Some properties of the $k^t - \epsilon$ turbulence model

The $k^t - \epsilon$ turbulence model is widely used in industry for solving the averaged Navier-Stokes equations (with the $k-\omega$-SST and also the Spalart-Allmaras models). The governing equations are recalled below. For an incompressible flow, the conservation of mass and of momentum are given by

$$\frac{\partial U_i}{\partial x_i} = 0$$  (1)

$$\frac{\partial(\rho U_i)}{\partial t} + \frac{\partial(\rho U_i U_j)}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \bar{P} + \frac{2}{3} \bar{\rho} k_i \right) + \frac{\partial}{\partial x_j} \left[ 2(\mu_t + \mu) \bar{S}_{ij} \right]$$  (2)

where the mean deformation tensor reads $\bar{S}_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$. The Reynolds tensor is closed thanks to the introduction of a turbulent viscosity $\mu_t$,

$$-\rho u_i' u_j' = 2\mu_t \bar{S}_{ij} - \frac{2}{3} \rho k_i \delta_{ij}$$  (3)

and this flow-dependent viscosity is determined from the turbulent kinetic energy $k_t$ and an approximate of the dissipation rate $\epsilon \approx \epsilon^h$ valid for high Reynolds number flow (the superscript $h$ will be however disregard to simplify notations) as follows,

$$\mu_t = \rho C_{\mu} \frac{k_t^2}{\epsilon}$$  (4)

Two additional transport equations are solved to compute $k_t$ and $\epsilon$,

$$\begin{cases} 
\frac{\partial(\rho k_i)}{\partial t} + \frac{\partial(\rho k_i U_j)}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \mu + \frac{\mu_t}{\sigma_k} \frac{\partial k_i}{\partial x_j} \right] + \mathcal{P} - \rho \epsilon \\
\frac{\partial(\rho \epsilon)}{\partial t} + \frac{\partial(\rho \epsilon U_i)}{\partial x_i} = \frac{\partial}{\partial x_j} \left[ \mu + \frac{\mu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_j} \right] + \frac{\epsilon}{k_t} (C_{\epsilon_1} \mathcal{P} - C_{\epsilon_2} \rho \epsilon) 
\end{cases}$$  (5)

where $\mathcal{P}$ is the production term of the turbulent kinetic energy $k_t$

$$\mathcal{P} = -\rho u_i' u_j' \frac{\partial U_i}{\partial x_j}$$  (6)

In this model, five calibration constants must be determined. For the standard formulation, they are numerically given by

$$C_{\mu} = 0.09 \quad C_{\epsilon_1} = 1.44 \quad C_{\epsilon_2} = 1.92 \quad \sigma_k = 1.0 \quad \sigma_\epsilon = 1.3$$  (7)

1. Production term

1. Show that the production term can be expressed as $\mathcal{P} = 2\mu_t \bar{S}_{ij}^2$

2. By considering the function $f(\lambda) = (u_i' + \lambda u_j')^2$, show that the Reynolds tensor must satisfy Schwarz’s inequality, that is

$$\bar{u_i'^2} \bar{u_j'^2} \leq \bar{u_i'^2} \bar{u_j'^2}$$
3. For a two-dimensional mean shear flow, \( \overline{U}_1 = \overline{U}_1(x_2) \) and \( \overline{U}_2 = \overline{U}_3 = 0 \), show that the \( k_t - \epsilon \) model imposes the following relationship

\[
9C^2 \frac{\sigma_{12}^2}{U_0} \leq \frac{\epsilon^2}{k_t^2} \sim \omega^2_t
\]

How can one interpret the quantity \( \omega_t \)?

2. Determination of the \( C_{\epsilon_2} \) constant

For grid-generated homogeneous turbulence, the production term \( P \equiv 0 \) is zero and turbulence is simply convected at a constant velocity \( \overline{U}_1 = U_0 \) and decays behind the grid. The two transport equations of the system (5) are reduced to

\[
\begin{align*}
U_0 \frac{\partial k_t}{\partial x_1} & \approx -\epsilon \\
U_0 \frac{\partial \epsilon}{\partial x_1} & \approx -C_{\epsilon_2} \frac{\epsilon^2}{k_t}
\end{align*}
\]

4. Show that the solution to this system of ordinary differential equations can be recast in the following form,

\[
k_t = k_{t0} \left[ 1 + (C_{\epsilon_2} - 1) \frac{\epsilon_0}{k_{t0} U_0} \right]^{-\frac{1}{C_{\epsilon_2}}} \quad \text{and} \quad \frac{\epsilon}{\epsilon_0} = \left( \frac{k_t}{k_{t0}} \right)^{C_{\epsilon_2}}
\]

where \( k_t = k_{t0} \) and \( \epsilon = \epsilon_0 \) at \( x_1 = 0 \).

5. Experimentally, refer to the measurements by Comte-Bellot & Corrsin (1966) among others, the following decaying law is observed \( (k/k_0) \sim (t/t_0)^{-n} \) with \( n \approx 1.3 \), in a frame convected at the velocity vitesse \( U_0 \), consequently \( t = x_1/U_0 \). Deduce a numerical estimate of the constant \( C_{\epsilon_2} \).

6. Provide the evolution of the length scale \( L = k_{t/2}/\epsilon \) and give an interpretation of this scale.

3. Calibration of the \( C_{\mu} \) and \( C_{\epsilon_1} \) constants

For turbulent shear flow, in particular in the log-law of a turbulent boundary layer where \( P \approx \rho \epsilon \), refer to Bradshaw et al. (1967), one also observes that

\[
-\frac{u_1' u_2'}{k_t} \approx 0.30
\]

To determine the value of \( C_{\mu} \), consider the turbulent boundary layer established in a plane channel flow, with \( \overline{U}_1 = \overline{U}_1(x_2), \overline{U}_2 = \overline{U}_3 = 0 \).

6. By examining the balance \( P \approx \rho \epsilon \), show that

\[
-\frac{u_1' u_2'}{k_t} = C_{\mu}^{1/2}
\]

and provide a numerical estimate of \( C_{\mu} \).
7. Provide the expressions of $d\overline{U}_1/dx_2$ and $\epsilon$ inside the log-law of a turbulent boundary layer. Show also that $v_t \approx \kappa u_\tau x_2$ and that $k_t \approx u_\tau^2/\sqrt{\kappa \mu}$.

8. By noting that the two differential equations of the system (5) simplify in the considered flow to

$$0 = P - \rho\epsilon \quad \text{and} \quad 0 = \frac{d}{dx_2} \left( \frac{\mu_t}{\sigma_\epsilon} \frac{d\epsilon}{dx_2} \right) + \frac{\epsilon}{k_t} (C_{\epsilon_1} P - C_{\epsilon_2} \rho \epsilon)$$

show that the compatibility relation must be satisfied,

$$\sigma_\epsilon C_{\mu}^{1/2} (C_{\epsilon_2} - C_{\epsilon_1}) = \kappa^2$$

where $\kappa$ is the von Kármán constant. Comment this result.