HOMEWORK #2

Remarks – (i) Note that you can reuse without demonstration all the results mentioned in the slides of the course (by only citing the considered slide number for instance); (ii) An essential step in your work is to provide valuable comments of your mathematical/numerical results.

Temperature fluctuations in the lowest layer of Earth's atmosphere

The turbulent kinetic energy budget, which has been established during the course, is revisited to include temperature effects. The Navier-Stokes equation, and the energy conservation written for the temperature T, are given by

$$\begin{cases} \dfrac{\partial(\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = -\nabla p + \nabla \cdot \bar{\bar{\boldsymbol{\tau}}} + \rho \boldsymbol{g} \\ \dfrac{\partial(\rho c_p T)}{\partial t} + \nabla \cdot (\rho c_p T \boldsymbol{u}) = -\nabla \cdot \boldsymbol{q} + \bar{\bar{\boldsymbol{\tau}}} : \nabla \boldsymbol{u} + \dfrac{Dp}{Dt} \end{cases}$$

where c_p is the specific heat at constant pressure ($dh = c_p dT$ for a perfect gas), $\boldsymbol{q} = -\lambda \nabla T$ (Fourier's law). We recall that the term $\bar{\bar{\boldsymbol{\tau}}} : \nabla \boldsymbol{u} = \tau_{ij}\, \partial u_i/\partial x_j$ represents the viscous dissipation. Finally, there is no volumetric heat source here ($\dot{q}_v = 0$), and the pressure term will be neglected in the RANS Equation.

1. Derive the equation for the mean temperature $\bar{T}$ by introducing the Reynolds decomposition $T = \bar{T} + \theta$ for the temperature. Show that the total mean heat flux $\bar{q}^t$ can be written as

$$\bar{q}_j^t = -\rho c_p \left(a \frac{\partial \bar{T}}{\partial x_j} - \overline{\theta u_j'} \right)$$

   What does the coefficient a represent? Provide its dimension. How could you simplified the expression of the total heat flux in turbulent regime?

2. Temperature fluctuations θ are involved in the Navier-Stokes equation through the buoyancy force $\rho \boldsymbol{g}$, which can be developed from a Taylor series of the density around the mean temperature,

$$\rho \boldsymbol{g} \simeq (\bar{\rho} + \rho') \boldsymbol{g} \simeq \bar{\rho}(1 - \beta \theta) \boldsymbol{g} \qquad \text{where} \qquad \beta = \left(-\frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_p \right)_{\bar{\rho}, \bar{T}}$$

   is the thermal expansion coefficient, and $g_i = -g\delta_{3i}$ is the gravity with $g \simeq 9.81$ m.s^{-2}. *In the following, the Boussinesq approximation will be applied to the Navier-Stokes equation. The density fluctuations will be thus taken into account in the buoyancy force only.*

   Calculate : (i) the expression of the thermal expansion coefficient for a perfect gas, (ii) the additional term in the Navier-Stokes equation written for the fluctuating velocity component u_i' and (iii) the additional term in the turbulent kinetic energy budget.

3. In order to quantify the importance of the buoyancy force effects, we can compare the additional term in the transport equation for k_t with the production term $\mathcal{P}$, by introducing the Richardson number Ri

$$\mathrm{Ri} \equiv -\frac{\bar{\rho} g \overline{\theta u_3'}/\bar{T}}{\mathcal{P}} \qquad \text{where} \qquad \mathcal{P} = -\bar{\rho}\, \overline{u_1' u_3'} \frac{\partial \bar{U}_1}{\partial x_3}$$

Show that if $d\bar{T}/dx_3 < 0$, the additional term acts as a production term for the turbulent kinetic energy. What is the sign of the Richardson number? Comment the dispersion of a pollutant for an unstable thermal stratification, that is when the temperature decreases with the altitude.

4. Repeat your analysis for the case of a stable thermal stratification.

5. A temperature inversion over the first few hundred meters is regularly observed in the atmospheric layer, leading to the trap of air pollution. Find a recent example of this phenomenon (internet, scientific paper written for a large audience), where the temperature inversion is documented and comment your finding in a few sentences.