HOMEWORK #3

Turbulent boundary layer model

The equation governing the inner part of the mean velocity profile of a turbulent boundary layer (introduced in classroom) is given by

$$\frac{d\overline{U}_1^+}{dx_2^+} = \frac{\tau_{12}^+}{1+\nu_t^+} \tag{1}$$

with $\tau_{12}^+ \simeq 1$ for $x_2/\delta \ll 1$ (the edge of the boundary layer with the wake region is not investigated here).

1. Solve numerically this equation for the case of a mixing-length model $l_m^+ = \kappa x_2^+$ to compute the turbulent viscosity ν_t^+. The value of the von Kármán constant is chosen to be equal to $\kappa \simeq 0.41$. In order to integrate Eq. (1), we can analytically derive the expression for the derivative $d\overline{U}_1^+/dx_2^+$, and then numerically integrate this equation from $x_2^+ = 0$ using a Runge-Kutta algorithm (with Matlab, see reverse side).

2. Estimate the constant B of the log-law by computing the expression

$$B = \overline{U}_1^+ - \frac{1}{\kappa}\ln x_2^+$$

from your numerical solution for $x_2^+ = 200, 300, 400$ and 500. Comment on this result.

3. Compare your numerical solution to the exact analytical solution of Eq. (1), provided by Hinze[2]

$$U_1^+ = \frac{1}{\kappa}\frac{1-\sqrt{1+4\,(\kappa x_2^+)^2}}{2\,\kappa x_2^+} + \frac{1}{\kappa}\ln\left[2\kappa x_2^+ + \sqrt{1+4\,(\kappa x_2^+)^2}\right]$$

4. Repeat the numerical integration with the following mixing-length model including a dumping function, proposed by Van Driest[4]

$$l_m^+ = \kappa x_2^+\left(1 - e^{-x_2^+/A_0^+}\right) \qquad \text{with} \qquad A_0^+ = 26 \tag{2}$$

5. Plot on the same graph both numerical solutions, as well as the viscous sublayer and the inner log laws. Comment.

6. Perform a Taylor series of the functions l_m^+ and $-\overline{u_1' u_2'}^+$ for both mixing-length models, and provide the behaviour of these two functions as $x_2^+ \to 0$.

7. By considering the incompressibility condition $\nabla\cdot\boldsymbol{u}' = 0$, provide the theoretical behaviour of $-\overline{u_1' u_2'}^+$ as $x_2^+ \to 0$.

8. Repeat the numerical integration using the Van Driest model (2) for solving Eq. (1), by using now the expression of Thomas & Hasani[3,5] for τ_{12}^+

$$\tau_{12}^+ \simeq 1 - 3\left(\frac{x_2}{\delta}\right)^2 + 2\left(\frac{x_2}{\delta}\right)^3$$

which improves the linear approximation[1] $\tau_{12}^+ \simeq 1 - (x_2/\delta)$ introduced in Chapter 3 (slide 57). The following numerical value $\mathrm{Re}^+ \simeq 2750$ can be used to compare to experimental data.

References

[1] Bailly, C. & Comte Bellot, G., 2015, *Turbulence* (in english), Springer, Heidelberg.

[2] Hinze, J. O., 1975, *Turbulence*, McGraw-Hill International Book Company, New York.

[3] Thomas, L. C. & Hasani, S. M. F., 1989, Supplementary boundary-layer approximations for turbulent flow, *Journal of Fluids Engineering* (Transactions of the ASME), **111**, 420-427.

[4] Van Driest, E. R., 1956, On turbulent flow near a wall, *Journal of Aeronautical Sciences*, **23**(11), 1007-1011.

[5] White, F., 2005, Viscous fluid flow, 3ed Ed., *McGraw-Hill, Inc.*, New-York (1st Ed. 1974).

Appendix

In order to integrate ordinary differential equation, you can directly used the *ode45.m* function from Matlab. You can also write your own script from the following 4th-order Runge-Kutta algorithm. To integrate the first-order differential equation $\partial \boldsymbol{U}/\partial t = \boldsymbol{F}(\boldsymbol{U}, t)$, consider

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^{n} + \Delta t \left(b_1 \boldsymbol{K}^1 + b_2 \boldsymbol{K}^2 + b_3 \boldsymbol{K}^3 + b_4 \boldsymbol{K}^4 \right)$$

where

$$\begin{cases} \boldsymbol{K}^1 = \boldsymbol{F}(\boldsymbol{U}^n, t^n) \\ \boldsymbol{K}^2 = \boldsymbol{F}(\boldsymbol{U}^n + a_{21}\boldsymbol{K}^1, t^n + c_2 \Delta t) \\ \boldsymbol{K}^3 = \boldsymbol{F}(\boldsymbol{U}^n + a_{32}\boldsymbol{K}^2, t^n + c_3 \Delta t) \\ \boldsymbol{K}^4 = \boldsymbol{F}(\boldsymbol{U}^n + a_{43}\boldsymbol{K}^3, t^n + c_4 \Delta t) \end{cases}$$

and with

$$\begin{array}{c|c} c_i & a_{ij} \\ \hline & b_i \end{array} \qquad\qquad \begin{array}{c|cccc} c_1 = 0 & 0 & & & \\ c_2 = 1/2 & 1/2 & & & \\ c_3 = 1/2 & 0 & 1/2 & & \\ c_4 = 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$