HOMEWORK #6

Expressions of the dissipation for homogeneous and isotropic turbulence

The dissipation rate of the turbulent kinetic energy is defined as

$$\rho\epsilon = \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}}$$

In this exercise, we consider alternative expressions of the dissipation, widely used to easily estimate numerically or experimentally this quantity, or to provide a particular physical meaning for ϵ. Assumptions used at each step of the calculations must be carefully specified.

1. Show that dissipation can be expressed as follows,

$$\rho\epsilon = \overline{\tau'_{ij}s'_{ij}} = \frac{1}{2}\overline{\tau'_{ij}\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)}$$

2. Show that for incompressible turbulence, the dissipation can be recast as

$$\epsilon = \frac{1}{2}\nu\overline{\left(\frac{\partial u'_i}{\partial x_j}+\frac{\partial u'_j}{\partial x_i}\right)^2} = \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}} + \nu\frac{\partial^2\overline{u'_iu'_j}}{\partial x_i\partial x_j}$$

Deduce that for homogeneous turbulence, the dissipation is exactly equal to ϵ^h

$$\epsilon = \epsilon^h \equiv \nu\overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}$$

3. Show that for homogeneous incompressible turbulence, the dissipation can be written as a function of the vorticity components,

$$\epsilon = \frac{1}{2}\nu\overline{\left(\frac{\partial u'_i}{\partial x_j}-\frac{\partial u'_j}{\partial x_i}\right)^2} = \nu\overline{\omega'_i\omega'_i}$$

4. Show that for isotropic turbulence, the incompressibility condition and the Kàrmàn & Howarth relation leads to $g = f + (r/2)f'$. Deduce the following relation for Taylor scales, $\lambda_f = \sqrt{2}\lambda_g$.

Then, show that for isotropic turbulence,

$$\epsilon = 3\nu\overline{\left(\frac{\partial u'_1}{\partial x_1}\right)^2} + 6\nu\overline{\left(\frac{\partial u'_1}{\partial x_2}\right)^2}$$

and finally, deduce that

$$\epsilon = \frac{15}{2}\nu\overline{\left(\frac{\partial u'_1}{\partial x_2}\right)^2} = 15\nu\overline{\left(\frac{\partial u'_1}{\partial x_1}\right)^2} \qquad \text{and} \qquad \epsilon = 15\nu\frac{\overline{u'^2}}{\lambda_g^2} = 30\nu\frac{\overline{u'^2}}{\lambda_f^2}$$

5. Show that for isotropic turbulence, the dissipation can be expressed from the one-dimensional velocity spectrum as

$$\epsilon = 30\nu\int_0^\infty k_1^2E_{11}^{(1)}(k_1)\,dk_1$$