Homework #8

# Second-order velocity structure function

The second-order velocity structure function is defined from the fluctuating velocity field by

$$F_{ij}(\boldsymbol{x},\boldsymbol{r}) = \overline{\left[u'_i(\boldsymbol{x}+\boldsymbol{r}) - u'_i(\boldsymbol{x})\right]\left[u'_j(\boldsymbol{x}+\boldsymbol{r}) - u'_j(\boldsymbol{x})\right]} \quad (1)$$

where for homogeneous turbulence, $F_{ij}(\boldsymbol{x},\boldsymbol{r}) = F_{ij}(\boldsymbol{r})$. Notations of the course are used, in particular for the correlation function $R_{ij}(\boldsymbol{r}) = \overline{u'_i(\boldsymbol{x})u'_j(\boldsymbol{x}+\boldsymbol{r})}$. Turbulence is assumed to be isotropic in this exercise.

1. Qualitatively and in a couple of sentences, what are the scales described by the structure function ?

2. Show that
$$F_{ij}(\boldsymbol{r}) = 2R_{ij}(0) - R_{ij}(\boldsymbol{r}) - R_{ij}(-\boldsymbol{r}) \quad (2)$$

3. Write the Kármán & Howarth relation and compute $R_{ii}(r)$. Confirm that $R_{ii}(0) = 3u'^2$.

4. Recall the dual link between $R_{ij}(\boldsymbol{r})$ and the spectral tensor $\phi_{ij}(\boldsymbol{k})$. Recall also the expression of $\phi_{ij}(\boldsymbol{k})$ as a function of the turbulent kinetic energy spectrum $E(k)$, and provide finally the expression of $\phi_{ii}(k)$.

5. By considering the spherical polar coordinates shown in Fig. 1, and noting that $\boldsymbol{k}\cdot\boldsymbol{r} = kr\cos\theta$, demonstrate that
$$R_{ii}(r) = 2\int_0^\infty E(k)\frac{\sin(kr)}{kr}dk \quad (3)$$

6. In a similar way by considering spherical polar coordinates in physical space with the polar axis $r_3$ chosen along the vector $\boldsymbol{k}$, demonstrate the inverse reciprocal relationship
$$E(k) = \frac{1}{\pi}\int_0^\infty R_{ii}(r)\,kr\,\sin(kr)dr \quad (4)$$

7. Compute the limit as $r \to 0$ in Eq. (3) to verify this relation.

8. Deduce from previous expressions that the structure function can be expressed as
$$F_{ii}(r) = 4\int_0^\infty E(k)\left[1 - \frac{\sin(kr)}{kr}\right]dk \quad (5)$$

9. In order to interpret the previous result, we seek to simplify the integral a little bit by considering the following approximation
$$h(\zeta) = 1 - \frac{\sin(\zeta)}{\zeta} \simeq \begin{cases} (\zeta/\pi)^2 & \zeta < \pi \\ 1 & \zeta \geq \pi \end{cases} \quad (6)$$
Justify this reasonable approximation.

10. Split integral (5) in two contributions, and provide an interpretation for both parts. Revisit also your answer to the first question.

11. What use of the structure function could you imagine for modeling, experiments or numerical simulation ?

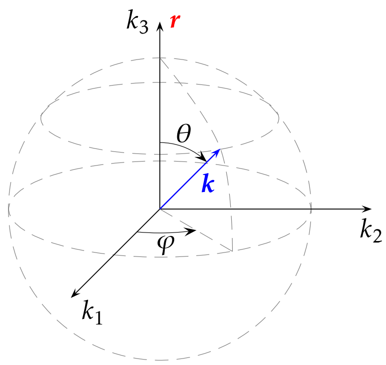


Figure 1 – Spherical polar coordinates in Fourier space with the polar axis $k_3$ chosen along the vector $\boldsymbol{r}$.