HOMEWORK #4

General remarks about the exercises :

(i) Note that you can reuse without demonstration all the results mentioned in the slides of the course (by only citing the considered slide number for instance) ;

(ii) An essential step in your work is **to provide valuable comments of your mathematical/numerical results.**

Rotating turbulence

In this exercise, the influence of a steadily rotating flow on a turbulence field is examined (applications in turbomachinary flows, in geophysical flows with Earth's atmosphere and oceans among others).

1. Write the Navier-Stokes equations, for an incompressible flow, and in a local frame $\mathcal{R}$ rotating with the flow, at a constant angular velocity $\boldsymbol{\Omega} = \Omega \boldsymbol{x}_3$ around the $\boldsymbol{x}_3$ axis. The general expression (2) of acceleration in a non-inertial reference frame is provided in Appendix.

2. Show that the centrifugal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$ can be put together with the pressure term as follows,
$$p_\star = p - \rho \frac{\Omega^2 r^2}{2}$$
where $r = \sqrt{x_1^2 + x_2^2}$ is the distance between the axis of rotation and a current point $\boldsymbol{x}$.

3. Write the Reynolds Averaged Navier-Stokes (RANS) equation.

4. From the transport equation for the Reynolds stress components (1) given below, write the transport equation for the turbulent kinetic energy k_t and comment your result. What can we state about the work of the Coriolis force ?

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{U}_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} = -\overline{u_j' u_k'} \frac{\partial \overline{U}_i}{\partial x_k} - \overline{u_i' u_k'} \frac{\partial \overline{U}_j}{\partial x_k} - \overline{u_j' (2\boldsymbol{\Omega} \times \boldsymbol{u}')_i} - \overline{u_i' (2\boldsymbol{\Omega} \times \boldsymbol{u}')_j}$$
$$- \frac{\partial \overline{u_i' u_j' u_k'}}{\partial x_k} - \frac{1}{\rho} \left(\overline{u_i' \frac{\partial p_\star'}{\partial x_j}} + \overline{u_j' \frac{\partial p_\star'}{\partial x_i}} \right) + \overline{u_i' (\nu \nabla^2 u_j')} + \overline{u_j' (\nu \nabla^2 u_i')} \quad (1)$$

In the following, we consider a mean shear flow $\overline{U}_1 = S x_2$ and $\overline{U}_2 = \overline{U}_3 \equiv 0$, submitted to a rotation at a constant angular vector $\boldsymbol{\Omega} = \Omega \boldsymbol{x}_3$. The turbulent field is assumed to be initially homogeneous, that is at time $t = 0$,
$$\overline{u_1'^2} = u_0^2, \quad \overline{u_2'^2} = \overline{u_3'^2} = u_0^2/2 \quad \text{and} \quad \overline{u_1' u_2'} = \overline{u_2' u_3'} = \overline{u_1' u_3'} = 0$$
where u_0 is a given constant.

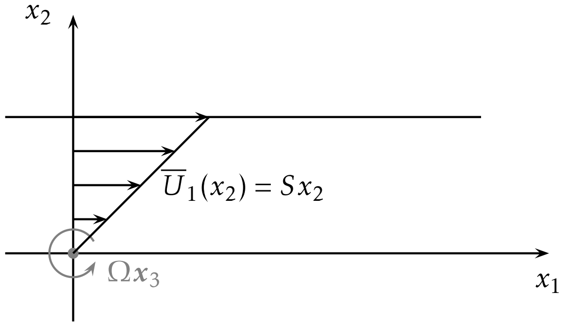

Figure 1 – Sketch of the rotating flow

5. Write the RANS equation for the mean flow (refer to Question 3). Show also that the continuity equation is unaffected by rotation.

6. The problem is now simplified in the framework of the Rapid Distorsion Theory (RDT) introduced by Batchelor & Proudman.[1] Only the influence of the mean flow on turbulence is retained (the so-called fast terms). Interactions of turbulence with itself (slow terms) as well as the viscous terms are neglected. As an example, all the terms of the second line are neglected in Eq. (1), only interactions of $\overline{u'_i u'_j}$ with the mean flow are kept. In this framework, write the transport equations for $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_3'^2}$ and $\overline{u'_1 u'_2}$

7. Write the transport equation for $\overline{u'_1 u'_2}$, and investigate the four following cases :
$\Omega = 0$, $0 < \Omega < S/2$, $\Omega = S/2$ and $\Omega > S/2$.

8. Provide a physical interpretation, by considering the rotation of a fluid particle.

References

[1] Batchelor, G. K. & Proudman, I., 1954, The effect of rapid distortion of a fluid in turbulent motion, *Q. J. Mech. Appl. Math.*, **7**(1), 121-152.

Appendix

The absolute velocity $\boldsymbol{u}(\boldsymbol{\xi}_{/\mathcal{R}_0})$ in a reference inertial frame $\mathcal{R}_0$, see figure 2, is given by

$$\boldsymbol{u}(\boldsymbol{\xi}_{/\mathcal{R}_0}) = \boldsymbol{u}_r(\boldsymbol{x}_{/\mathcal{R}}) + \left.\frac{d\boldsymbol{r}_0}{dt}\right|_{\mathcal{R}_0} + \boldsymbol{\Omega}_{\mathcal{R}/\mathcal{R}_0} \times \boldsymbol{x}$$

Introducing the shorthand notation $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathcal{R}/\mathcal{R}_0}$, the absolute acceleration $\boldsymbol{a}(\boldsymbol{\xi}_{/\mathcal{R}_0})$ in an inertial frame $\mathcal{R}_0$ is given by

$$\boldsymbol{a} = \boldsymbol{a}_r + \boldsymbol{a}_e + \boldsymbol{a}_c \qquad \text{with} \qquad \begin{cases} \boldsymbol{a}_r(\boldsymbol{x}_{/\mathcal{R}}) = \dfrac{D\boldsymbol{u}}{Dt} \\ \boldsymbol{a}_e = \left.\dfrac{d^2\boldsymbol{r}_0}{dt^2}\right|_{\mathcal{R}_0} + \left.\dfrac{d\boldsymbol{\Omega}}{dt}\right|_{\mathcal{R}/\mathcal{R}_0} \times \boldsymbol{x} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) \\ \boldsymbol{a}_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}(\boldsymbol{x}_{/\mathcal{R}}) \end{cases} \tag{2}$$

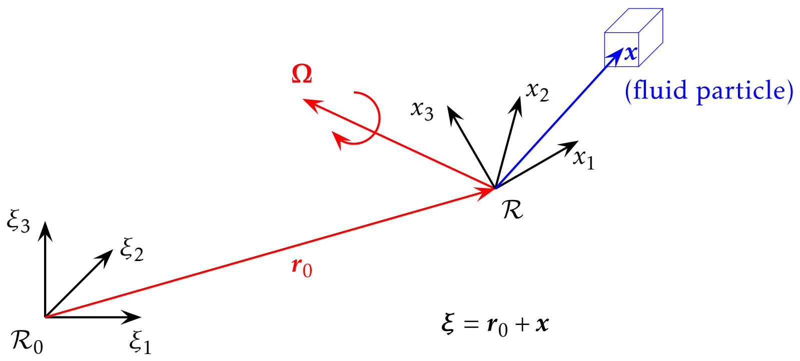

FIGURE 2 – Non-inertial frame $\mathcal{R}$ relative to the reference inertial frame $\mathcal{R}_0$