Velocity profile of a laminar boundary layer

Consider a laminar two-dimensional boundary layer developing along a flat plate in the presence of a uniform and steady external flow $U_e$.

Development of a zero-pressure-gradient boundary layer on a flat plate. The transition occurs for $Re = x_1 U_e / \nu \approx 3.2 \times 10^3$, that is $Re_\delta = \delta U_e / \nu \approx 2800$.

1. Recall without demonstration the governing equations for a laminar boundary layer, and very briefly the underlying assumptions.

2. Introduce the stream function $\psi$ defined by $U_1 = \partial \psi / \partial x_2$ and $U_2 = -\partial \psi / \partial x_1$, and show that the problem consists in solving the equation

$$\frac{\partial \psi}{\partial x_2} \frac{\partial^2 \psi}{\partial x_1 \partial x_2} - \frac{\partial \psi}{\partial x_1} \frac{\partial^2 \psi}{\partial x_2^2} = \nu \frac{\partial^3 \psi}{\partial x_2^3}$$

associated with the following boundary conditions,

$$\frac{\partial \psi}{\partial x_2} \bigg|_{x_2=0} = 0 \quad \frac{\partial \psi}{\partial x_1} \bigg|_{x_1=0} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial x_2} \bigg|_{x_2 \to \infty} = U_e$$

3. Recall very briefly the reasoning that leads to $\delta \sim \sqrt{\nu x_1 / U_e}$. The self similarity variable $\eta$ is then introduced

$$\eta = x_2 \sqrt{\frac{U_e}{\nu x_1}}$$

and consequently,

$$\psi(x_1, x_2) = \sqrt{U_e \nu x_1} f(\eta)$$

Show that the ordinary differential equation satisfied by $f$, known as the Blasius equation (1908), reads

$$2f''' + ff'' = 0$$

and provide the associated boundary conditions.
4. In order to solve numerically Eq. (1), this equation is recast into a first-order system (which can be solved using Matlab with ode45.m for example)

\[
\begin{aligned}
&f' = g \\
g' = h \\
h' = -\frac{1}{2}fh \\
\end{aligned}
\]

(2)

over the interval 0 ≤ η ≤ 10, with the following initial conditions at η = 0 : f(0) = 0, g(0) = 0 and h(0) = a. We then seek to determine the value of a by a so-called shooting method, in order to satisfy the outer boundary condition at η = 10, that is \( N(a) \equiv g - 1 = 0 \). A simple and efficiency method is to use a Newton algorithm,

\[ a^{n+1} = a^n - \frac{N(a^n)}{\frac{dN}{d\alpha}|_{a^n}} \]

and to observe that the derivative \( \frac{dN}{d\alpha} \) can be obtained by solving, in parallel to (2), the following variational system

\[
\begin{aligned}
&F' = G \\
&G' = H \\
&H' = -\frac{1}{2}(Fh + fH) \\
\end{aligned}
\]

where

\[
\begin{aligned}
F &= \frac{\partial f}{\partial \alpha} \\
G &= \frac{\partial g}{\partial \alpha} \\
H &= \frac{\partial h}{\partial \alpha} \\
\end{aligned}
\]

(3)

with the boundary conditions \( F(0) = 0, G(0) = 0 \) and \( H(0) = 1 \).

Solve both Eqs (2) and Eqs (3) to determine numerically \( a \) and thus the function \( f \).

5. From the computation of the \( f \) function, check the following properties for a zero-pressure-gradient laminar boundary layer,

\[
\begin{aligned}
\delta_{0.99} &\approx 4.91 \frac{x_1}{\text{Re}_{x_1}^{1/2}} \\
\delta^* &\approx 1.72 \frac{x_1}{\text{Re}_{x_1}^{1/2}} \\
\delta_{\theta} &\approx 0.664 \frac{x_1}{\text{Re}_{x_1}^{1/2}} \\
c_f &\approx \frac{\tau_w}{\frac{1}{2}\rho U_1^2} \approx 0.664 \frac{1}{\text{Re}_{x_1}^{1/2}} \\
\end{aligned}
\]

where \( \delta^* \) is the displacement thickness and \( \delta_{\theta} \) is the momentum thickness. Calculate the expression of the transverse velocity \( U_2 \). What value does \( U_2 \) take when \( \eta \to \infty \)? Provide an interpretation.

6. Show that

\[
U_1 \frac{\partial U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = -\frac{U_1^2}{2x_1} f f'''
\]

and then, by considering the integration of the momentum equation from the wall to a current point \( x_2 \), plot the evolution of the normalized shear stress \( \tau_{12}^{*} \equiv \tau_{12}/\tau_w \). Compare your numerical result to the expression given by Thomas & Hasani

\[
\tau_{12}^{*} \approx 1 - 3 \left( \frac{x_2}{\delta} \right)^2 + 2 \left( \frac{x_2}{\delta} \right)^3
\]

and to the linear approximation

\[
\tau_{12}^{*} \approx 1 - (x_2/\delta).
\]

7. In the presence of an external velocity \( U_{e1} = U_0 x_1^m \) inducing a pressure gradient, the Blasius equation (1) is modified as follows

\[
f''' + \frac{m+1}{2} f f'' + m(1-f'^2) = 0
\]

known as the Falkner & Skan equation. Using the same numerical procedure, compute the velocity profile for \( m = 1 \) and \(-0.09043\), and comment your results (Hint – the guest value for the Newton algorithm is of importance here, since this equation sometimes admits two solutions; this guest value can be estimated by continuation from the Blasius solution \( m = 0 \))
References


