A_w \mathrm{Re}^+ \\ l_m^+ = l_{m\text{outer}}^+ \tanh(l_{m\text{inner}}^+/l_{m\text{outer}}^+) \end{cases} \tag{7}$$

where the constant value is $A_w = 0.085$. Compute the mean velocity profile for the case $\mathrm{Re}^+ \simeq 2675$. Comment your results and the construction of the model (7).

Part 3 – Bonus

10. Retrieve the laminar solution (see figure in question 8) in following Appendix A.

References

[1] Bailly, C. & Comte Bellot, G., 2015, *Turbulence* (in english), Springer, Heidelberg.

[2] Cebeci, T. & Keller, H. B., 1971, Shooting and parallel shooting methods for solving the Falkner-Skan boundara leyer equation, *Comput. Phys.*, **7**, 289-300.

[3] Hinze, J. O., 1975, *Turbulence*, McGraw-Hill International Book Company, New York.

[4] Prigent, S.L. & Bailly, C., 2022, From shear stress to wall pressure spectra : a semi-analytical approach to account for mean pressure gradients in turbulent boundary layers, *Acta Acustica*, **6**, 43, 1-12.

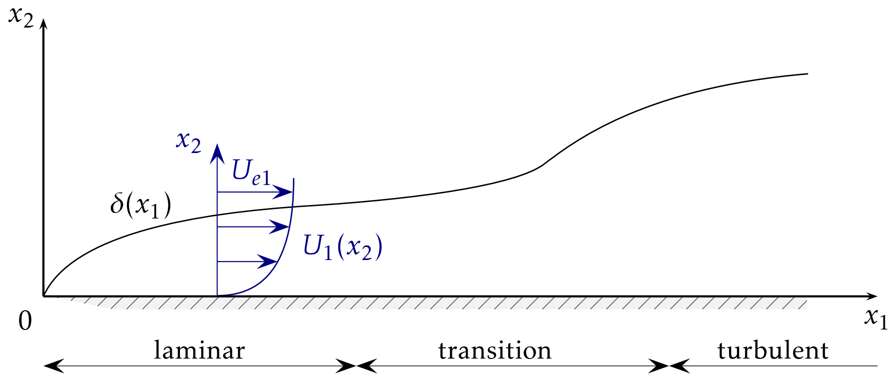
[5] Thomas, L. C. & Hasani, S. M. F., 1989, Supplementary boundary-layer approximations for turbulent flow, *Journal of Fluids Engineering* (Transactions of the ASME), **111**, 420-427.

[6] Van Driest, E. R., 1956, On turbulent flow near a wall, *Journal of Aeronautical Sciences*, **23**(11), 1007-1011.

[7] White, F., 2005, Viscous fluid flow, 3ed Ed., *McGraw-Hill, Inc.*, New-York (1st Ed. 1974).

Appendix A : Blasius solution

Consider a laminar two-dimensional boundary layer developing along a flat plate in the presence of a uniform and steady external flow U_{e1}.



Development of a zero-pression-gradient boundary layer on a flat plate. The transition occurs for $\mathrm{Re}_{x_1} = x_1 U_{e1}/\nu \simeq 3.2 \times 10^5$, that is $\mathrm{Re}_\delta = \delta U_{e1}/\nu \simeq 2800$.

1. Recall without demonstration the governing equations for a laminar boundary layer, and very briefly the underlying assumptions.

2. Introduce the stream function ψ defined by $U_1 = \partial\psi/\partial x_2$ and $U_2 = -\partial\psi/\partial x_1$, and show that the problem consists in solving the equation

$$\frac{\partial\psi}{\partial x_2}\frac{\partial^2\psi}{\partial x_1\partial x_2} - \frac{\partial\psi}{\partial x_1}\frac{\partial^2\psi}{\partial x_2^2} = \nu\frac{\partial^3\psi}{\partial x_2^3}$$

associated with the following boundary conditions,

$$\left.\frac{\partial\psi}{\partial x_2}\right|_{x_2=0} = 0 \qquad \left.\frac{\partial\psi}{\partial x_1}\right|_{x_2=0} = 0 \qquad \text{and} \qquad \left.\frac{\partial\psi}{\partial x_2}\right|_{x_2\to\infty} = U_{e1}$$

3. Recall very briefly the reasoning that leads to $\delta \sim \sqrt{\nu x_1/U_{e1}}$. The self similarity variable η is then introduced

$$\eta = x_2\sqrt{\frac{U_{e1}}{\nu x_1}} \qquad \text{and consequently,} \qquad \psi(x_1,x_2) = \sqrt{U_{e1}\nu x_1}\, f(\eta)$$

Show that the ordinary differential equation satisfied by f, known as the Blasius equation (1908), reads

$$2f''' + ff'' = 0 \tag{8}$$

and provide the associated boundary conditions.

4. In order to solve numerically Eq. (8), this equation is recast into a first-order system (which can be solved using Matlab with *ode45.m* for example)

$$\begin{cases} f' = g \\ g' = h \\ h' = -\frac{1}{2} f h \end{cases} \tag{9}$$

over the interval $0 \leq \eta \leq 10$, with the following initial conditions at $\eta = 0$: $f(0) = 0$, $g(0) = 0$ and $h(0) = \alpha$. We then seek to determine the value of α by a so-called shooting method, in order to satisfy the outer boundary condition at $\eta = 10$, that is $N(\alpha) \equiv g - 1 = 0$. A simple and efficiency method is to use a Newton algorithm,

$$\alpha^{n+1} = \alpha^n - N(\alpha^n) / \left. \frac{\partial N}{\partial \alpha} \right|_{\alpha^n}$$

and to observe that the derivative $\partial N / \partial \alpha$ can be obtained by solving, in parallel to (9), the following variational system[2]

$$\begin{cases} F' = G \\ G' = H \\ H' = -\frac{1}{2}(Fh + fH) \end{cases} \quad \text{where} \quad \begin{cases} F = \partial f / \partial \alpha \\ G = \partial g / \partial \alpha \\ H = \partial h / \partial \alpha \end{cases} \tag{10}$$

with the boundary conditions $F(0) = 0, G(0) = 0$ and $H(0) = 1$.
Solve both Eqs (9) and Eqs (10) to determine numerically α and thus the function f.

5. From the computation of the f function, check the following properties for a zero-pressure-gradient laminar boundary layer,

$$\frac{\delta_{0.99}}{x_1} \simeq \frac{4.91}{\mathrm{Re}_{x_1}^{1/2}} \qquad \frac{\delta_1}{x_1} \simeq \frac{1.72}{\mathrm{Re}_{x_1}^{1/2}} \qquad \frac{\delta_\theta}{x_1} \simeq \frac{0.664}{\mathrm{Re}_{x_1}^{1/2}} \qquad c_f = \frac{\tau_w}{\frac{1}{2}\rho U_{e1}^2} \simeq \frac{0.664}{\mathrm{Re}_{x_1}^{1/2}}$$

where δ_1 is the displacement thickness and δ_θ is the momentum thickness. Calculate the expression of the transverse velocity U_2. What value does U_2 take when $\eta \to \infty$? Provide an interpretation.

6. Show that

$$U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{U_{e1}^2}{2x_1} f f''$$

and then, by considering the integration of the momentum equation from the wall to a current point x_2, plot the evolution of the normalized shear stress $\tau_{12}^+ \equiv \tau_{12}/\tau_w$.